Classical and noncommutative ergodic theorems

Markus Haase and Sascha Trostorff

In this project we learn about ergodic theorems in the commutative and the non-commutative setting. In the simplest case, ergodic theorems deal with the problem of convergence as $n \to \infty$ of the sequence of arithmetic means

$$A_n := \frac{1}{n} \sum_{k=0}^{n-1} T^k$$

where T is a bounded linear operator on a Banach space E. One may ask about norm-, strong or weak convergence. If more structure is present, other convergence notions come into play, like almost-everywhere convergence in the case of L^p-spaces.
Ergodic theorems of this form are actually results about an operator *semigroup*, namely the semigroup of powers $\{T^n : n \in \mathbb{N}_0\}$ of a single operator. It is hence natural to widen the scope of our considerations to also include general group or semigroup actions.

If E is a classical function space — $\mathrm{C}(X)$ or L^p — we may speak of a commutative situation. It turns out that ergodic theorems on $\mathrm{C}(X)$ are usually difficult and relatively "rare", whereas there are a couple of important and quite general ergodic theorems on L^p. The simplest, but nevertheless very important results are the "mean ergodic theorems" on Hilbert spaces of von Neumann and Birkhoff–Alaoglu.

Turning to a non-commutative setting, we ask for ergodic theorems on C^*-algebras or, rather, on W^*-algebras (von Neumann algebras). Here, we focus on the classical result by Kovács–Szcs [3] and its generalization by Kmmerer–Nagel [5].

If an ergodic theorem holds, the space E decomposes into $E = \mathrm{fix}(T) \oplus E_0$ where $\mathrm{fix}(T)$ is the *fixed space* of the operator T (or the acting semigroup). It is an example of a decomposition into a highly structured part (no dynamics) and some other part E_0 which is in some sense negligible on the long run.
In the third part of the project we want to understand another important theorem of this kind, the so-called Jacobs-deLeeuw-Glicksberg theory. We confine first to the Hilbert space situation and then consider a non-commutative version due to Bátkai, Groh, Kunszenti-Kovács and Schreiber [1].

The main references are the papers cited above and the books [2, Chapters 8 & 16] and [4, Chapter 9].

This project is suited for 4 students.

References

[1] Bátkai, A., Groh, U., Kunszenti-Kovács, D., and Schreiber, M. Decomposition of operator semigroups on W*-algebras. *Semigroup Forum 84*, 1 (2012), 8–24.

[2] Eisner, T., Farkas, B., Haase, M., and Nagel, R. *Operator theoretic aspects of ergodic theory*, vol. 272 of *Graduate Texts in Mathematics*. Springer, Cham, 2015.

[3] Kovács, I., and Szűcs, J. Ergodic type theorems in von Neumann algebras. *Acta Sci. Math. (Szeged) 27* (1966), 233–246.

[4] Krengel, U. *Ergodic Theorems*, vol. 6 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.

[5] Kümmerer, B., and Nagel, R. Mean ergodic semigroups on W^*-algebras. *Acta Sci. Math. (Szeged) 41*, 1-2 (1979), 151–159.

Classical dynamics versus C*-dynamics

Bálint Farkas and Henrik Kreidler

As seen in the lectures, every classical dynamical system (X, G, α) induces a C*-dynamical system $(\mathrm{C}(X), G, \alpha)$ on the commutative unital C*-algebra $\mathrm{C}(X)$ and thereby gives rise to reduced and full crossed products $\mathrm{C}(X) \rtimes_{\mathrm{r}} G$ and $\mathrm{C}(X) \rtimes_{\mathrm{f}} G$. The goal of this project is to better understand the connection between these objects. How are properties of

(i) the classical dynamical system (X, G, α),

(ii) the C*-dynamical system $(\mathrm{C}(X), G, \alpha)$, and

(iii) the crossed products $\mathrm{C}(X) \rtimes_{\mathrm{r}} G$ and $\mathrm{C}(X) \rtimes_{\mathrm{f}} G$

related to each other?
In the first part of the project we will study basic concepts and examples in the theory of classical dynamical systems, called topological dynamics, using [1] and [3, Chapters 2-3]. In the second part we then translate properties of such dynamical systems to the induced C*-dynamical systems (with the help of [3, Chapter 4]) and crossed products (using [6] and [7]). Finally, depending on time and interest, we will examine how such translations give a new perspective and unlock new methods to address problems of topological dynamics (see, e.g., [2], [4] and [5]).

References

[1] J. Auslander, *Minimal Flows and their Extensions*, Noth-Holland (1988).

[2] N. Edeko, *On equicontinuous factors of flows on locally path-connected compact spaces.* Ergodic Theory Dynam. Systems (2020), 1-18.

[3] T. Eisner, B. Farkas, M. Haase and R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, Springer (2015).

[4] H. Kreidler, *The primitive spectrum of a semigroup of Markov operators*, Positivity 24 (2020), 287-312.

[5] V. Kühner, *What can Koopmanism do for attractors in dynamical systems?*, J. Anal. (2019). https://doi.org/10.1007/s41478-019-00211-2

[6] J. Tomiyama, *Invitation to C^*-algebras and Topological Dynamics*, World Scientific Publishing (1987).

[7] J. Tomiyama, *The interplay between Topological Dynamics and Theory of C^*-Algebras*, Lecture Note no. 2, Global Anal. Research Center Seoul (1992)

C*-uniqueness of group algebras

Martijn Caspers, Mario Klisse and Gerrit Vos

Let $\mathbb{C}[\Gamma]$ be the group algebra of a (discrete) group Γ. In the lecture notes it was discussed that there are two natural norms on $\mathbb{C}[\Gamma]$ such that the completion of $\mathbb{C}[\Gamma]$ with respect to these norms forms a C*-algebra. These yield the reduced and full (=universal/maximal) group C*-algebra of Γ that we denote by $C_r^*(\Gamma)$ and $C_u^*(\Gamma)$. It was also mentioned in the lecture notes that $C_r^*(\Gamma)$ and $C_u^*(\Gamma)$ could be isomorphic. Moreover, this is the case if and only if Γ is so-called amenable.
It could very well be that $\mathbb{C}[\Gamma]$ has other C*-completions than the reduced and universal group C*-algebra. Let us given an example. Consider the group $\mathbb{Z}$. Then we claim that $\mathbb{C}[\mathbb{Z}]$ has more than 1 C*-completion (even though $\mathbb{Z}$ happens to be amenable). Indeed, let $\mathcal{A}(\mathbb{T})$ be the $*$-subalgebra of $C(\mathbb{T})$ generated by the function $z : \mathbb{T} \to \mathbb{C} : \lambda \mapsto \lambda$. It is not difficult to show that $\mathbb{C}[\mathbb{Z}] \simeq \mathcal{A}(\mathbb{T})$ as a $*$-algebra. Further $C_r^*(\mathbb{Z}) \simeq C(\mathbb{T})$ so one completion is given by the reduced group C*-algebra (this was shown in the lecture notes and in fact this isomorphism also proves the previous sentence). Now let $\mathbb{T}_\uparrow$ be the elements in $\mathbb{T}$ with positive imaginary part. For $f \in \mathcal{A}(\mathbb{T})$ set the norm $\|f\| = \sup_{z\in\mathbb{T}_\uparrow} |f(z)|$ (since such a function f is analytic we see that if its norm is 0, it vanishes on a set with an accumulation point and therefore is 0). Let A be the completion of $\mathcal{A}(\mathbb{T})$ with respect to this norm. By continuity the identity map $\mathcal{A}(\mathbb{T}) \to \mathcal{A}(\mathbb{T})$ extends to a bounded map $C(\mathbb{T}) \to A$. This map is however not injective and so the C*-completions cannot be the same. We conclude that $\mathcal{A}(\mathbb{T}) \simeq \mathbb{C}[\mathbb{Z}]$ does not have a unique C*-completion.
The general aim of this project is to study a group Γ that does have a unique C*-completion of $\mathbb{C}[\Gamma]$. This was proved in [Sca]. The example uses crossed product techniques.

References

[Sca] E. Scarparo, *A torsion-free algebraically C*-unique group*, Rocky Mountain Journal of Mathematics **50** (5), 1813–1815.

Graph C*-algebras

Karen Strung

One of the most useful aspects of C*-algebra theory is its ability to interpret other mathematical or physical systems. Typically this involves input information from the system in question and an output C*-algebra. Analysis of the resulting C*-algebra gives us information about the original input system, and vice versa. The theory has been especially successful when the input is a directed graph. Associating a C*-algebra to a directed graph was first shown done by Watani, where directed graphs were associated to the incidence matrices of topological Markov chains and more general shifts of finite type. One then passes to a C*-algebra using methods of Cuntz and Krieger.

In this project we will see that determining properties of a graph C*-algebra is quite tractable in comparison to arbitrary C*-algebras, thanks to the combinatorial techniques available for the underlying graph. For example, one can read off the ideal structure directly from the graph. Conversely, one can also use the structure of the C*-algebra to identify information about the graph, for example, a graph C*-algebra is approximately finite (AF), then corresponding graph contains no loops.

A good reference is the book by Iain Reaburn [1].

References

[1] Iain Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, vol. 103, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005.

Irreducible representations and pure states

Christian Seifert and Fabian Gabel

In this project we consider irreducible representations and pure states of C^*-algebras. Recall that a representation of a C^*-algebra A on a Hilbert space H is a $*$-homomorphism $\pi\colon A \to B(H)$. Then π is called *irreducible* provided that for all closed subspaces $K \subseteq H$ satisfying $\pi(A)K \subseteq K$ we have $K \in \{\{0\}, H\}$. Moreover, a state φ on A is called *pure* provided that for all positive linear functionals $\psi\colon A \to \mathbb{C}$ satisfying $\psi \leq \varphi$ there exists $t \in [0,1]$ such that $\psi = t\varphi$.

In Lecture 5 we have seen that states correspond to cyclic representations. Our first aim in this project is to show that pure states correspond to irreducible representations. On our way we will characterise existence of irreducible subspaces in various ways.

Then we will focus on pure states and show a Krein–Milman type result that extremal points of the positive linear functionals with norm at most one are exactly the pure states. Here, we will also show existence of many pure states.

The above notion of irreducibility is a topological one, since we are focussing on closed subspaces. In a similar manner, one can also define *algebraic irreducibility* for representations if there are no non-trivial invariant subspaces. Surprisingly, these two concepts coincide, which will be the third part of our project.

The project is suitable for 3-4 students.

References

[1] B. Blackadar *Operator algebras*, Springer, Berlin, 2006. Section II.6.

[2] G. J. Murphy *C^*-algebras and operator theory*, Academic Press Inc., Boston, 1990. Chapter 5.

[3] G. Pedersen *C^*-algebras and their automorphism groups*, Academic Press Inc., London, 1979. Section 3.13.

Positive operators on C*-algebras

Ulrich Groh and Rainer Nagel

Positive operators on commutative C*–algebras have a rich spectral theory (traditionally called Perron-Frobenius theory) and an interesting and useful asymptotic behaviour. In this project we learn how these results can be extended to the non-commutative case. This will be done in the following steps.

1. Review of the classical Perron–Frobenius Theory ([7, Chap. 1]) and the main results for commutative C*–Algebras ([5, 6]).
2. Look at the selfadjoint part of a C*-algebra as an ordered Banach space ([4, Chap. 1.3 & 1.5]).
3. Find out what are the „right“ positive operators on these spaces ([8, 9]).
4. Discover symmetry properties of the spectra of such operators ([1]).
5. Describe, using the spectral properties, the asymptotic behaviour of these operators ([2]).

References

[1] U. Groh, *The peripheral point spectrum of schwarz operators on C*-algebras*, Math.Z. **176** (1981), 311 – 318.

[2] U. Groh, *Uniformly ergodic maps on C*-algebras*, Israel J. Math. **47** (1984), 227 – 235.

[3] U. Groh and B. Kümmerer, *Bibounded operators W*-algebras*, Math.Scand. **50** (1982), 269 – 285.

[4] G.K. Pedersen, *C*-algebras and their automorphism groups*, 2 ed., Academic Press, OPT 2018.

[5] H. H. Schaefer, *Invariant ideals of positive operators in $C(x)$, i*, Illinois J. Math. **11** (1967), no. 4, 703–715.

[6] H. H. Schaefer, *Invariant ideals of positive operators in $C(x)$, ii*, Illinois J. Math. **12** (1968), no. 4, 525–538.

[7] H. H. Schaefer, *Banach lattices and positive operators*, Springer–Verlag, Berlin AND Heidelberg AND New York, 1974.

[8] E. Størmer, *Positive linear maps of C^*–algebras*, Foundations of Quantum Mechanics and Ordered Linear Spaces (A. Hartkämter and H. Neumann, eds.), Lecture Notes in Physics, vol. 29, Springer, 1973, pp. 85 – 106.

[9] E. Størmer, *Positive linear maps of operator algebras*, Springer, 2013.

The reduced group C*-algebra of a free group and C*-simplicity

Christian Voigt and Xin Li

The aim of this project to study the structure of the reduced group C^*-algebra of the free group $\mathbb{F}_2$ on two generators. More precisely, we will show that $C^*_r(\mathbb{F}_2)$ is simple and has a unique trace. In order to do this we will first give a quick introduction to free groups in general, and then follow Powers' original argument [5], see also section VII. 7 in [2]. In a nutshell, this illustrates the difference between full and reduced crossed products in a very concrete example.

Time permitting, we may also take a look at C^*-simple groups more generally. Here, a discrete group G is called C^*-simple if its associated reduced group C^*-algebra $C^*_r(G)$ is simple. In recent years there has been remarkable progress on the task of determining which groups are C^*-simple, with a dynamical characterization in terms of the so-called Furstenberg boundary obtained by Kalantar and Kennedy in [4]. For more information and background we refer to [4], [1], [3].

References

[1] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa, *C^*-simplicity and the unique trace property for discrete groups*, Publ. Math. Inst. Hautes Études Sci. **126** (2017), 35–71. MR 3735864

[2] Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996. MR 1402012 (97i:46095)

[3] Pierre de la Harpe, *On simplicity of reduced C^*-algebras of groups*, Bull. Lond. Math. Soc. **39** (2007), no. 1, 1–26. MR 2303514

[4] Mehrdad Kalantar and Matthew Kennedy, *Boundaries of reduced C^*-algebras of discrete groups*, J. Reine Angew. Math. **727** (2017), 247–267. MR 3652252

[5] Robert T. Powers, *Simplicity of the C^*-algebra associated with the free group on two generators*, Duke Math. J. **42** (1975), 151–156. MR 374334

Tensor products of C*-algebras

Martijn Caspers, Mario Klisse and Gerrit Vos

Let A and B be C*-algebras. Then the tensor product $A \otimes B$ is a vector space that again has the structure of a $*$-algebra with

$$(a \otimes b)(c \otimes d) = (ac \otimes bd), \qquad (a \otimes b)^* = a^* \otimes b^*.$$

This project concerns the following question: is there a norm on $A \otimes B$ such that its closure becomes a C*-algebra? If so, is the norm unique? The answers to these questions are well-understood. It turns out that such a norm exists. Moreover, there is always a reduced and a maximal (=full/universal) norm and each other possible (reasonable) norm lies in between these two. It could be that the reduced and maximal norm are equal. In fact, if either A or B are so-called *nuclear* C*-algebras then this is always the case! The idea is to study this proof which is well-explaind in [Mur].

References

[Mur] G. Murphy, *C*-Algebras and Operator Theory*, Academic Press (book).

The Calkin algebra

Hendrik Vogt

It is known from basic functional analysis that the space $K(X)$ of compact operators on a Banach space X is a closed ideal in the space $B(X)$ of bounded linear operators. If H is a separable Hilbert space, then it turns out that $K(H)$ is the only non-trivial closed ideal in $B(H)$ (and that *every* non-trivial ideal in contained in $K(H)$).

In this project we are going to study the *Calkin algebra* $B(H)/K(H)$, for a separable Hilbert space H. By Theorem 4.23 from the ISem lecture notes this quotient is a C^*-algebra. It has a lot of interesting properties (where $\pi\colon B(H) \to B(H)/K(H)$ denotes the quotient map):

- $T \in B(H)$ is a Fredholm operator (i.e., $\dim\ker(T) < \infty$ and $\operatorname{codim}\operatorname{ran}(T) < \infty$) if and only if $\pi(T)$ is invertible.
- The essential spectrum $\sigma_{\text{ess}}(T)$ of a normal operator $T \in B(H)$ coincides with the spectrum of $\pi(T)$.
- Two normal operators $S, T \in B(H)$ have the same essential spectra if and only if they are unitarily equivalent modulo $K(H)$, i.e., $\pi(U^*SU) = \pi(T)$ for some unitary operator $U \in B(H)$.
- The previous bullet point does *not* hold if S, T are only *essentially normal*, i.e., if $\pi(S)$, $\pi(T)$ are normal elements in the Calkin algebra. In this case, S, T are unitarily equivalent modulo $K(H)$ if and only if the essential spectra coincide *and* $\lambda - S$, $\lambda - T$ have the same Fredholm index, for λ not in the essential spectrum. This is an important result due to Brown, Douglas and Fillmore and the starting point of BDF theory.
- A curious problem is the following: is every automorphism φ of $B(H)/K(H)$ an *inner* automorphism, i.e., is it of the form $\varphi(a) = uau^*$, for some unitary element u of the Calkin algebra? It turn out that this problem is *undecidable* in ZFC: it is neither provable nor disprovable that every automorphism is inner. (On the other hand, every automorphism of $B(H)$ is an inner automorphism!)

Starting point of the project is the book [1]. In Section IX.4 one can find a proof of the fact stated above that $K(H)$ is the only non-trivial closed ideal in $B(H)$. In Chapter XI one can find a lot of information about Fredholm operators, the Fredholm index and the essential spectrum, which are important introductory topics. The core of the project shall be decided among the participants: We can concentrate on the beginnings of BDF theory as described above (see [2, 3]), or we can try and tackle the problem of automorphisms of the Calkin algebra (see [4, 5]).

This project is suited for 3 to 4 students.

References

[1] J. B. Conway, *A course in functional analysis, Second edition.* Graduate Texts in Mathematics, 96. Springer-Verlag, New York, 1990.

[2] L. Brown, R. G. Douglas, P. A. Fillmore, Unitary equivalence modulo the compact operators and extensions of C^*-algebras. *Proceedings of a Conference on Operator Theory*, pp. 58–128. Lecture Notes in Math., Vol. 345, Springer, Berlin, 1973.

[3] K. R. Davidson, Essentially normal operators. *A glimpse at Hilbert space operators*, pp. 209–222, Oper. Theory Adv. Appl., 207, Birkhäuser Verlag, Basel, 2010.

[4] N. C. Phillips, N. Weaver, The Calkin algebra has outer automorphisms. *Duke Math. J.* **139** (2007), no. 1, 185–202.

[5] I. Farah, All automorphisms of the Calkin algebra are inner. *Ann. of Math. (2)* **173** (2011), no. 2, 619–661.

Universal C*-algebra of two projections

Amru Hussein

Problems involving pairs of projections play an important role in functional analysis, and some of these can be treated in a unified way by studying the C*-algebra generated by two projections. The aim of this project is to develop the theory for such C*-algebra using the methods learned in the ISem lectures and elaborating mainly the two articles [1] and [2], where the article [1] by Halmos is considered to be a classic in functional analysis on two subspace problems in Hilbert spaces.

The representation theory for such C*-algebras exhibits that there is a representation by functions on $[0, 1]$ with values in the 2×2-matrices. The question when two pairs of projection $\{P, Q\}$ and $\{P', Q'\}$ can be rotated into each other by one unitary transform U, that is $UPU^* = P'$ and $UQU^* = Q'$, can also be analyzed in the context of the theory of C*-algebras.

The methods used are closely related to the ones known from the lecture phase.

This project is suited for 3 to 4 students.

References

[1] P. R. Halmos. Two subspaces. Trans. Amer. Math. Soc., 144:381–389, 1969.

[2] I. Raeburn and A. M. Sinclair. The C*-algebra generated by two projections. Math. Scand., 65:278–290, 1989.

[3] ISem 24 Lecture Notes, 2020/21.

von Neumann algebras

Julian Großmann, Karsten Kruse and Jan Meichsner

The project will deal with a special kind of C^*-algebra, a so-called *von Neumann algebra* or W^**-algebra*. In Theorem 5.19 of the lecture notes we learnt that any C^*-algebra may be seen as a norm-closed $*$-subalgebra of $B(H)$ for some Hilbert space H. However, there exist reasonable coarser locally convex topologies on $B(H)$ which one can consider for several reasons. A von Neumann algebra is a $*$-subalgebra of $B(H)$ that is closed with respect to one (and surprisingly actually all of) those coarser topologies.

The first part of the project will therefore deal with an introduction in such locally convex topologies as well as an algebraic characterisation of the closure with respect to them, the so-called *von Neumann bicommutant theorem* ([1, Thm. 1.2.1], [2, Thm. I.7.1] or [3, Satz 5]).

Afterwards, and depending on the interests of the participants, we will deal with a more advanced abstract characterisation of von Neumann algebras showing that they can be seen as C^*-algebras with a predual, see [4, Thm. 1.16.7], and/or a characterisation of commutative von Neumann algebras ([4, Thm. 1.18.1]).

This project is suited for 3 to 4 students.

References

[1] W. Arveson. *An Invitation to C^*-Algebras.* volume 39 of *Grad. Texts in Math.* Springer-Verlag, New York, 1976. doi: 10.1007/978-1-4612-6371-5

[2] K. R. Davidson, *C^*-algebras by Example.* American Mathematical Society, Providence, RI, 1996. doi: 10.1090/fim/006

[3] J. v. Neumann, Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren. *Math. Ann.*, 102(1): 370–427, 1930 (in German). doi: 10.1007/BF01782352

[4] S. Sakai. *C^*-Algebras and W^*-Algebras.* volume 60 of *Ergeb. Math. Grenzgeb. (2).* Springer-Verlag, Berlin, Heidelberg and New York, 1971. (republished in *Classics Math.*, 1998. doi: 10.1007/978-3-642-61993-9)

Quantum families of maps

Piotr Mikołaj Sołtan

The first fundamental theorem of C*-algebras characterizes commutative unital C*-algebras as algebras of the form $\mathrm{C}(X)$ with X a compact topological space. Clearly a non-commutative C*-algebra A is not of this form, but one can encounter some fascinating mathematics by treating A *as if* it were an algebra of functions on some space. We say that A is the algebra of continuous functions on a *quantum space*. This is the basic idea of *non-commutative geometry* and *non-commutative topology*. The aim of this project is to study the non-commutative analog of families of continuous maps between quantum spaces continuously and, in particular, the non-commutative analog of the space space $\mathrm{C}(X,Y)$ of continuous maps between two topological spaces.
The specific goals of the project are to

1. extend the Gelfand-Naimark theorem to a a categorical statement, as indicated in [1, Section 3.12],
2. define quantum families of maps and the analog of $\mathrm{C}(X,Y)$,
3. prove existence of the quantum space of maps between two quantum spaces (subject to additional assumptions),
4. provide interesting examples of quantum families of maps and quantum spaces of maps.

Time permitting, one can also carry this further to include the study of additional structure on C*-algebras associated to some quantum families of maps.
The theory of quantum families of maps was initiated in [4] and continued in [3] as well as other publications. It is thoroughly based on the theory of C*-algebras developed in [1, 2]
This project is suited for 3–4 students.

References

[1] ISem 24 Lecture Notes, 2020/21.

[2] G.K. Pedersen: C*-*algebras and their automorphism groups.* Academic Press, 1979.

[3] P.M. Sołtan: Quantum families of maps and quantum semigroups on finite quantum spaces. *J. Geom. Phys.* **59** (2009), 354–368.

[4] S.L. Woronowicz: Pseudospaces, pseudogroups and Pontriagin duality. *Mathematical problems in theoretical physics (Proc. Internat. Conf. Math. Phys., Lausanne, 1979)*, Lecture Notes in Phys. **116** (1980), pp. 407–412.