Γ

$\Gamma = (V,E)$, $|V| = n$, $|E| = m$, no

$\varepsilon \in M_n(\{0,1\})$ adjacency matrix

① $\xrightarrow{C^*}$

$$C^*(\Gamma) := C^*\left(\begin{array}{l} p_v \text{ projections}, v \in V \\ s_e \text{ part. isom.}, e \in E \end{array} \;\middle|\; \begin{array}{l} p_v p_w = 0 \quad v \neq w \\ s_e^* s_e = p_{r(e)} \\ \sum\limits_{\substack{e \in E \\ s(e)=v}} s_e s_e^* = p_v \;,\; s^{-1}(v) \neq \emptyset \end{array} \right)$$

$$\Rightarrow \sum_{v \in V} p_v = 1 \;, \quad s_e^* s_f = \delta_{ef}\, p_{r(e)}$$

$C^*(\Gamma)$

② Compact QGroups

Def. [Woronowicz 80's]: Let $n \in \mathbb{N}$. (A,u) Compact Matrix QG, if

(i) A unital C^*-algebra generated by u_{ij}, $i,j \in \{1,\dots,n\}$

(ii) $u = (u_{ij})$, $\bar{u} = (u_{ij}^*)$ invertible

(iii) $\Delta: A \to A \otimes_{\min} A$, $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ $*$-hom.

Fundamental Theorem [Gelfand-Naimark & Woronowicz 80's]: (A,u) CMQG.
A commutative $\Longleftrightarrow$ $A \cong C(G)$, $G \subseteq GL_n(\mathbb{C})$ compact group

Ex. [Sh. Wang 90's]: $C(S_n^+) := C^*(u_{ij}, 1 \leq i,j \leq n \mid u_{ij} \text{ proj.}, \sum_k u_{ik} = \sum_k u_{kj} = 1)$

$\Downarrow$

$C(S_n)$ $\Rightarrow$ $S_n \subseteq S_n^+$ "more q. permutations"

③ G^+_{aut}

$\sigma: V \hookrightarrow\!\!\!\rightarrow V$, $(i,j) \in E \Leftrightarrow (\sigma(i), \sigma(j)) \in E$

$$\mathrm{Aut}(\Gamma) = \{\sigma \in S_n \mid \sigma \varepsilon \sigma^{-1} = \varepsilon\}$$
$$= \{\sigma \in S_n \mid \sigma \varepsilon = \varepsilon \sigma\} \subseteq S_n$$

$$\Rightarrow C(\mathrm{Aut}(\Gamma)) = C(S_n) / \langle u\varepsilon = \varepsilon u \rangle \Leftrightarrow u_{ji} u_{\ell k} = 0 \text{ if } \varepsilon_{ik} \neq \varepsilon_{j\ell}$$

[Banica 2005] $C(G^+_{aut}(\Gamma)) := C(S_n^+) / \langle u\varepsilon = \varepsilon u \rangle$ Ex: $\mathrm{Aut}(\boxtimes) = S_4$, $G^+_{aut}(\boxtimes) = S_4^+$

$$\Rightarrow \begin{array}{ccccc} S_n & \subseteq & & S_n^+ & \\ \cup\! \mid & & & \cup\! \mid & \\ \mathrm{Aut}(\Gamma) & \subseteq & G^*_{aut}(\Gamma) \subseteq & G^+_{aut}(\Gamma) & \end{array}$$

"more q. automorphisms" [Bichon 2003]

④ QSym

classical group action on a space

$\mathring{\alpha}: G \times X \to X$ & Prop.

$\Rightarrow \alpha: C(X) \to C(G \times X) \cong C(G) \otimes C(X)$
$\quad f \mapsto f \circ \mathring{\alpha}$ & Prop.

Def.: $(C(G),u)$ CMQG acts on $C^*(\Gamma)$ from left & right, if

$\alpha: C^*(\Gamma) \to C(G) \otimes C^*(\Gamma)$ $*$-hom $\beta: \dots$

$p_i \mapsto \sum_{k=1}^{n} u_{ik} \otimes p_k$ & $\sum u_{ki} \dots$

$s_{e_j} \mapsto \sum_{\ell=1}^{m} u_{s(e_j)s(e_\ell)} u_{r(e_j)r(e_\ell)} \otimes s_{e_\ell}$ …

⑤ $G^+_{aut}(\Gamma) = QSym(C^*(\Gamma))$

THM [Schmidt, W. 2018]: $\Gamma = (V,E)$ finite graph, no multiple edges.

(a) $G^+_{aut}(\Gamma) \curvearrowright C^*(\Gamma)$

(b) $G \curvearrowright C^*(\Gamma)$ CMQG $\Rightarrow$ $G \subseteq G^+_{aut}(\Gamma)$

Hence, $QSym(C^*(\Gamma)) = G^+_{aut}(\Gamma)$. "$C^*(\Gamma)$ preserves q. symmetry"

Proof: (b) $u_{ij} \in C(G)$ proj., $\sum_k u_{ik} = \sum u_{kj} = 1$ ✓

$$\sum_k u_{r(e_j)k} \otimes p_k = \alpha(p_{r(e_j)}) = \alpha(s_{e_j}^* s_{e_j}) = \sum_{\ell,\ell'} (u_{s(e_j)s(e_\ell)} u_{r(e_j)r(e_\ell)})^* (u_{s(e_j)s(e_{\ell'})} u_{r(e_j)r(e_{\ell'})}) \otimes \underbrace{s_{e_\ell}^* s_{e_{\ell'}}}_{\delta_{\ell\ell'} p_{r(e_\ell)}}$$

$$\Rightarrow u_{r(e_j)k} = \sum_{i \in V, (i,k) \in E} (u_{s(e_j)i} u_{r(e_j)k})^* (u_{s(e_j)i} u_{r(e_j)k})$$

$$u_{r(e_j)k} \Big(\sum_i u_{s(e_j)i}\Big) u_{r(e_j)k} = \sum_{i \in V} (u_{s(e_j)i} u_{r(e_j)k})^* (u_{s(e_j)i} u_{r(e_j)k})$$

$\Rightarrow u_{s(e_j)i} u_{r(e_j)k} = 0$, $\varepsilon_{ik} = 0$

⑥ Questions

Q1: Properties of $\Gamma \rightsquigarrow$ Properties of $C^*(\Gamma) \rightsquigarrow$ Prop. of $G^+_{aut}(\Gamma)$

Q2: $K_0(C(S_n^+)) = \mathbb{Z}^{(n-1)^2+1}$, $K_1(C(S_n^+)) = \mathbb{Z}$ [Chr. Voigt 05]
$K_i(G^+_{aut}(\Gamma)) = ?$ $C^*_{red}(G^+_{aut}(\Gamma))$ simple?
[more: 1710.06199, Sect. 3]

Q3: $C^*(\text{Hypergraph}) = ?$ G^+_{aut}? $QSym = G^+_{aut}$?

THANK YOU!

Quantum symmetry groups of graph C*-algebras
arXiv: 1706.08833, Canadian Math. Bull.

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