QUANTUM SYMMETRIES OF FINITE GRAPHS

– A SURVEY –

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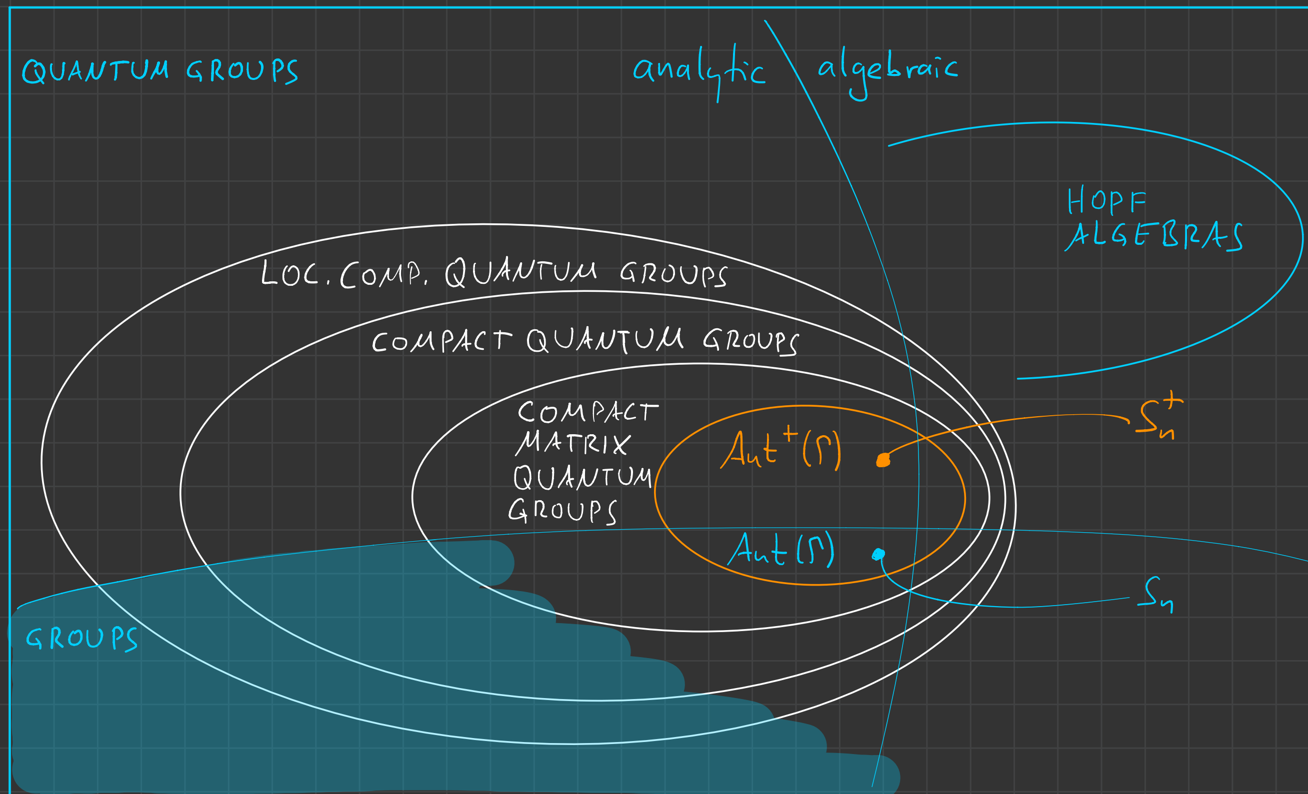
THE CONTEXT

THE OBJECTS

SOME RESULTS

- QSYM(Γ)
- QIT
- $C^*(\Gamma)$

CONTEXT: QUANTUM GROUP NOTIONS



CONTEXT: QUANTUM MATHEMATICS

CLASSICAL	NONCOMMUTATIVE
TOPOLOGY	C^*-ALGEBRAS
MEASURE THEORY	VON NEUMANN ALG.
PROBABILITY TH.	FREE PROB., QU. PROB
DIFF. GEOMETRY	NONCOMM. GEOMETRY
(LOC. COMP.) GROUPS	(LOC. COMP.) QU. GROUPS
INFORMATION TH.	QU. INFORMATION TH.
COMPLEX ANALYSIS	FREE ANALYSIS

THE CONTEXT

THE OBJECTS

SOME RESULTS

- QSYM(Γ)
- QIT
- $C^*(\Gamma)$

THE OBJECTS: COMPACT QUANTUM GROUPS

Def. [Woronowicz 1980s]: $n \in \mathbb{N}$. $G = (A, u)$ compact matrix qu. group, (CMQG)

if

(i) $A = C^*(1, u_{ij},\ 1 \leq i,j \leq n)$

(ii) $u = (u_{ij})_{i,j=1,\dots,n}$, $\bar{u} = (u_{ij}^*)_{i,j=1,\dots,n} \in M_n(A)$ invertible

(iii) $\Delta : A \longrightarrow A \otimes_{min} A$, $u_{ij} \longmapsto \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ $*$-hom.

Fundamental Thms [Woronowicz 1980s]: (A, u) CMQG.

1) A commutative $\xLeftrightarrow{+\text{[Gelfand-Naimark 1940s]}}$ $\exists\, G \subseteq GL_n(\mathbb{C})$ compact group: $A \cong C(G)$

2) Haar integration: $\exists!\, h : A \to \mathbb{C}$ Haar state $((id \otimes h) \circ \Delta = (h \otimes id) \circ \Delta = h)$

3) Algebraization: $\exists\, (A_0, \Delta_{|A_0}, S, \varepsilon) \subseteq A$ dense Hopf algebra

4) Categorial representation theory: Tannaka-Krein / Schur-Weyl type duality

THE OBJECTS: QUANTUM PERMUTATIONS

Def. [Wang 1990s]: $S_n^+ := (C(S_n^+), u)$ free symmetric qu. group

$$C(S_n^+) := C^*\Big(1, u_{ij},\ 1 \le i,j \le n \ \Big|\ u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1\Big)$$

"magic unitary"

Check: S_n^+ is a CMQG ✓

> **Def.** [Woronowicz 1980s]: $n \in \mathbb{N}$. $G = (A, u)$ compact matrix qu. group, (CMQG)
> if (i) $A = C^*(1, u_{ij},\ 1 \le i,j \le n)$
> (ii) $u = (u_{ij})_{i,j=1,\dots,n}$, $\bar{u} = (u_{ij}^*)_{i,j=1,\dots,n} \in M_n(A)$ investible
> (iii) $\Delta : A \longrightarrow A \otimes_{min} A$, $u_{ij} \longmapsto \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ $*$-hom.

Check: $S_n \subseteq S_n^+$ more quantum permutations!

$\sigma \in M_n(\mathbb{C})$ magic unitary $\iff$ σ permutation matrix

hence $C(S_n^+) \longrightarrow\!\!\!\!\!\rightarrow C(S_n)$
$u_{ij} \longmapsto ev_{ij}$ $\quad (ev_{ij}(\sigma) = \sigma_{ij})$

OPEN: $\exists\, S_n \subsetneq G \subsetneq S_n^+$? i.e.: $C(S_n^+) \overset{\not\cong}{\longrightarrow\!\!\!\!\!\rightarrow} B \overset{\not\cong}{\longrightarrow\!\!\!\!\!\rightarrow} C(S_n)$, $w_{ij} \mapsto \sum_k w_{ik} \otimes w_{kj}$ $*$-hom.
$u_{ij} \longmapsto w_{ij} \longmapsto ev_{ij}$

THE OBJECTS: QUANTUM PERMUTATIONS

permutations $\qquad\qquad$ quantum permutations

$$S_4 \quad \subsetneq \quad S_4^+$$

$$\cup \qquad\qquad\qquad\qquad „\cup“$$

$\boxed{\begin{array}{l} \exists\, \pi : C(S_4^+) \longrightarrow C^*(1, p, q \text{ projections}) \\ \qquad u \longmapsto \begin{pmatrix} 0 & p & 0 & 1-p \\ 0 & 1-p & 0 & p \\ q & 0 & 1-q & 0 \\ 1-q & 0 & q & 0 \end{pmatrix} \\ \Longrightarrow C(S_4^+) \text{ noncommutative } (pq \neq qp) \\ \Longrightarrow C(S_4^+) \overset{\not\cong}{\twoheadrightarrow} C(S_4) \end{array}}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}$$

THE OBJECTS: QUANTUM SYMMETRY OF n POINTS

n points: $X_n = \{1, \dots, n\}$, $C(X_n) = C^*(p_1, \dots p_n \text{ proj.} \mid \sum_k p_k = 1)$

$S_n \curvearrowright X_n$: $\alpha : S_n \times X_n \longrightarrow X_n$, $(\sigma, i) \longmapsto \sigma(i)$

$S_n^+ \curvearrowright X_n$: $\alpha : C(X_n) \longrightarrow C(S_n^+) \otimes C(X_n)$ $*$-hom.

$$p_i \longmapsto \sum_k u_{ik} \otimes p_k =: p_i'$$

$$\left[p_i'^2 = \sum_{k,l} u_{ik} u_{il} \otimes \underbrace{p_k p_l}_{=\delta_{kl} p_k} = \sum_k \underbrace{u_{ik}^2}_{=u_{ik}} \otimes p_k = p_i' \,,\; p_i'^* = p_i' \,,\; \sum p_k' = 1 \right]$$

S_n^+ maximal with this action! $\Longrightarrow$ $S_n^+ = QSym(n \text{ points})$

$$\left[\sum_k v_{ik}^2 \otimes p_k = \sum_{k,l} v_{ik} v_{il} \otimes p_k p_l = p_i'^2 = p_i' = \sum_k v_{ik} \otimes p_k \implies v_{ik}^2 = v_{ik} \right]$$

THE OBJECTS: QUANTUM AUTOMORPHISM GROUPS OF GRAPHS

$\Gamma = (\{1,\dots,n\}, E)$ finite graph with adj. matrix $\varepsilon \in M_n(\{0,1\})$

$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n$ autom. group (symmetries of Γ)

Def. [Banica 2005]: $\mathrm{Aut}^+(\Gamma)$ quantum automorphism group

$$C(\mathrm{Aut}^+(\Gamma)) := C^*\left(u_{ij} \;\middle|\; u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1,\ u\varepsilon = \varepsilon u\right)$$

$$\boxed{\begin{array}{ll} \mathrm{Aut}^+(\Gamma) & \text{[Banica 2005]} \\ \cup\!\!\mid & \\ \mathrm{Aut}^*(\Gamma) & \text{[Bichon 2003]} \\ \cup\!\!\mid & \\ \mathrm{Aut}(\Gamma) & \end{array}}$$

$$\begin{array}{ccc} \mathrm{Aut}^+(\Gamma) & \subseteq & S_n^+ \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathrm{Aut}(\Gamma) & \subseteq & S_n \end{array}$$

$$\begin{tikzcd} C(\mathrm{Aut}^+(\Gamma)) \arrow[d, two heads] & C(S_n^+) \arrow[l, two heads] \arrow[d, two heads] \\ C(\mathrm{Aut}(\Gamma)) & C(S_n) \arrow[l, two heads] \end{tikzcd}$$

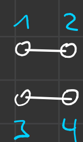
Γ has quantum symmetries $:\Longleftrightarrow$ $\mathrm{Aut}(\Gamma) \subsetneq \mathrm{Aut}^+(\Gamma)$ $\Longleftrightarrow$ $C(\mathrm{Aut}^+(\Gamma))$ noncomm.

THE OBJECTS: QUANTUM AUTOMORPHISM GROUPS OF GRAPHS

$\Gamma = (\{1,\dots,n\}, E)$ finite graph with adj. matrix $\varepsilon \in M_n(\{0,1\})$

$$C(\mathrm{Aut}^+(\Gamma)) := C^*\Big(u_{ij} \;\Big|\; u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1,\ u\varepsilon = \varepsilon u\Big)$$

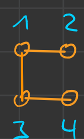
Ex.1: $\Gamma =$  n points $\qquad \mathrm{Aut}^+(\Gamma) = S_n^+ \neq S_n$, $n \geq 4$, has qsym

Ex.2: $\Gamma =$ 

$$\varepsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$C(\mathrm{Aut}^+(\Gamma)) \twoheadrightarrow C^*(p, q \text{ proj.})$, has qsym

$$u \longmapsto \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

Ex.3: $\Gamma =$ 

$$\varepsilon = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad u = \begin{pmatrix} u_{11} & 0 & 1-u_{11} & 0 \\ 0 & u_{11} & 0 & 1-u_{11} \\ 1-u_{11} & 0 & u_{11} & 0 \\ 0 & 1-u_{11} & 0 & u_{11} \end{pmatrix}$$

$\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}(\Gamma) = \mathbb{Z}_2$, no qsym

THE CONTEXT

THE OBJECTS

SOME RESULTS

- $\mathrm{QSYM}(\Gamma)$
- QIT
- $C^*(\Gamma)$

SOME RESULTS: EXISTENCE OF QSYM, SPECIFIC EXAMPLES

NO QSYM: odd graphs O_k, Hamming $H(n,3)$, Johnson $J(n,2)$, Kneser $K(n,2)$, Petersen, Moore (diameter 2), cubic distance-transitive (order ≥ 10), Shrikhande, Paley P_9, P_{13}, P_{17}, some circulant graphs, ...

[Banica, Bichon, Chenevier, Schmidt]

$\Gamma_{Petersen} =$  Petersen graph

[Banica–Bichon 2007]: has qsym?
[Schmidt 2018]: no!

Proof: general fact: $u\varepsilon = \varepsilon u \iff (u_{ii'} u_{ks} = 0 \text{ if } \varepsilon_{ik} \neq \varepsilon_{i's})$

properties of $\Gamma_{Petersen}$: (1) given $\Lambda \Rightarrow$ ∘-×-∘
(2) given ∘-×-∘ $\Rightarrow \exists! \Lambda$

Case 1: Let i, i', ℓ, ℓ' be vertices with i ∘-×-∘ ℓ, i' ∘-×-∘ ℓ'

$\overset{(2)}{\Longrightarrow}$ $\exists!\ k, k'$:

$\Longrightarrow u_{ii'} u_{\ell\ell'} = u_{ii'} \left(\sum_s u_{ks}\right) u_{\ell\ell'} \overset{(2)}{=} u_{ii'} u_{kk'} u_{\ell\ell'}$
$\overset{[\dots]}{=} u_{ii'} u_{kk'} u_{\ell\ell'} u_{ii'} = u_{ii'} u_{\ell\ell'} u_{ii'}$

$\Longrightarrow u_{ii'} u_{\ell\ell'}$ selfadjoint, i.e. $u_{ii'} u_{\ell\ell'} = u_{\ell\ell'} u_{ii'}$

$\Longrightarrow C(\mathrm{Aut}^+(\Gamma_{Petersen}))$ commutative $\Longrightarrow$ no qsym. □

QSYM: complete graphs, complete bipartite, cycles, cube, folded cube FQ_n (n odd), Clebsch, 4x4 rook's, crown graphs, Hamming $H(n,k)$ ($k>3$), Higman-Sims, some undirected trees, ...

[Banica, Bichon, Fulton, Schmidt]

OPEN: $J(6,3)$, $J(n,k)$ ($k\geq 3$), Paley P_k ($k>17$), Tutte 12-cage, ... qsym?

SOME RESULTS: EXISTENCE OF QSYM, TOOLS

[Schmidt 2020]: Γ has two disjoint automorphisms $\Rightarrow$ Γ has qsym

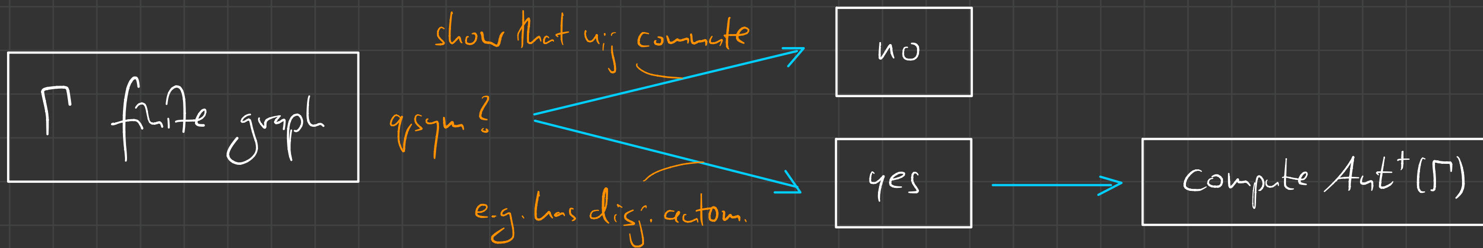
$\tau_1, \tau_2 \in \mathrm{Aut}(\Gamma)$ disjoint $:\Leftrightarrow$ $\forall i \in V: \begin{pmatrix} \tau_1(i) \neq i \Rightarrow \tau_2(i) = i \\ \tau_2(i) \neq i \Rightarrow \tau_1(i) = i \end{pmatrix}$

idea:
$$\begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow u \mapsto \begin{pmatrix} p & 1-p & 0 & & 0 \\ 1-p & p & & & \\ 0 & & q & 1-q & 0 \\ & & 1-q & q & \\ 0 & & 0 & & 1 \end{pmatrix}$$

[Nechita, Schmidt, W. 2020]: Sinkhorn type algorithm for testing "$C(\mathrm{Aut}^+(\Gamma))$ noncomm."

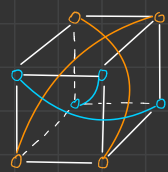
[Eder, Levandovskyy, Schanz, Schmidt, Steenpass, W. 2019]: Computer algebra approach (Gröbner basis)

SOME RESULTS: EXISTENCE OF QSYM, $\mathrm{Aut}^+(\Gamma) = ?$



[Bichon 2003, 2004]: $\mathrm{Aut}^+(\Gamma \sqcup \ldots \sqcup \Gamma) = \mathrm{Aut}^+(\Gamma) \wr_* S_n^+$ (disjoint product)

[Banica–Bichon 2007, Chassaniol 2016]: similarly for direct, Cartesian, lexicographic products

[Schmidt 2020]: $FQ_n =$  folded cube, n odd $\Rightarrow \mathrm{Aut}^+(FQ_n) = SO_n^{-1}$

$$C(SO_n^{-1}) := C^*\Big(u_{ij} : u_{ij} = u_{ij}^*,\ u^t u = u u^t = 1,\ u_{ij}u_{kl} = \begin{cases} u_{kl}u_{ij} & i \neq k \text{ and } j \neq l \\ -u_{kl}u_{ij} & i = k \text{ or } j = l \end{cases},\ \sum_{\sigma \in S_n} u_{\sigma(1)1} \cdots u_{\sigma(n)n} = 1\Big)$$

strict

OPEN: Aut^+ of FQ_n, n even, 4×4 rook's, Higman–Sims, … ?

$\exists \Gamma$: $\mathrm{Aut}^+(\Gamma) \neq \mathrm{Aut}(\Gamma) \cong A_n$ alternating group? $A_n^+ := ?$

SOME RESULTS: PROBABILISTIC STATEMENTS

	have symmetries	have quantum symmetries
graphs	$\mathbb{P} \to 0$ as $N \to \infty$ [Erdös–Renyi 1963]	$\mathbb{P} \to 0$ as $N \to \infty$ [Lupini–Mancinska–Roberson 2017]
trees	$\mathbb{P} \to 1$ as $N \to \infty$ [Erdös–Renyi 1963]	$\mathbb{P} \to 1$ as $N \to \infty$ [Junk–Schmidt–W. 2019]

clearly $\mathrm{Aut}^+(\Gamma) = \{e\} \Longrightarrow \mathrm{Aut}(\Gamma) = \{e\}$

OPEN: "$\Longleftarrow$" ? $\quad \exists \Gamma : \mathrm{Aut}(\Gamma) = \{e\},\ \mathrm{Aut}^+(\Gamma) \neq \{e\}$?

[Chirvasitu–Wasilewski 2020]: $\mathbb{P} \to 0$ also for quantum graphs

THE CONTEXT

THE OBJECTS

SOME RESULTS

- QSYM(Γ)
- QIT
- $C^*(\Gamma)$

SOME RESULTS: QUANTUM ISOMORPHISMS OF GRAPHS

& [Lupini–Mančinska–Roberson 2020]

Def. [Atserias–Mančinska–Roberson–Šámal–Severini–Varvitsiotis 2019]: $\Gamma_i = (V_i, E_i)$, $|V_i| = n$, $i = 1,2$

$\Gamma_1 \cong_q \Gamma_2 \;:\Longleftrightarrow\; \exists\, \pi : C(S_n^+) \to A$, A some C^*-algebra: $\pi(u)\,\varepsilon_1 = \varepsilon_2\,\pi(u)$

> $\Gamma = (\{1,\dots,n\}, E)$ finite graph with adj. matrix $\varepsilon \in M_n(\{0,1\})$
> $Aut(\Gamma) := \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n$ autom. group (symmetries of Γ)
>
> **Def.** [Banica 2005]: $Aut^+(\Gamma)$ quantum automorphism group
> $C(Aut^+(\Gamma)) := C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1,\ u\varepsilon = \varepsilon u)$

e.g.: $v \in M_n(M_m(\mathbb{C}))$, $v_{ij} = v_{ij}^* = v_{ij}^2$, $\sum_k v_{ik} = \sum_k v_{kj} = 1$, $v\varepsilon_1 = \varepsilon_2 v$

$m = 1$: $v \in S_n$, $\varphi := Ad(v) : \Gamma_1 \xrightarrow{\cong} \Gamma_2$ $\big(i \sim j \Longleftrightarrow \varphi(i) \sim \varphi(j)\big)$

$\Gamma_1 \cong \Gamma_2 \;\substack{\Longrightarrow \\ \not\Longleftarrow}\; \Gamma_1 \cong_q \Gamma_2$

There are graphs which are quantum isomorphic but not isomorphic!

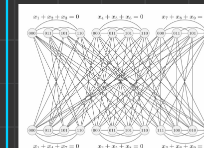


Figure 1: $X(\mathcal{F})$ for the Mermin magic square game [7].

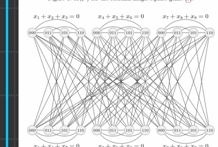


Figure 2: $X(\mathcal{F}_0)$ for the Mermin magic square game [7].

1712.01820

OPEN: smaller examples? $(16 \leq |V| \leq 23\ ?)$

SOME RESULTS: LINKS WITH QUANTUM INFORMATION

nonlocal game:
- given $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$, $|V_1| = |V_2|$
- referee gives $v_A, v_B \in V_1 \dot\cup V_2$ to Alice & Bob
- Alice & Bob reply with $w_A, w_B \in V_1 \dot\cup V_2$
- win, if: (1) $|\{v_A, v_B, w_A, w_B\} \cap V_i| = 2$, $i = 1, 2$

  (2) the 2 vertices from Γ_1 are linked $\Longleftrightarrow$ the 2 vertices from Γ_2 are linked

Fact: Alice & Bob win classically $\Longleftrightarrow$ $\Gamma_1 \cong \Gamma_2$

[Atserias–Mančinska–Roberson–Šámal–Severini–Varvitsiotis 2019]: Alice & Bob win with quantum strategy $\Longleftrightarrow$ $\Gamma_1 \cong_q \Gamma_2$

SOME RESULTS: LINKS WITH QUANTUM INFORMATION

[Atserias–Mančinska–Roberson–Šámal–Severini–Varvitsiotis 2019]: $\Gamma \cong_q \Gamma'$ ⇝ nonlocal games
arXiv 2016

[Lupini–Mančinska–Roberson 2020]
arXiv 2017
[Musto–Reutter–Verdon 2018] & Vicary

$\Gamma \cong_q \Gamma'$ ⇝ $Aut^+(\Gamma)$

& qu. orbitals, linear binary constraints systems, ...

& qu. functions, Frobenius algebras, LBCS, ...

& qu. Latin squares, ...

[Soltan 2019]: C^*(synchronous game)

[Helton–Meyer–Paulsen–Satriano 2019]: graph homomorphism game,
[Brannan–Ganesan–Harris 2020] quantum graph hom. game, bisynchronous,
[Paulsen–Rahaman 2019]
[Eifler 2020] nonlocal game on quantum metric spaces

[Roberson–Schmidt 2020]: Γ has 3 disj. autom. $\Longrightarrow$ Γ has nonlocal symmetry

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$ $\not\Uparrow$

Γ has 2 disj. autom. $\xRightarrow{\text{[Schmidt 2020]}}$ Γ has quantum symmetry

SOME RESULTS: QUANTUM LOVASZ THEOREM

H, Γ graphs. $\varphi: H \to \Gamma$ graph homomorphism $:\Longleftrightarrow$ $(i \sim j \Rightarrow \varphi(i) \sim \varphi(j))$

[Lovasz 1967]: $\Gamma_1 \cong \Gamma_2 \Longleftrightarrow \forall H$ graph: $|\{\varphi: H \to \Gamma_1 \text{ hom.}\}| = |\{\varphi: H \to \Gamma_2 \text{ hom.}\}|$

Q: $\Gamma_1 \cong \Gamma_2 \overset{?}{\Longleftrightarrow} \forall H$ planar graph: $|\{\varphi: H \to \Gamma_1 \text{ hom.}\}| = |\{\varphi: H \to \Gamma_2 \text{ hom.}\}|$

$\not\Uparrow$

[Mancinska-Roberson 2019]: $\Gamma_1 \cong_q \Gamma_2 \Longleftrightarrow \forall H$ planar graph: $|\{\varphi: H \to \Gamma_1 \text{ hom.}\}| = |\{\varphi: H \to \Gamma_2 \text{ hom.}\}|$

$\leadsto$ "quantum group techniques" show that planar graph hom. counts do not characterize graph isomorphism!

SOME RESULTS: REPRESENTATION THEORY OF $\mathrm{Aut}^+(\Gamma)$

Rep. theory of compact matrix quantum groups: intertwiner spaces [Woronowicz 80s]
Tannaka–Krein

Rep. theory of S_n^+: $\mathrm{Mor}_{S_n^+}(u^{\otimes k}, u^{\otimes \ell})$
[Banica 90s]

$:= \{ T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes \ell} \text{ lin.} \mid T u^{\otimes k} = u^{\otimes \ell} T \}$

$\downarrow$

"easy" q.u. groups $\quad = \mathrm{span}\{ T_p \mid p \text{ noncrossing / planar partition of } \{1, \dots, k+\ell\} \}$

Rep. theory of $\mathrm{Aut}^+(\Gamma)$: $\mathrm{Mor}_{\mathrm{Aut}^+(\Gamma)}(u^{\otimes k}, u^{\otimes \ell}) = \mathrm{span}\{ T_\varphi \mid \varphi \text{ hom. from planar graphs to } \Gamma \}$
[Mančinska–Roberson 2019]
([Chassaniol 2019])

[M.-R.]: graph iso game / q. iso of graphs $\longleftrightarrow$ quantum Lovasz Thm $\longleftrightarrow$ rep. th. of $\mathrm{Aut}^+(\Gamma)$

QIT $\qquad$ ALG. COMB. $\qquad$ QG

THE CONTEXT

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SOME RESULTS

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SOME RESULTS: QUANTUM SYMMETRIES OF GRAPH ALGEBRAS

$$C^*(\Gamma) := C^*\Big(p_v \text{ proj.}, v \in V;\ s_e \text{ partial isom.}, e \in E \ \Big|\ s_e^* s_e = p_{r(e)},\ \sum_{s(e)=v} s_e s_e^* = p_v\Big)$$

where $r, s: E \longrightarrow V$ are range and source map (if $s^{-1}(v) \neq \emptyset$)

[Schmidt-W. 2018]: $QSym(C^*(\Gamma)) = Aut^+(\Gamma)$, i.e. $\Gamma \longmapsto C^*(\Gamma)$ respects symmetries

[1.) $Aut^+(\Gamma) \curvearrowright C^*(\Gamma)$, i.e. have left (+right) action $C^*(\Gamma) \longrightarrow C(Aut^+(\Gamma)) \otimes C^*(\Gamma)$

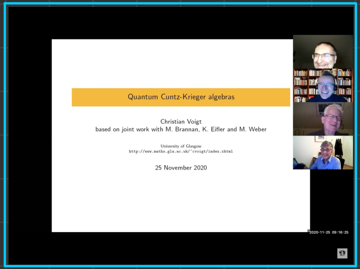
$$p_v \longmapsto \sum_{k \in V} u_{vk} \otimes p_k, \qquad s_e \longmapsto \sum_{f \in E} u_{s(e)s(f)} u_{r(e)r(f)} \otimes s_f$$

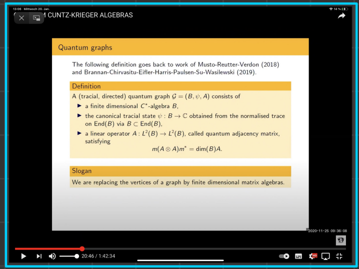
2.) $G \curvearrowright C^*(\Gamma)$ left + right as above $\Longrightarrow u_{ij} \in C(G)$ satisfy relations of $C(Aut^+(\Gamma))$]

[Banica-Skalski 2013]: quantum symmetry of $C^*(\Gamma)$ in the sense of orthogonal filtrations / spectral triples

[Joardar-Mandal 2018]: NCG, KMS states, ...

SOME RESULTS: QUANTUM GRAPHS



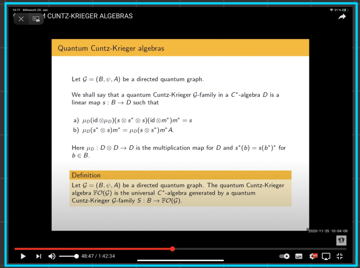


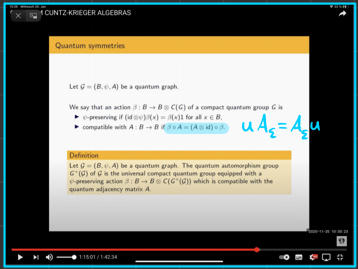
graph $(\{1, \dots, n\}, \varepsilon)$: $\mathbb{C}^n \xrightarrow{\varepsilon} \mathbb{C}^n$

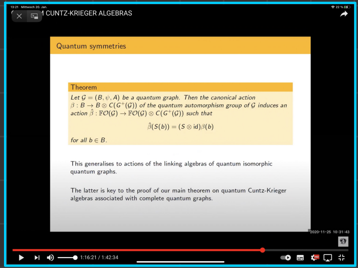
quantum graph $((B, \psi), A_\varepsilon)$: $\bigoplus_{a=1}^{d} M_{N_a}(\mathbb{C}) \xrightarrow{A_\varepsilon} \bigoplus_{a=1}^{d} M_{N_a}(\mathbb{C})$

[Weaver 2012]
[Duan-Severini-Winter 2013]
[Musto-Reutter-Verdon 2019]
[Brannan-Chirvasitu-Eifler-Harris-Paulsen-Su-Wasilewski 2020]

SOME RESULTS: QUANTUM GRAPH C^*-ALGEBRAS







$$"\ \mathbb{F}\mathcal{O}(\mathcal{G}) = C^*(\,S : B \to \mathbb{F}\mathcal{O}(\mathcal{G}) \mid SS^*S = S, \quad S^*S = SS^*A_\varepsilon\,)\ "$$

$$\mathbb{F}\mathcal{O}(\mathcal{G}) = C^*\Big(S_{ij}^{(a)},\ i,j \in \{1,\dots,N_a\},\ a \in \{1,\dots,d\} \;\Big|\; \sum_{r,s} S_{ir}^{(a)} (S_{sr}^{(a)})^* S_{sj}^{(a)} = S_{ij}^{(a)},\ \sum_{\ell} (S_{\ell i}^{(a)})^* S_{\ell j}^{(a)} = \sum_{b,r,s} A_{ija}^{rsb} \sum_{\ell} S_{r\ell}^{(b)} (S_{r\ell}^{(b)})^* \Big)$$

$\mathrm{Aut}^+(\mathcal{G}) \subseteq \mathrm{Aut}^+(B,\psi)$ just like $\mathrm{Aut}^+(\Gamma) \subseteq S_n^+ = \mathrm{Aut}^+(\{n \text{ points}\})$ for graphs Γ

$\mathrm{Aut}^+(\mathcal{G}) \curvearrowright \mathbb{F}\mathcal{O}(\mathcal{G})$ (but not $\mathrm{QSym}(\mathbb{F}\mathcal{O}(\mathcal{G})) = \mathrm{Aut}^+(\mathcal{G})$)

[Brannan-Eifler-Voigt-W. 2020]

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THANKS