Orthogonal vs. Unitary

in the case of "easy" quantum groups

Abstract: We consider quantum subgroups of Wang's free orthogonal quantum group on the one hand and of his free unitary quantum group on the other. In the first case, the generators of the underlying C*-algebras are selfadjoint which is dropped in the latter case. We compare these two cases along the lines of so called "easy" quantum groups and we observe that the step from the orthogonal to the unitary case is huge. This is a survey talk on the landscape of "easy" quantum groups with a particular emphasis on the differences between the orthogonal and the unitary case.

Moritz Weber, Saarland University, Quantum Groups Seminar, 21 June 2021

Orthogonal vs. Unitary – warm up

real $\mathbb{R}$	$x = \overline{x}$	vs.	complex $\mathbb{C}$	$x \neq \overline{x}$
self-adjoint	$x = x^*$	vs.	non-selfadjoint	$x \neq x^*$
orthogonal	$u^t u = u u^t = 1$	vs.	unitary	$u^* u = u u^* = 1$
	$u = (u_{ij}) \in M_n(\mathbb{R})$			$u = (u_{ij}) \in M_n(\mathbb{C})$
	$u_{ij} = \overline{u_{ij}}$			$u_{ij} \neq \overline{u_{ij}}$
$G \subseteq O_N^+$		vs.	$G \subseteq U_N^+$	

Liberation / quantization

$G = (A, u)$ CMQG $:\Leftrightarrow$

- $A = C^*(1, u_{ij}, i,j = 1, \dots, N)$
- $u = (u_{ij})$, $\bar{u} = (u_{ij}^*)$ invertible
- $\Delta : A \longrightarrow A \otimes_{min} A$ *-Hom.
  $u_{ij} \longmapsto \sum_k u_{ik} \otimes u_{kj}$

compact group $\xrightarrow{\text{liberation}}$ compact matrix quantum group [Woronowicz 1980's]

$G \subseteq GL_N(\mathbb{C})$

G' "liberated" version of G, if

$G = (C(G), u)$ $\quad u = (u_{ij})_{i,j=1,\dots,N}$

$u_{ij} : G \longrightarrow \mathbb{C}$

$(g_{kl}) \longmapsto g_{ij}$

$G' = (A, u)$, $u = (u_{ij})_{i,j=1,\dots,N}$

$u_{ij} \in A = C^*(u_{ij}, i,j = 1, \dots N)$

with $A / \langle ab = ba, a,b \in A \rangle \cong C(G)$

(no deformation)

Q: How to find liberations? How many ways?

Example: O_N^+ [Sh. Wang 1990's]

orthogonal group $\xrightarrow{\text{liberation}}$ free orthogonal quantum group

$O_N \subseteq GL_N(\mathbb{C})$

O_N^+ "liberated" version of O_N

$$C(O_N) \cong C^*\Big(u_{ij},\ i,j=1,\dots N \ \Big|\ u_{ij}=u_{ij}^*,\ \sum_k u_{ik}u_{jk} = \sum_k u_{ki}u_{kj} = \delta_{ij},\ u_{ij}u_{kl} = u_{kl}u_{ij}\Big)$$

$$C(O_N^+) := C^*\Big(u_{ij},\ i,j=1,\dots N \ \Big|\ u_{ij}=u_{ij}^*,\ \sum_k u_{ik}u_{jk} = \sum_k u_{ki}u_{kj} = \delta_{ij}\Big)$$

with $C(O_N^+) / \langle ab=ba,\ a,b\in A\rangle \cong C(O_N)$

Q: Other liberations?

Example: U_N^+ [Sh. Wang 1990's]

$G = (A, u)$ CMQG $:\Leftrightarrow$
- $A = C^*(1, u_{ij}, i,j = 1, \dots, N)$
- $u = (u_{ij})$, $\bar{u} = (u_{ij}^*)$ invertible
- $\Delta : A \to A \otimes_{min} A$ $*$-Hom.

  $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$

unitary group $\xrightarrow{\text{liberation}}$ free unitary quantum group

$U_N \subseteq GL_N(\mathbb{C})$

U_N^+ "liberated" version of U_N

$$C(U_N) \cong C^*\Big(u_{ij},\ i,j = 1, \dots N \ \Big|\ \cancel{u_{ij} = u_{ij}^*},\ \sum_k u_{ik} u_{jk}^* = \sum_k u_{ki}^* u_{kj} = \delta_{ij},\ u_{ij} u_{kl}^\varepsilon = u_{kl}^\varepsilon u_{ij},\ \varepsilon \in \{1, *\}\Big)$$

$$C(U_N^+) := C^*\Big(u_{ij},\ i,j = 1, \dots N \ \Big|\ \sum_k u_{ik} u_{jk}^* = \sum_k u_{ki}^* u_{kj} = \sum_k u_{ik}^* u_{jk} = \sum_k u_{ki} u_{kj}^* = \delta_{ij}\Big)$$

with $C(U_N^+) / \langle ab = ba,\ a, b \in A \rangle \cong C(U_N)$

Note: $C(U_N^+) / \langle u_{ij} = u_{ij}^* \rangle = C(O_N^+)$

Q: Other liberations?

Example: S_N^+ [Sh. Wang 1990's]

Symmetric group $\xrightarrow{\text{liberation}}$ free symmetric quantum group

$S_N \subseteq GL_N(\mathbb{C})$

S_N^+ "liberated" version of S_N

$$C(S_N) \cong C^*\Big(u_{ij},\ i,j=1,\dots N \,\Big|\, u_{ij}=u_{ij}^*=u_{ij}^2,\ \sum_k u_{ik}=\sum_k u_{ki}=1,\ u_{ij}u_{kl}=u_{kl}u_{ij}\Big)$$

$$C(S_N^+) := C^*\Big(u_{ij},\ i,j=1,\dots N \,\Big|\, u_{ij}=u_{ij}^*=u_{ij}^2,\ \sum_k u_{ik}=\sum_k u_{ki}=1\Big)$$

with $C(S_N^+)/\langle ab=ba,\ a,b\in A\rangle \cong C(S_N)$

Q: Other liberations?

Let's be more conceptual!

$S_N, O_N, U_N \xrightarrow{\text{liberation}} S_N^+, O_N^+, U_N^+$

Q: Is there a conceptual way "$G \subseteq GL_N(\mathbb{C}) \rightsquigarrow G^+$"?

Q: Other liberations of S_N, O_N, U_N?

Q: Orthogonal vs. Unitary: difference between $u_{ij} = u_{ij}^*$ vs. $u_{ij} \neq u_{ij}^*$?

Tannaka-Krein for CMQG [Woronowicz 1980's]

$G = (A, u = (u_{ij})_{i,j=1,\dots,N})$ CMQG $\Longrightarrow$ $\mathcal{R} := \mathrm{Rep}(G)$ good tensor category

$\exists\, G$ CMQG : $\mathcal{R} = \mathrm{Rep}(G)$ $\Longleftarrow$ $\mathcal{R}$ good tensor category

$\mathcal{R} = \mathrm{Rep}(G)$: (pseudoabelian completion of)

- objects : $u^k := u^{k_1} \otimes \dots \otimes u^{k_m}$, $k_1, \dots, k_m \in \{\circ, \bullet\}$, $u^\circ := (u_{ij})$, $u^\bullet := (u_{ij}^*)$
- morphisms: $\mathrm{Hom}(u^k, u^\ell) = \{T : (\mathbb{C}^N)^{\otimes |k|} \to (\mathbb{C}^N)^{\otimes |\ell|}$ lin. $\mid T u^k = u^\ell T\}$
- morphisms closed under $T \otimes S$, $T \circ S$, T^*

Orthogonal vs. Unitary: $u_{ij} = u_{ij}^* \Rightarrow u^\circ = u^\bullet$, i.e. $\circ = \bullet$

Tannaka-Krein for CMQG [Woronowicz 1980's]

$G=(A,u=(u_{ij})_{i,j=1,\dots,N})$ CMQG $\Rightarrow$ $\mathcal{R}:=\mathrm{Rep}(G)$ good tensor category

$\exists\, G$ CMQG : $\mathcal{R}=\mathrm{Rep}(G) \Leftarrow$ $\mathcal{R}$ good tensor category

$\mathcal{R}=\mathrm{Rep}(G)$: (pseudoabelian completion of)

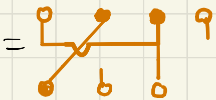
- objects : $u^k := u^{k_1}\otimes \dots \otimes u^{k_m}$, $k_1,\dots,k_m\in\{\circ,\bullet\}$, $u^\circ := (u_{ij})$, $u^\bullet := (u_{ij}^*)$
- morphisms: $\boxed{\mathrm{Hom}(u^k,u^\ell)=\{T:(\mathbb{C}^N)^{\otimes|k|}\to(\mathbb{C}^N)^{\otimes|\ell|} \text{ lin.} \mid Tu^k=u^\ell T\}}$
- morphisms closed under $T\otimes S$, $T\circ S$, T^*

Orthogonal vs. Unitary: $u_{ij}=u_{ij}^* \Rightarrow u^\circ = u^\bullet$

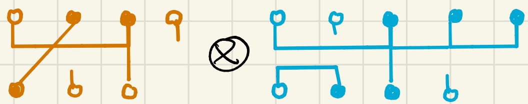
[Banica-Speicher 2009]: $\boxed{G \text{ "easy" QG} :\Leftrightarrow \mathrm{Hom}(u^k,u^\ell)=\mathrm{Span}\{T_p \mid p\in\mathcal{C}(k,\ell)\}}$

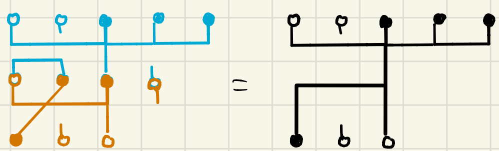
"Easy" quantum groups [Banica–Speicher 2009]

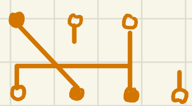
[Banica–Speicher 2009]: $\boxed{G \text{ "easy" QG} :\Longleftrightarrow \mathrm{Hom}(u^k,u^\ell) = \mathrm{Span}\{T_p \mid p\in\mathcal{C}(k,\ell)\}}$

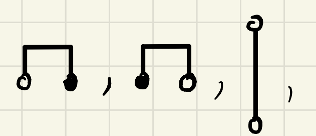
$\mathcal{C}$ category of partitions $:\Longleftrightarrow$ $\mathcal{C} \subseteq \bigcup_{\substack{k,\ell\in\{\circ,\bullet\}^m \\ m\in\mathbb{N}_0}} \{p\in P(k,\ell) \mid p =$  $\begin{matrix}|k| \\ |\ell|\end{matrix}\}$,

$\mathcal{C}$ closed under

,

,



&  $\in \mathcal{C}$

"Easy" quantum groups [Banica–Speicher 2009]

[Banica–Speicher 2009]: $\boxed{G \text{ "easy" QG} :\Longleftrightarrow \mathrm{Hom}(u^k,u^\ell) = \mathrm{Span}\{T_p \mid p \in \mathcal{C}(k,\ell)\}}$

$p \in P(k,\ell)$, $k \in \{\circ,\bullet\}^m$, $\ell \in \{\circ,\bullet\}^n$

$$T_p : (\mathbb{C}^N)^{\otimes m} \to (\mathbb{C}^N)^{\otimes n}, \quad e_{i_1} \otimes \dots \otimes e_{i_m} \mapsto \sum_{j_1,\dots,j_n} \overbrace{\delta_p(i,j)}^{\in\{0,1\}} e_{j_1} \otimes \dots \otimes e_{j_n}$$

$T_p \in \mathrm{Hom}(u^k,u^\ell)$, i.e. $T_p(u^{k_1} \otimes \dots \otimes u^{k_m}) = (u^{\ell_1} \otimes \dots \otimes u^{\ell_n}) T_p$

Ex:

$T_{\sqcap_{\circ\bullet}}(1) = \sum_j e_j \otimes e_j$	$\delta_{ij} = \sum_k u_{ik} u_{jk}^*$	$\left(T_{\sqcap_{\circ\bullet}} = (u \otimes \bar{u}) T_{\sqcap_{\circ\bullet}}\right)$
$T_{\times}(e_{i_1} \otimes e_{i_2}) = e_{i_2} \otimes e_{i_1}$	$u_{ij} u_{kl} = u_{kl} u_{ij}$	$\left(T_{\times}(u \otimes u) = (u \otimes u) T_{\times}\right)$
$T_{\mid}(e_i) = e_i$	$u_{ij} = u_{ij}^*$	$\left(T_{\mid} u = \bar{u} T_{\mid}\right)$

"Easy" quantum groups [Banica–Speicher 2009]

[Banica–Speicher 2009]: $\boxed{G \text{ "easy" QG} :\Longleftrightarrow \mathrm{Hom}(u^k,u^l)=\mathrm{Span}\{T_p \mid p\in\mathcal{C}(k,l)\}}$

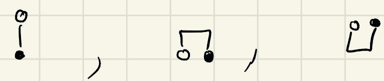
O_N : $C^*\big(u_{ij},\ i,j=1,\dots N \mid u_{ij}=u_{ij}^*,\ \sum_k u_{ik}u_{jk}=\sum_k u_{ki}u_{kj}=\delta_{ij},\ u_{ij}u_{kl}=u_{kl}u_{ij}\big)$

{all pair partitions} = <

> = [Brauer 1930s]

O_N^+ : $C^*\big(u_{ij},\ i,j=1,\dots N \mid u_{ij}=u_{ij}^*,\ \sum_k u_{ik}u_{jk}=\sum_k u_{ki}u_{kj}=\delta_{ij}\big)$

{noncrossing pair partitions} = <



> [Banica 1990s]

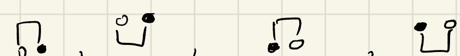
U_N : $C^*\big(u_{ij},\ i,j=1,\dots N \mid \cancel{u_{ij}=u_{ij}^*},\ \sum_k u_{ik}u_{jk}^*=\sum_k u_{ki}^*u_{kj}=\delta_{ij},\ u_{ij}u_{kl}^\varepsilon=u_{kl}^\varepsilon u_{ij},\ \varepsilon\in\{1,*\}\big)$

{pair partitions connecting ∘ & • (on one line)} = <

>

U_N^+ $C^*\big(u_{ij},\ i,j=1,\dots N \mid \sum_k u_{ik}u_{jk}^*=\sum_k u_{ki}^*u_{kj}=\sum_k u_{ik}^*u_{jk}=\sum_k u_{ki}u_{kj}^*=\delta_{ij}\big)$

{noncrossing pair partitions connecting ∘ & • (...)} = <



>

"Easy" quantum groups [Banica-Speicher 2009]

[Banica-Speicher 2009]: $\boxed{G \text{ "easy" QG} :\Leftrightarrow \mathrm{Hom}(u^k, u^\ell) = \mathrm{Span}\{T_p \mid p \in \mathcal{C}(k,\ell)\}}$

O_N : { all pair partitions }

O_N^+ : { noncrossing pair partitions }

U_N : { pair partitions connecting ∘ & • (on one line) }

U_N^+ : { noncrossing pair partitions connecting ∘ & • (...) }

S_N : { all partitions }

S_N^+ : { noncrossing partitions }

Observe:
- liberation : remove crossings
- Orthogonal vs. Unitary: $u_{ij} = u_{ij}^* \Rightarrow u^\circ = u^\bullet$

Q: how many?

Q: big difference?

Free liberations

"easy" groups $G \leq O_N$: categories with $\times$, $\uparrow$ $\in \mathcal{C}$ $\rightsquigarrow$ 6

free liberation $G \leq O_N^+$: categories with only noncrossing partitions $\rightsquigarrow$ 7

Orthogonal	Unitary
"easy" groups: $S_N,\ S_N \tilde{\times} \mathbb{Z}_2,\ B_N,\ B_N \tilde{\times} \mathbb{Z}_2,\ \ H_N,\ O_N$	
free liberations: $S_N^+,\ S_N^+ \tilde{\times} \mathbb{Z}_2,\ B_N^+,\ B_N^+ \tilde{\times} \mathbb{Z}_2,\ B_N^+ \tilde{\times} \mathbb{Z}_2,\ H_N^+,\ O_N^+$ [Banica-Speicher, W.]	

Free liberations

"easy" groups $G \leq O_N$: categories with , $\in \mathcal{C}$ $\rightsquigarrow$ 6

free liberation $G \leq O_N^+$: categories with only noncrossing partitions $\rightsquigarrow$ 7

unitary case : ∞ many, arising from product constructions $\rightsquigarrow \infty$

$G = (A, (u_{ij}))$ CMQG. $G \tilde{\times} \mathbb{Z}_k := \left(C^*(u_{ij} z \mid i,j = 1, \ldots N) \subseteq A \otimes_{max} C^*(\mathbb{Z}_k) \, , \, (u_{ij} z) \right)$

$$G \tilde{*} \mathbb{Z}_k := \left(C^*(u_{ij} z \mid i,j = 1, \ldots N) \subseteq A * C^*(\mathbb{Z}_k) \, , \, (u_{ij} z) \right)$$

$$G \tilde{*}_r \mathbb{Z}_k := \left(C^*(u_{ij} z \mid i,j = 1, \ldots N) \subseteq A * C^*(\mathbb{Z}_k) / \langle u_{ij} z^r = z^r u_{ij} \rangle \, , \, (u_{ij} z) \right)$$

$H_N := \mathbb{Z}_2 \wr S_N := \mathbb{Z}_2^{\oplus N} \rtimes S_N \rightsquigarrow H_N^{(d)+} := \mathbb{Z}_d \wr_* S_N^+$ [Bichon]

Orthogonal	Unitary
"easy" groups: S_N, $S_N \tilde{\times} \mathbb{Z}_2$, B_N, $B_N \tilde{\times} \mathbb{Z}_2$, H_N, O_N	"easy" groups: $S_N \tilde{\times} \mathbb{Z}_k$, $B_N \tilde{\times} \mathbb{Z}_k$, $C_N \tilde{\times} \mathbb{Z}_k$, $(\mathbb{Z}_d \wr S_N) \tilde{\times} \mathbb{Z}_k$, $O_N \tilde{\times} \mathbb{Z}_k$, U_N
free liberations: S_N^+, $S_N^+ \tilde{\times} \mathbb{Z}_2$, B_N^+, $B_N^+ \tilde{\times} \mathbb{Z}_2$, $B_N^+ \tilde{*} \mathbb{Z}_2$, H_N^+, O_N^+ [Banica–Speicher, W.]	free liberations: $(G_N^+ \tilde{\times}_{(r)} \mathbb{Z}_d) \tilde{\times} \mathbb{Z}_k$, $G_N^+ \in \{S_N^+, B_N^+, C_N^+, H_N^{(d)+}, O_N^+\}$ [Tarrago–W.]

Half-Liberations

"easy" groups $G \le O_N$: categories with , $\in \mathcal{C}$ $\rightsquigarrow$ 6

free liberation $G \le O_N^+$: categories with only noncrossing partitions $\rightsquigarrow$ 7

half-liberation $G \le O_N^+$: categories with , $\in \mathcal{C}$ and $\notin \mathcal{C}$

$\Updownarrow$ $u_{ij} u_{kl} u_{mn} = u_{mn} u_{kl} u_{ij}$

$S_N \subsetneq G \subsetneq S_N^+$: $\emptyset$

$O_N \subsetneq G \subsetneq O_N^+$: O_N^*

OPEN: "non-easy"?

$\langle \quad \rangle \rightsquigarrow 1$

Half-Liberations

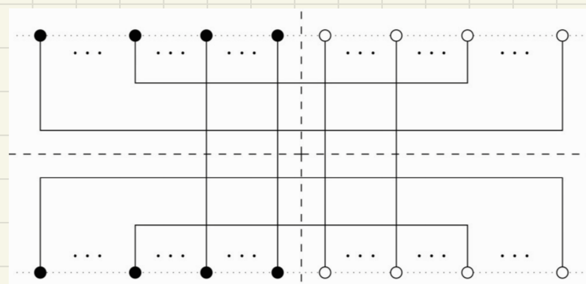
"easy" groups $G \le O_N$: categories with , $\in \mathcal{C}$ $\rightsquigarrow$ 6

free liberation $G \le O_N^+$: categories with only noncrossing partitions $\rightsquigarrow$ 7

half-liberation $G \le O_N^+$: categories with , $\in \mathcal{C}$ and $\notin \mathcal{C}$

unitary case $(U_N \le G \le U_N^+)$: $D \subseteq (\mathbb{N}_0, +)$ subsemigroup (+ $\mathbb{Z}_d$ - family)

$I_D = \langle$



k $j \notin D$ $i \in D$ 0

, $k \in \mathbb{N} \setminus D$, $\rangle$

if $0 \notin D$

Orthogonal

"easy" groups: S_N, $S_N \tilde{\times} \mathbb{Z}_2$, B_N, $B_N \tilde{\times} \mathbb{Z}_2$, H_N, O_N

free liberations: S_N^+, $S_N^+ \tilde{\times} \mathbb{Z}_2$, B_N^+, $B_N^+ \tilde{\times} \mathbb{Z}_2$, $B_N^+ \tilde{*} \mathbb{Z}_2$, H_N^+, O_N^+

[Banica-Speicher, W.]

other liberations:

$S_N \subsetneq G \subsetneq S_N^+$: $\emptyset$ OPEN: "non-easy"?

$O_N \subsetneq G \subsetneq O_N^+$: O_N^* [Banica-Speicher]

Unitary

"easy" groups: $S_N \tilde{\times} \mathbb{Z}_k$, $B_N \tilde{\times} \mathbb{Z}_k$, $C_N \tilde{\times} \mathbb{Z}_k$, $(\mathbb{Z}_d \wr S_N) \tilde{\times} \mathbb{Z}_k$, $O_N \tilde{\times} \mathbb{Z}_k$, U_N

free liberations: $(G_N^+ \tilde{*}_{(r)} \mathbb{Z}_d) \tilde{\times} \mathbb{Z}_k$, $G_N^+ \in \{S_N^+, B_N^+, C_N^+, H_N^{(d)+}, O_N^+\}$

[Tarrago-W.]

other liberations:

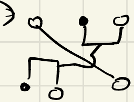
$U_N \subseteq G \subseteq U_N^+$: family indexed by $D \subseteq (\mathbb{N}_0, +)$, $\mathbb{Z}_d$

$O_N \subseteq G \subseteq U_N^+$: see below OPEN: products?

[Mang-W.]

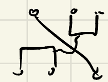
Hyperoctahedral liberations

$G=(A,u)$ group-theoretical, if u_{ij} normal, $\underbrace{u_{ij}u_{ij}^* \text{ central projection}}$, $S_N \subseteq G$

$\iff$  is in Hom

orthogonal case: $u_{ij}=u_{ij}^*$. $G \cong \mathbb{Z}_2^{*N}/\mathcal{U} \rtimes S_N$,

$\mathcal{U} \trianglelefteq \mathbb{Z}_2^{*N}$ invariant under permutation of generators

G "easy" with category $\mathcal{C}$,  $\in \mathcal{C} \Rightarrow G \subseteq H_N^+$.

In all but two cases also $H_N \subseteq G$

unitary case: $G \cong \mathbb{Z}_k^{*N}/\mathcal{U} \rtimes S_N$

Orthogonal

"easy" groups: S_N, $S_N \tilde{\times} \mathbb{Z}_2$, B_N, $B_N \tilde{\times} \mathbb{Z}_2$, H_N, O_N

free liberations: S_N^+, $S_N^+ \tilde{\times} \mathbb{Z}_2$, B_N^+, $B_N^+ \tilde{\times} \mathbb{Z}_2$, $B_N^+ \tilde{\ast} \mathbb{Z}_2$, H_N^+, O_N^+

[Banica-Speicher, W.]

other liberations:

$S_N \subsetneq G \subsetneq S_N^+$: $\emptyset$ OPEN: "non-easy"?

$O_N \subsetneq G \subsetneq O_N^+$: O_N^* [Banica-Speicher]

$H_N \subseteq G \subseteq H_N^+$ ("hyperoctahedral"):

H_N^*, $\mathbb{Z}_2^{*N}/\mathcal{U} \bowtie S_N$ [Raum-W.]

Unitary

"easy" groups: $S_N \tilde{\times} \mathbb{Z}_k$, $B_N \tilde{\times} \mathbb{Z}_k$, $C_N \tilde{\times} \mathbb{Z}_k$, $(\mathbb{Z}_d \wr S_N) \tilde{\times} \mathbb{Z}_k$, $O_N \tilde{\times} \mathbb{Z}_k$, U_N

free liberations: $(G_N^+ \tilde{\ast}_{(r)} \mathbb{Z}_d) \tilde{\times} \mathbb{Z}_k$, $G_N^+ \in \{S_N^+, B_N^+, C_N^+, H_N^{(d)+}, O_N^+\}$

[Tarrago - W.]

other liberations:

$U_N \subseteq G \subseteq U_N^+$: family indexed by $D \subseteq (\mathbb{N}_0, +)$, $\mathbb{Z}_d$

$O_N \subseteq G \subseteq U_N^+$: see below OPEN: products? [Mang-W.]

hyperoctahedral: $\mathbb{Z}_d^{*N}/\mathcal{U} \bowtie S_N$, OPEN: further? [Maaßen]

Non-hyperoctahedral liberations

$\mathcal{C}$ hyperoctahedral $:\Leftrightarrow$ $\sqcap\!\sqcap\!\sqcap \in \mathcal{C}$, $\uparrow\uparrow \notin \mathcal{C}$, i.e. $G \subseteq H_N^+$

$\mathcal{C}$ non-hyperoctahedral, associated "easy" QG $G \subseteq O_N^+$: 13 cases

unitary case:

block sizes / block color sums / total color sum / color distances between subsequent legs of same block / ... of two crossing blocks

$O_N \subseteq G \subseteq U_N^+$

f	v	s	l	k	x
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash m\mathbb{Z}$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm N$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash\pm N$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash\{0\}\backslash\pm N$
$\{1,2\}$	$\pm\{0,1,2\}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
$\{1,2\}$	$\pm\{0,1,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
$\{1,2\}$	$\pm\{0,1\}$	$um\mathbb{Z}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
$\{1,2\}$	$\pm\{0,1,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm E$
$\{1,2\}$	$\pm\{0,1\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash\pm E$
$\mathbb{N}$	$\mathbb{Z}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm E$

$u \in \mathbb{N}_0$
$m \in \mathbb{N}$
$D \subseteq \{0, \ldots, \lfloor \frac{m}{2} \rfloor\}$
$E \subseteq \mathbb{N}_0$
$N \subseteq (\mathbb{N}_0, +)$ subsemigroup

Orthogonal

"easy" groups: S_N, $S_N \tilde{\times} \mathbb{Z}_2$, B_N, $B_N \tilde{\times} \mathbb{Z}_2$, H_N, O_N

free liberations: S_N^+, $S_N^+ \tilde{\times} \mathbb{Z}_2$, B_N^+, $B_N^+ \tilde{\times} \mathbb{Z}_2$, $B_N^{\#+} \tilde{\times} \mathbb{Z}_2$, H_N^+, O_N^+

[Banica - Speicher , W.]

other liberations:

$S_N \subsetneq G \subsetneq S_N^+$: $\varnothing$ OPEN: "non-easy" ?

$O_N \subsetneq G \subsetneq O_N^+$: O_N^* [Banica-Speicher]

$H_N \subseteq G \subseteq H_N^+$ ("hyperoctahedral"):

H_N^* , $\mathbb{Z}_2^{*N}/\mathcal{U} \rtimes S_N$

[Raum - W.]

non-hyperoctahedral : 13 cases

[Banica - Speicher , W.]

Non-easy : [Gromada, Gromada-W., Maaßen]

Unitary

"easy" groups: $S_N \tilde{\times} \mathbb{Z}_k$, $B_N \tilde{\times} \mathbb{Z}_k$, $C_N \tilde{\times} \mathbb{Z}_k$, $(\mathbb{Z}_d \wr S_N) \tilde{\times} \mathbb{Z}_k$, $O_N \tilde{\times} \mathbb{Z}_k$, U_N

free liberations: $(G_N^+ \tilde{\times}_{(r)} \mathbb{Z}_d) \tilde{\times} \mathbb{Z}_k$, $G_N^+ \in \{S_N^+, B_N^+, C_N^+, H_N^{(d)+}, O_N^+\}$

[Tarrago - W.]

other liberations:

$U_N \subseteq G \subseteq U_N^+$: family indexed by $D \subseteq (\mathbb{N}_0, +)$, $\mathbb{Z}_d$

$O_N \subseteq G \subseteq U_N^+$: see below OPEN: products?

[Mang - W.]

hyperoctahedral : $\mathbb{Z}_d^{*N}/\mathcal{U} \rtimes S_N$, OPEN: further?

[Maaßen]

non-hyperoctahedral:

f	v	s	l	k	x
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash m\mathbb{Z}$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm N$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash\pm N$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash\{0\}\backslash\pm N$
$\{1,2\}$	$\pm\{0,1,2\}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
$\{1,2\}$	$\pm\{0,1,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
$\{1,2\}$	$\pm\{0,1\}$	$um\mathbb{Z}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
$\{1,2\}$	$\pm\{0,1,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm E$
$\{1,2\}$	$\pm\{0,1\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash\pm E$
$\mathbb{N}$	$\mathbb{Z}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash(\pm D+m\mathbb{Z})$
$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash\pm E$

OPEN: products?

[Mang - W.]

$C^*(x_1, \dots x_n \mid \sum x_i^2 = 1,\ x_i = x_i^*)$ "=" C(orth. sphere) Banica – Goswami

Partition quantum spaces, St. Jung – W.

Maaßen – Flake: Deligne $\longleftrightarrow$ categ. of part.

ss

Rep(S_N)

Rep(S_t), $t \in \mathbb{C}$

$N^{\#\text{loops}}$, $N \in \mathbb{C}$

OPEN: $A_n \subseteq S_n$ alternating group $\rightsquigarrow A_n^+$?

Amaury Freslon: more colors on $u^\circ, u^\bullet$

Quantum automorphism groups of finite graphs $\mathrm{Aut}^+(\Gamma) \subseteq S_N^+$

Banica, Bichon, Simon Schmidt, Mancinska, Roberson