Easy quantum groups and free probability

a short introduction, written by Moritz Weber

We begin with a quick outline of main ideas in operator algebras bevor we pass to quantum groups and the more specific class of easy quantum groups. We then give a short introduction into free probability theory and its links to easy quantum groups.

C^*-algebras and von Neumann algebras

It is a fundamental philosophy in operator algebras to view certain objects as "noncommutative function algebras". As a motivation, consider a compact, topological space X. Let $C(X)$ be the algebra of complex valued continuous functions over X. Then, $C(X)$ fulfills the axioms of a commutative **C^*-algebra**. Conversely, every commutative C^*-algebra is exactly of this form, by Gelfand-Naimark's theorem. In this sense, noncommutative C^*-algebras may be viewed as function algebras over "noncommutative topological spaces".

Due to a construction by Gelfand-Naimark-Segal, every abstractly defined C^*-algebra may be represented concretely as a norm closed *-subalgebra of the algebra $B(H)$ of bounded linear operators on a Hilbert space. Closely related objects are **von Neumann algebras**, i.e. *-subalgebras of $B(H)$ which are closed in the strong operator topology (or equivalently in the weak operator topology). While C^*-algebras correspond to "noncommutative topology", von Neumann algebras may be seen as a "noncommutative measure theory". Indeed, the commutative von Neumann algebras are exactly given by the algebras $L^\infty(X,\mu)$ of measurable, bounded functions over a measurable space X.

Quantum groups

One of the basic tools in mathematics in general is the study of symmetries of the arising objects. In topology, this is expressed by the investigation of actions of a group G on a topological space X, hence of a map $\alpha : G \times X \to X$ with good properties. Passing to the dual structure – i.e. to function algebras – we obtain a *-homomorphism $C(X) \to C(G\times X)$ by composition with α (let us assume that G and X are compact). The latter algebra is isomorphic to $C(G) \otimes C(X)$. Now, if we replace $C(X)$ by a noncommutative C^*-algebra following the above philosophy of noncommutative spaces – shouldn't we also replace $C(G)$ by some noncommutative object?

This is exactly what Woronowicz did in the late 1980's, developing the theory of **(compact) quantum groups**, based on the theory of C^*-algebras. Note that other concepts of quantum groups have been around earlier. Let us give an idea of Woronowicz's approach starting with the classical case. A compact group G comes with a group law $G \times G \to G$. Passing to $C(G)$ yields a *-homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$ with certain properties. Again, we used the isomorphism $C(G\times G) \cong C(G) \otimes C(G)$. Now, a compact quantum group is by definition a (possibly noncommutative) unital C^*-algebra A together with a *-homomorphism (the comultiplication) $\Delta : A \to A \otimes_{\min} A$ with certain properties. Note that we could also dualize the existence of inverse elements and of the neutral element (like in Hopf algebra theory), but this plays only a minor role in Woronowicz's theory.

We observe that every compact group is a compact quantum group, but the converse is not true. Thus, the class of (compact) quantum groups extends the class of compact groups, providing more symmetries in the realm of noncommutative geometry and operator algebras.

Easy quantum groups

A particular subclass of compact quantum groups is the one of **compact matrix quantum groups**. They may be seen as quantum analogs of compact groups $G \subset M_n(\mathbb{C})$. In this setting, S. Wang gave a definition of a quantum version S_n^+ of the permutation group $S_n \subset M_n(\mathbb{C})$ in the 1990's. Furthermore, he defined the free orthogonal quantum group O_n^+ and the free unitary quantum group U_n^+. His approach was largly extended and put into a systematic framework by Banica and Speicher in 2009. They introduced the so called **easy quantum groups**. These are compact matrix quantum groups $S_n \subset G \subset O_n^+$ which are determined by the combinatorics of set theoretical **partitions**.

The main idea is the following: Consider a finite set $\{1,\dots,k,k+1,\dots,k+l\}$ and divide it into several disjoint subsets. To each such partition p, we associate a linear map T_p which gives rise to an intertwining map of tensor powers of fundamental (co-)representations of our quantum group. By a Tannaka-Krein type result (proven by Woronowicz), the intertwiner spaces completely determine the quantum group. Now, easy quantum groups are those, whose intertwiner spaces may be indexed by partitions. In other words, easy quantum groups carry an intrinsic combinatorial structure. This is why they might also be called **partition quantum groups**.

Free probability

When Murray and von Neumann introduced von Neumann algebras in the 1940's, they also revealed a way of producing many examples. Let G be a discrete group. We consider the group algebra $\mathbb{C}G$ with the multiplication given by the convolution and define an involution turning it into a *-algebra. Representing it faithfully on the Hilbert space $\ell^2(G)$ using the (left) regular representation and then taking the weak closure, we obtain the group von Neumann algebra LG. This yields many examples of von Neumann algebras, but even today it is quite unclear which of them are actually isomorphic. It is a famous open problem whether the free group von Neumann algebras (in fact, they are even factors) $L\mathbb{F}_n$ and $L\mathbb{F}_m$ are isomorphic for $n \neq m$ or not.

In the 1980's, Voiculescu started to tackle this question using techniques from probability theory. His approach is based on the following observation: The space of random variables $L^\infty(\Omega,\mu)$ equipped with the expectation E has a quite algebraic structure – it is an algebra together with a linear functional. Independence yields a factorization rule for variables under the expectation which allows to compute mixed moments ($E(X^nY^m)$ etc.) from the single moments ($E(X^n), E(Y^m)$ etc.). Now, the algebra $L\mathbb{F}_n$ also comes with a linear functional (actually, it is a tracial state) and there is a factorization rule which resembles the one of independence. Yet, as highly noncommutative structures are involved, it gives a completely new kind of independence – namely **free independence** (or **freeness**). This was the birth of **free probability theory** which nowadays is a well-established, highly developed theory with a lot of impact. Important partial results have been obtained on the isomorphism problem for free group factors and their structure, although the final answer is still unknown. But more strikingly, many links to other fields of mathematics have been revealed over the years, such as random matrix theory, combinatorics, complex analysis and the representation theory of large groups, besides operator algebras of course. This puts free probability right in the center of various very active research communities.

Free probability and easy quantum groups

Now, what are the symmetries arising in free probability? There are many hints that easy quantum groups provide the right framework. For instance, in classical probability, distributional invariance under permutations is equivalent to independence (over the tail algebra) and identical distribution. Such a **de Finetti** result also holds in free probability, as proven by Köstler and Speicher: Invariance under quantum permutations (i.e. under the action of S_n^+) is equivalent to freeness (over the tail von Neumann algebra). This result has been refined using other easy quantum groups.

Furthermore, as compact quantum groups come with a Haar state (dualizing the existence of a Haar measure in the group case), we can do free probability on the easy quantum groups themselves. It turns out that the few distributions (arising as **laws of characters**) that have been explicitly computed so far, are well-known players in free probability. The question is whether easy quantum groups could help to find other natural distributions in free probability and maybe to classify them.

A third link to free probability comes from the **von Neumann algebras associated to easy quantum groups** that are obtained using the GNS construction with the Haar state. Surprisingly, they share many properties with the free group factors which opens the door to work on some of the very basic questions in free probability, now from the quantum group perspective.

Exposition of my research results (concerning easy quantum groups)

The **classification of easy quantum groups** was first addressed by Banica, Speicher and later also by Curran. As a combination of their results and mine, we know that there are exactly seven free easy quantum groups (corresponding to noncrossing partitions; somehow the extremal noncommutative case) and six easy groups (the extremal classical case). Also the half-liberated case (in some sense a soft step from groups to quantum groups) is classified. Furthermore, the investigations of Banica, Curran and Speicher are refined yielding a complete list of all easy quantum groups in the so called non-hyperoctahedral case.

The classificiation in the **hyperoctahedral case** however was tackled in a series of papers in joint work with Raum. We prove that it splits into two subcases: the one where easy quantum groups behave more like groups, the other one where they are closer to being “free” or “combinatorial” objects. In the first case, we can explicitly decompose any quantum group into a semidirect (quantum) product of a reflection group with the permutation group. Hence, the ingredients are classical, but the combination of them is a quantum group. Moreover, we show that the problem of classifying all easy quantum groups falling into this class contains the problem of classifying all varieties of groups – hence, this class is extremely rich! This was very unexpected in the early days of the theory. Let us also mention that our result goes beyond easy quantum groups, as it even holds for more general quantum groups. This is the first proof of the idea that easy quantum groups should be an “easy” step into the world of quantum groups, helping to finally understand also other quantum groups.

In the second case, the case of easy quantum groups which behave more like free quantum groups, we prove that there is precisely a one parameter series of quantum groups. Summarizing, easy quantum groups $S_n \subset G \subset O_n^+$ roughly split into three classes – the non-hyperoctahedral ones (exactly 13 cases), the group like hyperoctahedral ones (huge class), and the non-group like hyperoctahedral ones (one series indexed by the natural numbers). **This completes the classification of all easy quantum groups**. In joint work in progress with Tarrago, we are currently extending the framework of easy quantum groups to quantum groups $S_n \subset G \subset U_n^+$. Developing the theory for these **unitary easy quantum groups** is straightforward, but our first classification results show that this class is way richer than in the orthogonal case.

In another article, Freslon and myself found a combinatorial way of understanding the representation theory of easy quantum groups (the **fusion rules**). By Woronowicz's Tannaka-Krein result, all information about an easy quantum group should be visible in its combinatorial data – in principle. Our result verifies this philosophy in the case of the fusion rules in a very concrete and precise way. In large parts of the article, we use nothing but combinatorics on partitions. The conclusion about the resulting quantum group structure only comes at the very end. This work has been extended by Freslon in a further article deriving several operator algebraic properties of the quantum groups, such as the Haagerup property of the enveloping von Neumann algebras. All this is an explicit case study of how relatively complicated operator algebraic properties may be described by relatively simple combinatorial means.

A very, very incomplete list of literature

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