Quantum Symmetry

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In mathematics, symmetry is usually captured using the formalism of groups. However, the developments of the past few decades revealed the need to go beyond groups: to quantum groups. We explain the passage from spaces to quantum spaces, from groups to quantum groups and from symmetry to quantum symmetry, following an analytical approach.

1 Symmetry and groups

Symmetry can be observed in everyday life as well as in mathematics. One prominent example from nature is the symmetry of a butterfly: Reflecting a picture of a (perfect) butterfly at the axis along its body yields the same picture again – we say that the butterfly is symmetric (Fig. 1).

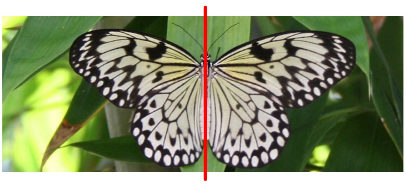

Figure 1: The symmetry of a butterfly.

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1.1 Symmetry in geometry

We now turn to a more mathematical understanding of symmetry. In geometry, the equilateral triangle, the square or the circle are basic objects. How to describe their symmetries? Again, we could think of their symmetry axes. In case of the triangle, there are three, the square has four while the circle has infinitely many (Fig. 2).

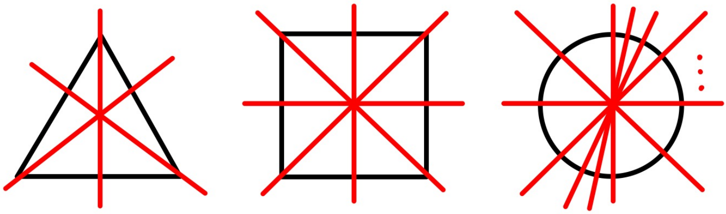

Figure 2: The reflection axes of equilateral triangle, square and circle.

We can also consider more abstract objects such as n points or a graph on n vertices. As for the points, we observe that permuting all points arbitrarily is an operation which gives back the same n points. For a graph[2], the situation is a bit more delicate – only those permutations are allowed which satisfy the rule: “If two points are connected by a line before applying the permutation, then so should they after permuting the points” (Fig. 3).

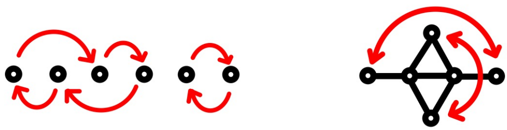

Figure 3: Permuting n points or n vertices.

The latter examples begin to reveal our limits in formulating the notion of symmetry. If we think of way more complicated objects which might even have no pictorial representation – what should be “symmetry” then? We need a precise mathematical formalism in order to capture symmetry: groups.

[2] Let’s take a finite, undirected graph with no multiple edges, to be precise. As you will see, the more technical math of this snapshot is banned to the footnotes – you may simply skip all the footnotes, if you don’t care about too many technical details.

1.2 Groups

In mathematics, a *group* $G = (G, \circ)$ is a set G together with a map $\circ : G \times G \to G$ such that for all $a, b, c \in G$:

1. We have *associativity*: $(a \circ b) \circ c = a \circ (b \circ c)$.
2. There is a *neutral element* $e \in G$ with $a \circ e = e \circ a = a$. Think of it as a symmetry operation which does not change anything.
3. For every $a \in G$, there is an *inverse element* $a^{-1} \in G :\ a \circ a^{-1} = a^{-1} \circ a = e$. Hence, we are able to reverse our symmetry operation – think of reversing some reflection.

Let us consider a couple of examples of groups:

Example 1: The set of integers $\mathbb{Z}$ together with the operation $a \circ b := a + b$ is a group with neutral element $e := 0$ and inverse element $a^{-1} := -a$ for any $a \in \mathbb{Z}$.

Example 2: For any $k \in \mathbb{N}$, the set $\mathbb{Z}_k := \{0, 1, 2, \ldots, k-1\}$ becomes a group via $a \circ b := a + b \mod k$. It is called the *cyclic group (of order k)*.[3]

Example 3: The set consisting only in one element $\{e\}$ is a group, the *trivial group*. The map $\circ$ is defined as $e \circ e := e$.

Example 4: If S_n is the set of all bijective maps σ from $\{1, \ldots, n\}$ to itself, defining $\circ$ as the composition of maps, we obtain the *symmetric group*.[4]

Example 5: Given $n \in \mathbb{N}$, consider the set of all $n \times n$ matrices $M_n(\mathbb{R})$ with real-valued entries; so $A \in M_n(\mathbb{R})$ is given by n^2 numbers $a_{ij} \in \mathbb{R}$ with $i, j \in \{1, \ldots, n\}$. We define the product $A \cdot B$ of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ as the matrix $C = (c_{ij})$ whose i-j-th entry is given by:

$$c_{ij} := \sum_{k=1}^{n} a_{ik} b_{kj} := a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj} \tag{1}$$

We define the transpose of $A = (a_{ij})$ as $A^t := (a_{ji})$. We put $E_n := (\delta_{ij}) \in M_n(\mathbb{R})$ where δ_{ij} is the Kronecker Delta, i.e. $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Now, $O_n \subset M_n(\mathbb{R})$ is the set of all matrices $A \in M_n(\mathbb{R})$ such that:

$$A \cdot A^t = A^t \cdot A = E_n \tag{2}$$

With the above matrix multiplication, O_n becomes the *orthogonal group*.[5]

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[3] We obtain $a \circ b$ as follows: First compute $a + b \in \mathbb{Z}$ as in Example 1; then find a number $c \in \mathbb{Z}_k$ such that $(a + b) - c = km$ for some $m \in \{0, 1\}$; finally, put $a \circ b = c$. In other words: We identify k and 0. The neutral element is $e := 0$ and the inverse element is $a^{-1} = k - a$.

[4] The neutral element is the identity map $\mathrm{id} : \{1, \ldots, n\} \to \{1, \ldots, n\}, \mathrm{id}(x) := x$; the inverse element for $\sigma \in S_n$ is given by the inverse of the map σ.

[5] The neutral element is E_n and the inverse element is A^t. Note that $A \cdot A^t = E_n$ amounts to $\sum_{k=1}^{n} a_{ik} a_{jk} = \delta_{ij}$, whereas $A^t \cdot A = E_n$ is equivalent to $\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij}$.

1.3 Actions of groups

Coming back to our goal to put the concept of symmetry on a more mathematical ground, we need to speak about *actions of groups*. So, if X is a set and G is a group, we say that G *acts on* X *(from the left)*[6], if there is a map $\alpha : G \times X \to X$ such that:

1. We have $\alpha(e, x) = x$ for the neutral element $e \in G$ and any $x \in X$. Again, the neutral element "does not change anything".
2. We have $\alpha(ab, x) = \alpha(a, \alpha(b, x))$ for all $a, b \in G$ and all $x \in X$. Thus, first acting with b and then with a is the same as first combining a and b to a new element ab in G – and then acting with this new element.

As a first example, note that the cyclic group $\mathbb{Z}_3$ of order three acts on the equilateral triangle by rotating the triangle by 120°[7], even the symmetric group S_3 acts on the equilateral triangle by permuting the three sides. One may observe that $\mathbb{Z}_3 \subsetneq S_3$ holds, i.e. $\mathbb{Z}_3$ is a subgroup of S_3.[8] Secondly, we see that $\mathbb{Z}_4$ acts on the square by a rotation by 90° but S_4 does not since for instance permuting only the upper two vertices but not the lower two does not give back a square. As a third example, the symmetric group S_n acts on n points by permutation of the points.[9]

We note that actions of groups model exactly the symmetry operations of our imagination!

1.4 Groups as a formalism for symmetry

We may now ask: What is the "maximal" group G acting on a given set X? In other words: Is there a group G acting on X such that any other group H acting on X is a subgroup of G? If we may find such a group G and if it is unique, we call it the *symmetry group* $\mathrm{Sym}(X)$ *of* X.[10]

One can check that the symmetry group of the equilateral triangle is S_3 – the cyclic group $\mathbb{Z}_3$ also acts on the triangle, but it is not maximal. As for the square, again, the cyclic group $\mathbb{Z}_4$ is not maximal, but in this case S_4 is too big

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[6] For an action from the right, replace $\alpha(ab, x) = \alpha(a, \alpha(b, x))$ by $\alpha(ab, x) = \alpha(b, \alpha(a, x))$.

[7] More precisely, $\alpha(a, \cdot)$ rotates the triangle by a times 120°, for $a \in \mathbb{Z}_3$.

[8] A group H is a subgroup of a group G if $H \subset G$ as sets and the operation $\circ$ on G restricts to the group operation of H. As for $\mathbb{Z}_3 \subset S_3$, let $\sigma_3 : \{1, 2, 3\} \to \{1, 2, 3\}$ be the map in S_3 given by $\sigma_3(1) := 2$, $\sigma_3(2) := 3$ and $\sigma_3(3) := 1$. Then $\{e, \sigma_3, \sigma_3^{-1}\} \subset S_3$ is a group with $\circ$ coming from S_3 – it is a subgroup of S_3 which is isomorphic to $\mathbb{Z}_3$ since we may label its points as $\{0, 1, 2\}$ and recover the group operation $\circ$ of $\mathbb{Z}_3$.

[9] Let $X = \{1, \ldots, n\}$ and $G = S_n$. Define $\alpha(\sigma, x) := \sigma(x)$ for $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

[10] It is not clear whether such a group always exists and whether or not it is unique.

– it does not act on the square. The answer is, that the symmetry group of the square is the dihedral group D_4.[11]

Instead of considering the circle, let us consider its higher-dimensional analog, the *real sphere*:

$$S^{n-1}_{\mathbb{R}} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \dots + x_n^2 = 1\} \subset \mathbb{R}^n \qquad (3)$$

For $n = 2$, the sphere is the circle and for $n = 3$, it is the ball. Now, the symmetry group of the real sphere $S^{n-1}_{\mathbb{R}}$ is the orthogonal group O_n.[12]

The symmetry group of n points is clearly S_n, but what about a graph Γ? Note that we may view the equilateral triangle, the square and also n points as graphs, [13] so the situation of graphs covers all the previous examples except for the sphere. Given a finite graph $\Gamma = (V, E)$ with vertices $V = \{1, \dots, n\}$ and edges $E \subset V \times V$, [14] we say that a map $\sigma \in S_n$ is an automorphism of the graph, if we have: (i, j) is an edge if and only if $(\sigma(i), \sigma(j))$ is an edge. In other words, we permute the vertices, but if two vertices i and j are connected by an edge, then they ought to be connected also after applying the permutation σ (and vice versa). The set of all such automorphisms forms a subgroup of S_n and we define the *automorphism group of* Γ[15] as:

$$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \mid \text{for } i, j \in V : (i, j) \in E \iff (\sigma(i), \sigma(j)) \in E\} \subset S_n \qquad (4)$$

1.5 Take away message no. 1

We conclude that the take away message of this chapter is:

Symmetry = groups

Note that symmetry can serve as a mean to distinguish objects: If X and Y are two objects with $\mathrm{Sym}(X) \neq \mathrm{Sym}(Y)$, then $X \neq Y$. So, distinguishing the geometrical objects X and Y becomes a problem in group theory.[16]

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[11] The dihedral group has eight elements whereas $\mathbb{Z}_4$ has only four.

[12] A matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ acts on a vector $(x_1, \dots, x_n) \in \mathbb{R}^n$ via matrix multiplication $y := Ax \in \mathbb{R}^n$, where $y_i := \sum_{j=1}^n a_{ij} x_j$. Using $\sum_{k=1}^n a_{ki} a_{kj} = \delta_{ij}$ it is easy to check that $Ax \in S^{n-1}_{\mathbb{R}}$ whenever $x \in S^{n-1}_{\mathbb{R}}$. Thus, $\alpha(A, x) := Ax$ defines an action of O_n on $S^{n-1}_{\mathbb{R}}$.

[13] In fact, as undirected graphs: For the triangle, take $V = \{1, 2, 3\}$ and $E = V \times V$; for the square, take $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 3), (3, 4), (1, 4)\}$; the n points are the graph $V = \{1, \dots, n\}$ and $E = \emptyset$.

[14] Hence, we assume that our graph has no multiple edges.

[15] Fun question: Which is the smallest $n > 1$ such that there is an undirected graph $\Gamma = (\{1, \dots, n\}, E)$ with $\mathrm{Aut}(\Gamma)$ being the trivial group?

[16] See also the fundamental group in topology, the crystallographic groups in chemistry or many other examples of such a strategy.

2 Quantum spaces

Let's go quantum now! The main feature of "quantum theories", on the mathematical side, is noncommutativity, so let's take a look at it first.

2.1 Noncommutativity

Recall the commutativity law for real numbers: Given two numbers $a, b \in \mathbb{R}$, we have:

$$ab = ba \tag{5}$$

In other words, the multiplication in $\mathbb{R}$ is commutative. Is every multiplication commutative? Well, we have encountered matrices $A = (a_{ij}) \in M_n(\mathbb{R})$ before and we have defined a multplication on it. We may easily find examples [17] of matrices $A, B \in M_n(\mathbb{R})$ such that:

$$A \cdot B \neq B \cdot A \tag{6}$$

We observe that the multiplication in $M_n(\mathbb{R})$ is *noncommutative*! This fact is of crucial importance for instance in quantum physics: On the atomic level, it makes a difference whether the position is measured first and then the momentum – or vice versa. Expressed as matrices, this means that the two observables do not commute.

2.2 A bit of topology

Another ingredient for our journey to the quantum world comes from topology and is a bit less intuitive; nevertheless, let us take a quick glance. Recall that we call a function $f : [0,1] \to \mathbb{R}$ on an interval $[0,1]$ (everywhere) *continuous*, if for any $x \in [0,1]$ "moving a bit to the left or right does not change $f(x)$ too much". With tools from topology, this may be put into a more robust mathematical framework [18] and it may be generalized to functions $f : X \to \mathbb{R}$ on arbitrary spaces X. Some of these spaces X behave nicer than others and they are called *compact spaces*. Examples of compact spaces are all finite subsets of $\mathbb{R}$, all intervals $[a, b]$ in $\mathbb{R}$ and all finite unions of them. The set of all positive real numbers however, or $\mathbb{R}$ itself are not compact.

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[17] For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 5 & 0 \end{pmatrix}$ we have $A \cdot B = \begin{pmatrix} 11 & 1 \\ 23 & 3 \end{pmatrix} \neq \begin{pmatrix} 4 & 6 \\ 5 & 10 \end{pmatrix} = B \cdot A$.

[18] A topology on Y is a set of subsets $U \subset Y$ which is closed under certain operations; in the case of $\mathbb{R}$, you may simply think of arbitrary unions of open intervals (a, b). Now, a map $f : X \to Y$ is continuous, if the preimage $f^{-1}(U)$ belongs to the topology of X whenever U belongs to the topology of Y.

2.3 Algebras of functions

We have met the multiplication of numbers as well as of matrices. We can define a multiplication on yet another set. Consider:

$$C([0,1]) := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\} \tag{7}$$

This set admits a multiplication defined as a pointwise operation:

$$(f \cdot g)(x) := f(x) \cdot g(x) \tag{8}$$

We also have a pointwise addition, and we may multiply any function f with any number $\lambda \in \mathbb{R}$. Checking some compatibility rules for these operations, we infer that $C([0,1])$ is an *algebra*, the algebra of (continuous) functions. But this set has even more structure! Namely, we have a norm:

$$||f||_\infty := \max\{|f(x)| \mid x \in [0,1]\} \tag{9}$$

A norm is a kind of a generalization of the absolut value $|x|$ of a number $x \in \mathbb{R}$. More generally, we may define for any compact set X:

$$C(X) := \{f : X \to \mathbb{R} \mid f \text{ is continuous}\} \tag{10}$$

In fact, compactness of X guarantees that we may define a norm $|| \cdot ||_\infty$ as above.[19]

Summarizing, for a compact space X, the set $C(X)$ has the nice structure of a C^**-algebra*, i.e. it is an algebra[20] equipped with a norm satisfying a few compatibility and topological properties. Furthermore, the multiplication is commutative, as clearly $f \cdot g = g \cdot f$. Now, let's be courageous and consider algebras satisfying all axioms of a C^*-algebra – without the requirement that the multiplication is commutative. For instance, the set of all $n \times n$-matrices $M_n(\mathbb{R})$ is such a *noncommutative* C^*-algebra.

Then, a "Fundamental Theorem in C^*-algebras"[1, II.2.2.4] characterizes exactly the commutative C^*-algebras: A (unital) C^*-algebra is commutative if and only if it is equal[21] to $C(X)$ for some compact space X. This is a nice but quite abstract theorem. Let's take a breath and a step back.

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[19] Recall that $\mathbb{R}$ is not compact, and indeed, the set $\{|f(x)| \mid x \in \mathbb{R}\}$ might be unbounded for certain continuous functions $f : \mathbb{R} \to \mathbb{R}$.

[20] To be precise, we shall consider complex-valued functions and complex unital algebras; however, in order to keep it simple, let's pretend to work over the real numbers only.

[21] or rather isomorphic

2.4 "Quantum mathematics"

So, this Fundamental Theorem in C^*-algebras – what is it about? Isn't it just abstract nonsense? Well, this is one of these points where mathematics splits from its nature as a toolbox for natural sciences – and becomes a science of its own. We can take the above mentioned theorem as an abstract theorem – or as the starting point for a whole new philosophy!

Just recall, what we have so far: Given a compact space X, its algebra of functions $C(X)$ is a commutative C^*-algebra. Conversely, any commutative C^*-algebra is of the form $C(X)$ for some compact space X. So, apparently, we may identify X with $C(X)$ in some sense, right? Now, let's be abstract mathematicians: Once we have done this identification – what are these mysterious *noncommutative* C^*-algebras then? Somehow, in our imagination, we can identify them with *compact quantum spaces*! What does this mean? On a *technical level*, we only work with commutative or noncommutative C^*-algebras; but on an *intuitive level* we work with compact spaces or compact quantum spaces extending the above identification.

This thought looks completely crazy on the first sight, but it turns out to be an extremely powerful machine for our imagination and it is just the starting point of a whole "noncommutative dictionnary":[22]

classical theory		*noncommutative version*
topology (compact spaces)	$\longleftrightarrow$	C^*-algebras
measure theory	$\longleftrightarrow$	von Neumann algebras
probability theory	$\longleftrightarrow$	free probability theory
differential geometry	$\longleftrightarrow$	noncommutative geometry

2.5 Take away message no. 2

We summarize this chapter in form of the following take away message:

Compact spaces = commutative algebras of functions
Compact quantum spaces = noncommutative algebras of functions

[22] C^*-algebras [1, 3] were introduced by Israel Gelfand and Mark Naimark in the 1940's; von Neumann algebras [1, 5] were introduced by Francis Murray and John von Neumann in the 1930's; free probability theory [10, 17] was initiated by Dan-Virgil Voiculescu in the 1980's; noncommutative geometry [2, 4] is a project by Alain Connes starting in the 1980's. In principle, one could also add "free analysis" [6, 16] (originating from the 1970's) by Joseph Taylor, as a noncommutative analog of complex analysis; and "quantum information theory" [12, 19] (1980's) as an analog of information theory – but the philosophy behind these two theories is less based on "noncommutative algebras of functions"; however, they should be seen as the "quantum versions" of the corresponding classical theories, having links to the other listed theories.

3 Quantum symmetry and quantum groups

Given a compact space X, its symmetries shall be encoded in form of a group. Now, passing to a compact quantum space – how about its symmetries? It turns out, that groups are not enough to describe them; again, we need to go quantum: to quantum groups!

3.1 Compact quantum groups

Let us recall two main ingredients of this snapshot: Firstly, a group is a set G together with a map

$$\circ : G \times G \to G \tag{11}$$

satisfying certain axioms. Secondly, the passage from the classical world to the quantum world was performed according to the following recipe: Take a compact space X; consider the algebra $C(X)$ of continuous functions on X; extract interesting properties of this algebra; consider algebras which share all of these properties with $C(X)$ (i.e. C^*-algebras), with the only difference that we allow the multiplication to be noncommutative – we obtain a theory of compact quantum spaces!

Now, let's apply the same recipe to compact groups G – groups which are compact as a set! So, we first pass to $C(G)$. What are interesting properties of this algebra (besides those for a general $C(X)$, where X is a compact space)? A compact group is not only a compact set G, but we also have the map $\circ$. How does it behave on the level of $C(G)$? By composition with $\circ$, we obtain the following map:

$$\Delta : C(G) \to C(G \times G) \tag{12}$$

$$\Delta(f)(a, b) := f(a \circ b) \tag{13}$$

But, how to pass from $C(G)$ to more general algebras A? Luckily, we have that $C(G \times G)$ is isomorphic to the tensor product $C(G) \otimes C(G)$ of $C(G)$ with itself. Thus, if A is any (possibly noncommutative) C^*-algebra possessing a map

$$\Delta : A \to A \otimes A \tag{14}$$

which shares some characteristic features with $\Delta : C(G) \to C(G) \otimes C(G)$, we may speak of it as a *compact quantum group*! A quick check then yields: Every compact group is a compact quantum group – but not the converse. Thus, compact quantum groups are an honest generalization of compact groups.[23]

[23] In fact, we also have a "Fundamental Theorem in compact quantum groups"[15, 5.1.3] Let (A, Δ) be a compact quantum group; in particular A is a C^*-algebra. Then A is commutative if and only if A is isomorphic to $C(G)$ for some compact group G.

3.2 Quantum symmetries in noncommutative geometry

In perfect analogy to the classical situation (of Chapter 1), we may define actions of compact quantum groups on quantum spaces and we may define the quantum symmetry group QSym of such a quantum space.[24] Let us consider a concrete example. We view S_n as permutation matrices. [25] There is a quantum version of the symmetric group S_n called the *free symmetric quantum group* S_n^+. [18] We may imagine S_n^+ as matrices $A = (a_{ij})$ with matrix-valued entries $a_{ij} \in M_m(\mathbb{R})$ satisfying the same conditions as permutation matrices. [26]

One can check that the free symmetric quantum group S_n^+ is the quantum symmetry group – of n points! [20] Wait, wasn't this S_n? True, but the question is: In which class are we trying to find our object modelling the symmetries of n points? Within the category of groups, S_n is the right object; however, within the category of quantum groups, S_n^+ is the correct one.[27] The funny thing is, that S_n can be seen as a subgroup of S_n^+. [28] This means that we have more possibilities for "quantum permuting" n points than in the classical world! In fact, comparing two examples of permutation matrices in S_4 and in S_4^+ respectively, we observe:

$$\begin{array}{cc} \text{in } S_4 & \text{in } S_4^+ \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \end{array} \tag{15}$$

While the example in S_4 maps the second point to the fourth position, the example in S_4^+ maps it "partially to the third position and partially to the fourth one". This intuition is a bit shorthand, but not so far from the truth.

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[24] However, depending on the choice of certain regularity assumptions for the actions, we might obtain different definitions of QSym.

[25] Given $\sigma \in S_n$, define $A_\sigma := (\delta_{i\sigma(j)}) \in M_n(\mathbb{R})$. Identifying a point $k \in \{1, \ldots, n\}$ with the vector e_k consisting in 1 at the k-th entry and 0 otherwise, we observe that the matrix $A_\sigma := (\delta_{i\sigma(j)})$ acts as $A_\sigma e_k = e_{\sigma(k)}$. Thus, $\sigma \mapsto A_\sigma$ is a representation of S_n as matrices. Observe that every matrix $A_\sigma = (a_{ij})$ satisfies $\sum_{k=1}^n a_{ik} = \sum_{k=1}^n a_{kj} = 1$.

[26] For the entries $a_{ij} \in M_m(\mathbb{R})$, we require $\sum_{k=1}^n a_{ik} = \sum_{k=1}^n a_{kj} = E_m$. Moreover, we make the additional technical assumption $a_{ij} = a_{ij}^2 = a_{ij}^t$.

[27] There is a canonical way to view n points as a quantum space following $X \mapsto C(X)$. [20]

[28] You basically have to check that if $m = 1$, then the conditions for the elements $a_{ij} \in M_m(\mathbb{R})$ characterize permutation matrices. So, S_n^+ is like S_n with higher-dimensional m.

3.3 Take away message no. 3

Groups are an appropriate formalism in order to capture symmetries of classical spaces. However, passing to quantum spaces, we should use quantum groups rather than groups. Our noncommutative dictionnary may be extended to:

classical theory		*noncommutative version*
compact groups	$\longleftrightarrow$	compact quantum groups

And our last take away message is:

Quantum symmetry = Quantum groups

Let us remark, that there are cases where $\mathrm{QSym}(X) \neq \mathrm{QSym}(Y)$ holds but $\mathrm{Sym}(X) = \mathrm{Sym}(Y)$. So, quantum groups may help to distinguish X and Y in cases when groups fail.[29]

3.4 Disclaimer: algebraic vs. analytical approach

In this snapshot, we chose an analytical/topological approach [15, 11] to quantum groups, but there is also an algebraic one. [7, 8, 9] The point is, that you have a choice which kind of algebras you want to "quantize". So, while we associated the algebra of continuous functions $C(G)$ to a group G and extracted its properties in order to define what a quantum group is, one could also associate the algebra of polynomials over G or some universal envelopping algebra. Depending on this a priori choice, one obtains different approaches to quantum groups.[30]

This aspect, that the deformation of the algebras associated to groups is a major ingredient of the theory of quantum groups makes it impossible to give a general and overall definition of what a quantum group is. This is different from the classical situation, where you first give a definition of a group and then define a compact group as a group with some additional structure – in the case of compact quantum groups, the property "compact" is intrinsic, it is already part of the definition! Hence, we may define quantum groups based on deformations of its algebra of polynomials or quantum groups based on deformations of its algebra of continuous functions – and we obtain two different definitions of quantum groups. We are only in the beginning of investigating the links between these different approaches.

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[29] You may find examples for instance amongst quantum automorphism groups of graphs, [14, 13] see Section 1.4 for the classical counterpart. By the way, did you find the smallest $n > 1$ such that there is an undirected graph on n points having trivial automorphism group? It is $n = 6$, an example being $\{(1,2),(2,3),(3,4),(4,5),(3,5),(4,6)\}$.

[30] The analytical/topological one has been initiated by Stanisław Woronowicz; [21, 22] the algebraic one has Vladimir Drinfeld and Michio Jimbo as two of its most famous forefathers.

4 Acknowledgements

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