Compact matrix quantum groups and their representation theory

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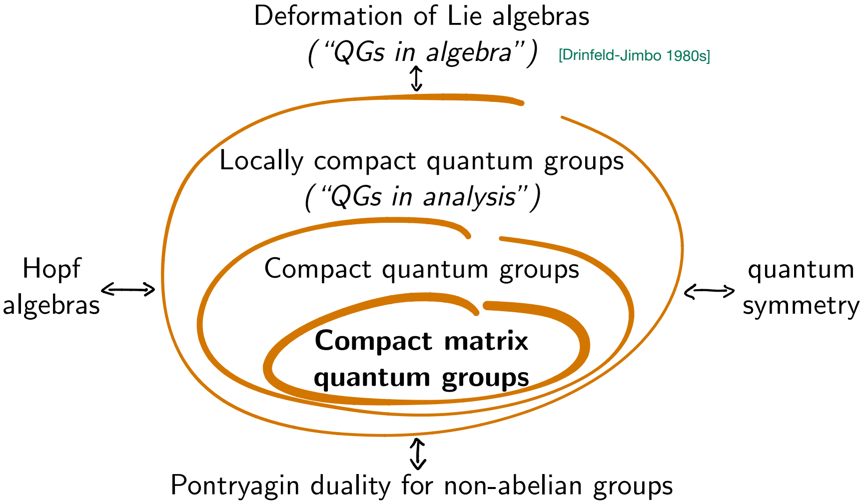
CMQG: how to find them on an imprecise map



CMQG: in the context of "quantum mathematics"

Classical	Quantum	
Topology	C^*-Algebras	[Gelfand-Naimark 1940s]
Measure Theory	Von Neumann Algebras	[Murray-von Neumann 1940s]
Probability Theory	Free Probability Theory	[Voiculescu 1980s]
	& Quantum Probability	[Accardi, Hudson-Parthasarathy 1970s]
Differential Geometry	Noncommutative Geometry	[Connes 1980s]
(Loc. Comp.) Groups	(Loc. Comp.) Quantum Groups	[Woronowicz 1980s]
Information Theory	Quantum Information Theory	[Feynmann, Deutsch 1980s]
Complex Analysis	Free Analysis	[J.L.Taylor 1970s]

Philosophy behind Quantum Mathematics:

commutative algebras $\Longleftrightarrow$ **classical situation**

noncommutative algebras $\Longleftrightarrow$ **quantum situation**

CMQG: BY DEFINITION

Definition [Woronowicz 1980s]

Let $N \in \mathbb{N}$. $G = (A, u)$ is a compact matrix quantum group (CMQG) $:\Longleftrightarrow$

- *A is a unital C^*-algebra with $A = C^*(u_{ij}, 1 \mid i,j \in \{1, \dots, N\})$*
- *$u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ invertible $N \times N$-matrices in $M_N(A)$*
- *$\Delta : A \to A \otimes A, u_{ij} \mapsto \sum_{k=1}^N u_{ik} \otimes u_{kj}$ *-homomorphism*

Let $G \subseteq GL_N(\mathbb{C})$ be a compact group.
Put $A := C(G) := \{f : G \to \mathbb{C} \mid f \text{ cont.}\}$ and $u_{ij} : C(G) \to \mathbb{C}$, $u_{ij}(g) := g_{ij}$.
Then (A, u) is a compact matrix quantum group:

- $A = C^*(u_{ij}, 1 \mid i,j \in \{1, \dots, N\})$ [Stone-Weierstrass]
- $u = (u_{ij}), \bar{u} = (u_{ij}^*) \in M_N(C(G))$ invertible [$u(g) = g$]
- $\Delta : C(G) \to C(G) \otimes C(G), u_{ij} \mapsto \sum_{k=1}^N u_{ik} \otimes u_{kj}$ [matrix multipl.]

Hence, compact matrix quantum groups generalize $G \subseteq GL_N(\mathbb{C})$

Theorem (Gelfand-Naimark type) [Woronowicz 1980s]

$G = (A, u)$ CMQG with $N \in \mathbb{N}$. Then:

$$A \text{ commutative} \iff \exists G \subseteq GL_N(\mathbb{C}) \text{ compact group} : A \cong C(G)$$

Proof.

"$\Longrightarrow$" Gelfand-Naimark. "$\Longleftarrow$" $A := C(G)$, $u_{ij} : C(G) \to \mathbb{C}$, $g \mapsto g_{ij}$. □

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Theorem (Existence of Haar state) [Woronowicz 1980s]

Every CMQG (A, u) possesses a Haar state $h : A \to \mathbb{C}$, i.e. h with:

$$(\mathrm{id}_A \otimes h)\Delta(a) = (h \otimes \mathrm{id}_A)\Delta(a) = 1_A h(a)$$

Proof.

Convolve any state $\frac{1}{n}\sum_{k=1}^{n} \varphi^{*k} \to h$, $n \to \infty$, where $\varphi * \varphi := (\varphi \otimes \varphi) \circ \Delta$ □

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Theorem (Link with Hopf algebras) [Woronowicz 1980s]

There is a dense Hopf *-algebra $A_0 \subseteq A$ with

$$\Delta_{|A_0} : A_0 \to A_0 \otimes A_0, \quad \varepsilon(u^\alpha_{ij}) = \delta_{ij}, \quad S(u^\alpha_{ij}) = (u^\alpha_{ji})^*.$$

Proof.

Put $A_0 := \{\text{matrix elements } u^\alpha_{ij} \text{ of fin.-dim. rep.}\}$ and use h for density. □

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CMQG $\longleftrightarrow$ Hopf *-algebra with Haar integration

Example (Symmetric quantum group) [Wang 1990s]

$S_N^+ := (A_S(N), u)$ *CMQG with*

$$A_S(N) := C^*(u_{ij}, 1 \mid u_{ij} = u_{ij}^2 = u_{ij}^*, \sum_k u_{ik} = \sum_k u_{kj} = 1) \quad \twoheadrightarrow \quad C(S_N)$$

$$S_N \quad \subseteq \quad S_N^+ \quad \textit{quantum permutations}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \quad \text{“}\ni\text{”} \quad \begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}$$

Example (Symmetric/free orthogonal/free unitary QGs) [Wang 1990s]

$S_N \subseteq S_N^+ := (A_S(N), u)$, $O_N \subseteq O_N^+ := (A_O(N), u)$ and $U_N \subseteq U_N^+ := (A_U(N), u)$ CMQGs with:

$$A_S(N) := C^*(u_{ij}, 1 \mid u_{ij} = u_{ij}^2 = u_{ij}^*, \sum_k u_{ik} = \sum_k u_{kj} = 1) \quad \twoheadrightarrow \quad C(S_N)$$

$$A_O(N) := C^*(u_{ij}, 1 \mid u_{ij} = u_{ij}^*, \sum_k u_{ik} u_{jk} = \sum_k u_{ki} u_{kj} = \delta_{ij}) \quad \twoheadrightarrow \quad C(O_N)$$

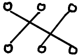
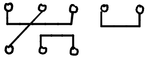
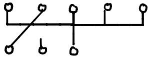
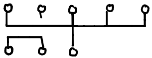
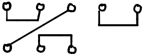
$$A_U(N) := C^*(u_{ij}, 1 \mid u, \bar{u} \text{ unitary}) \quad \twoheadrightarrow \quad C(U_N)$$

algebraic relations → **solutions in** $\mathbb{C}$ ⟶ **group**

algebraic relations → **solutions in** $M_N(\mathbb{C})$ ⟶ **quantum group**

- Have quantum versions $S_N \subseteq S_N^+$, $O_N \subseteq O_N^+$, $U_N \subseteq U_N^+, \ldots$
- Associated "reduced" C^*-algebras $C_{red}(G)$ and von Neumann algebras $L(G)$ are interesting OPEN: $L(O_N^+) \cong L(O_M^+)$?
  [Banica, Vaes, Vergnioux, Brannan, Freslon,...]
- S_N^+, O_N^+, U_N^+ yield quantum symmetries for free probability or Connes's noncommutative geometry [Köstler, Speicher, Curran, Banica, Goswami,...]
- S_N^+ is a Calabi-Yau algebra of dimension 3 [Bichon, Franz, Gerhold,...]
- (Hochschild) cohomological dimensions of S_N^+, O_N^+, U_N^+ are 3
  [Thom, Bichon, Franz, Gerhold, Das, Kula, Skalski,...]
- L^2-Betti numbers of S_N^+, O_N^+ and U_N^+ known:
  $\beta_p^{(2)} = 0$ except $\beta_1^{(2)}(U_N^+) = 1$ [Vergnioux, Collins, Härtl, Thom, Bichon, Raum, Kyed, Vaes, Valvekens,...]

Rep. Theory of CMQG: Schur-Weyl ...

(quantum) group	representation category	diagrams
U_N	permutations (Schur-Weyl)	
O_N	pair partitions (Brauer diagrams)	
S_N	all partitions of sets	
S_N^+	noncrossing partitions	
O_N^+	noncrossing pair partitions	

Theorem (Tannaka-Krein duality) [Woronowicz 1980s]

Let $\mathcal{R}$ be a tensor category.

$$\exists G \text{ CMQG: } \mathrm{Rep}(G) = \mathcal{R} \quad \Longleftrightarrow \quad \mathcal{R} \text{ with good structure}$$

What is this "good structure"? What is Rep(G)?

Let $G = (A, u)$ be a CMQG (of Kac type), where $u = (u_{ij}) \in M_N(A)$.

- Objects Rep(G) := {fin. dim. unitary rep.}
  $\mathrm{Rep}(G) \ni u^r = \sum_{ij} e_{ij} \otimes u^r_{ij} \in M_{n_r}(\mathbb{C}) \otimes A$ unitary, $\sum_k u^r_{ik} \otimes u^r_{kj} = \Delta(u^r_{ij})$
- Rep(G) equiped with $\otimes$: define $u^r \otimes u^s \in M_{n_r}(\mathbb{C}) \otimes M_{n_s}(\mathbb{C}) \otimes A$
- $\mathrm{Mor}(r,s) := \{T : \mathbb{C}^{n_r} \to \mathbb{C}^{n_s} \text{ lin.} \mid Tu^r = u^s T\}$ intertwiners
- Mor closed under $\otimes$, composition, involution, ...

If $u_{ij} = u^*_{ij}$, only need $\mathrm{Mor}(k,l) := \{T : (\mathbb{C}^N)^k \to (\mathbb{C}^N)^l \text{ lin.} \mid Tu^{\otimes k} = u^{\otimes l} T\}$.

CMQGs $\longleftrightarrow$ "Woronowicz" tensor categories

Theorem (Tannaka-Krein duality) [Woronowicz 1980s]

Let $\mathcal{R}$ be a tensor category.

$\exists G$ *CMQG:* $\mathrm{Rep}(G) = \mathcal{R} \iff \mathcal{R}$ *"Woronowicz" tensor category*

CMQGs $\longleftrightarrow$ **"Woronowicz" tensor categories**

$\updownarrow$

"Combinatorial" descriptions

Meta Conjecture

Every CMQG G possesses a combinatorial description of its rep. theory:

$\mathrm{Rep}(G) = \text{combinatorial category} + \text{fiber functor}$

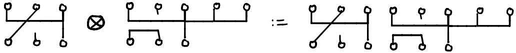
(1) "easy" QGs: a combinatorial category ...

$P(k,l) :=$ {partitions of sets on k upper and l lower points}

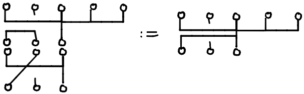
Definition [Banica-Speicher 2009]

A category of partitions is a set $\mathcal{C} \subseteq \bigcup_{k,l \in \mathbb{N}_0} P(k,l)$ which is closed under

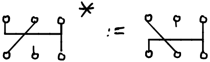
tensor products



composition



involution



and containing  *and* 

Example

(a) all partitions, *(b) pair partitions,* *(c) noncrossing partitions (NC)*
(d) noncrossing pair partitions, *(e)* $\{p \in NC \mid$ blocks of size 1 or 2$\}$

Examples of combinatorial descriptions: (1) "easy" QGs: ... and a fiber functor

Definition of $T_p : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$ as follows.

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) := \sum_{j_1,\ldots,j_l} \delta_p(i_1,\ldots,i_k; j_1,\ldots,j_l) e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Then: $T_p u^{\otimes k} = u^{\otimes l} T_p \quad \longrightarrow \quad$ relations on the u_{ij}

crossings $\longleftrightarrow$ **commutativity relations**

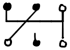
Examples of combinatorial descriptions: (1) "easy" QGs: Definition

Definition [Banica-Speicher 2009]

G CMQG with $S_N \subseteq G \subseteq O_N^+$. G "easy": $\Longleftrightarrow$
$\exists \mathcal{C}$ *category of partitions*: $\mathrm{Mor}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{T_p \mid p \in \mathcal{C} \cap P(k,l)\}$

"easy" QG $\longleftrightarrow$ categories of partitions

Extensions to $u_{ij} \neq u_{ij}^*$: Need $u^w = u^{\otimes(\circ\circ\bullet\circ)} := u \otimes u \otimes \bar{u} \otimes u$ etc.
Consider categories of *two-colored* partitions.



Definition [Tarrago-W. 2016]

G CMQG with $S_N \subseteq G \subseteq U_N^+$. G "easy": $\Longleftrightarrow$
$\exists \mathcal{C}$ *categ. of two-col. part.*: $\mathrm{Mor}(u^{\otimes w}, u^{\otimes v}) = \mathrm{span}\{T_p \mid p \in \mathcal{C} \cap P^{\circ\bullet}(w,v)\}$

Examples of combinatorial descriptions: ① "easy" QGs: some aspects

Theorem [Banica-Speicher 2009, Banica-Curran-Speicher 2010, W. 2013, Raum-W. 2016]

Orthogonal "easy" QG (i.e. $S_N \subseteq G \subseteq O_N^+$) are completely classified:

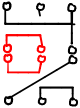
	Categories of partitions	Quantum groups
groups	$\{$all partitions$\}$, $\{\lvert b\rvert = 2\}$, $\{\lvert b\rvert = 1$ or $2\}$, $\{\lvert b\rvert$ even$\}$, $\{\lvert p\rvert$ even$\}$, $\{\lvert p\rvert$ even, $\lvert b\rvert = 1$ or $2\}$	S_N, O_N, B_N, $\mathbb{Z}_2 \wr S_N, S_N \times \mathbb{Z}_2$, $B_N \times \mathbb{Z}_2$
"free"	$\{NC\}$, $\{NC, \lvert b\rvert = 2\}$, $\{NC, \lvert b\rvert = 1$ or $2\}$, $\{NC, \lvert b\rvert$ even$\}$, $\{NC, \lvert p\rvert$ even$\}$, $\{NC, \lvert p\rvert$ even, $\lvert b\rvert = 1, 2\}$, $\{+$ leg dist. even$\}$	S_N^+, O_N^+, B_N^+, $\mathbb{Z}_2 \wr_* S_N^+, S_N^+ \times \mathbb{Z}_2$, $B_N^+ \times \mathbb{Z}_2, B_N^+ * \mathbb{Z}_2$
"half-lib."	$\{\lvert b\rvert = 2$, leg dist. even$\}$, ...	$O_N^*, B_N^*, H_N^*, \ldots$
	use $(\mathbb{Z}_2)^{*N} \twoheadrightarrow \Gamma$	$\hat{\Gamma} \bowtie S_N$
	use	

Theorem [Tarrago-W. 2018, Gromada 2018, Mang-W. 2019, 2020]

Unitary "easy" QG (i.e. $S_N \subseteq G \subseteq U_N^+$) are partially classified:

	Categories of partitions	Quantum groups
groups	$\{$all two-col. partitions$\}$, $\{+\lvert b\rvert = 2$ and $\}$, rules on block sizes and colorings	S_N, U_N, $S_N \times \mathbb{Z}_k, \ldots$,
"free"	way more noncrossing ones	S_N^+, U_N^+, $S_N^+ \times \mathbb{Z}_k, S_N^+ * \mathbb{Z}_k$, $(S_N^+ * \mathbb{Z}_d) \times \mathbb{Z}_k, \ldots$
	use $D \subseteq (\mathbb{N}_0, +)$	many U_N^* versions
half-lib.	rules on block sizes and colorings and crossings	?

- Classification program shows: class is very rich (in particular the unitary case)
  OPEN: Full classification?
- Read irreducible rep. and fusion rules from partitions [Freslon-W. 2016]
- New product constructions for CMQG designed first for partitions and then generalized [Gromada-W. 2019, Gromada 2020]
- Deligne interpolation categories: Replace loop parameter $t \in \mathbb{N}$ by $t \in \mathbb{C}$ and obtain $\mathrm{Rep}(S_t, t \in \mathbb{C})$.
  "Easy" QG: Many further examples
  OPEN: investigation of many examples?



[Flake-Maaßen 2020]

- Extremal traces on limit algebras [Wahl 2020]
  OPEN: investigation of many examples?
- **OPEN:** investigation of many associated operator algebras?

Examples of combinatorial descriptions: ② Variants and generalizations of “easy” QGs

a Freslon’s partition QGs: [Freslon 2017, 2019]

Use color set $\mathcal{O}$ with involution $x \mapsto \bar{x}$; then $\mathcal{O}$-colored partitions/categ.

For $\mathcal{O} = \{\circ, \bullet\}$ and $\circ \mapsto \bullet$: $G \subseteq U_N^+$. All $S_N^+ \subseteq G \subseteq U_N^+$ in

For $\mathcal{O} = \{\circ, \bullet\}$ and $\circ \mapsto \circ$: $G \subseteq O_{N_\circ}^+ * O_{N_\bullet}^+$. All $S_N^+ \subseteq G \subseteq O_N^+ * O_N^+$ in

For general $\mathcal{O}$: $G \subseteq U_{N_1}^+ * \cdots * U_{N_k}^+$

OPEN: All $S_{N_\circ}^+ \times S_{N_\bullet}^+ \subseteq G \subseteq O_{N_\circ}^+ * O_{N_\bullet}^+$ or for general $\mathcal{O}$?

b “easy” QGs with 3D partitions: [Cébron-W. 2016]

Given $N = N_1 \cdot N_2 \cdot \ldots \cdot N_m$ and $p \in P(k,l)$ we have:

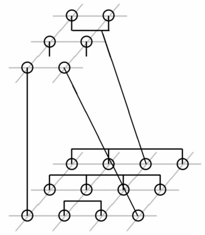
$T_p : (\mathbb{C}^{N_1 \cdot N_2 \cdot \ldots \cdot N_m})^{\otimes k} \to (\mathbb{C}^{N_1 \cdot N_2 \cdot \ldots \cdot N_m})^{\otimes l}$

Viewing $p \in P(km, lm)$ as a 3D-partition, we have:

$T_p : (\mathbb{C}^{N_1} \otimes \ldots \otimes \mathbb{C}^{N_m})^{\otimes k} \to (\mathbb{C}^{N_1} \otimes \ldots \otimes \mathbb{C}^{N_m})^{\otimes l}$

For $N := N_1 = \ldots = N_m$: $S_N \subseteq G \subseteq O_{N^m}^+$.

OPEN: Unitary case; links with other CMQGs etc?



Examples of combinatorial descriptions:
(2) Variants and generalizations of "easy" QGs

(c) *"quizzy" QGs:* [Banica]

Insert weights $\varepsilon : P_{\text{even}} \to \{-1, +1\}$ in the assignment $p \mapsto T_p$:

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) := \sum_{q \geq p} \varepsilon(q) \sum_{j_1, \ldots, j_l} \delta_{=p}(i_1, \ldots, i_k; j_1, \ldots, j_l) e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Use categories of partitions with this twisted fiber functor.
Obtain twists of O_N, U_N and others.

(d) *"super-easy" QGs and 2-parameter deformations:* [Banica-Skalski 2011, Banica 2017]

Study $u = J\bar{u}J^{-1}$ with J diagonal: r blocks of $\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$ and s times 1.

2-parameters: partitions with two colors on r-part and one color on s-part.
Fiber functor similar to $p \mapsto T_p$ or $\{-1, 0, +1\}$-twists as for "quizzy" QGs.
Obtain versions of O_N^+, S_N^+ etc. such as $O^+(r, s)$, $S^+(r, s)$ etc.
OPEN: Which values are allowed for twists of $p \mapsto T_p$?

Examples of combinatorial descriptions: (3) $O^+(Q)$ and $U^+(Q)$

Definition [Wang-VanDaele, Banica 1990s]

Let $Q \in \mathrm{GL}_N(\mathbb{C})$. *Define* $O^+(Q)$ *and* $U^+(Q)$ *via:*

$$A_U(Q) := C^*(u_{ij}, 1 \mid u, Q\bar{u}Q^{-1} \text{ unitary})$$

$$A_O(Q) := C^*(u_{ij}, 1 \mid u = Q\bar{u}Q^{-1} \text{ unitary}) \qquad Q\bar{Q} = c1, c \in \mathbb{R}$$

For $Q = \mathrm{id}$: $O_N^+ = O^+(\mathrm{id})$ and $U_N^+ = U^+(\mathrm{id})$.
For $Q = J$: "super-easy"/2-parameter deformation.

Take the categories of partitions of $O_N^+ = O^+(\mathrm{id})$ and $U_N^+ = U^+(\mathrm{id})$.
Deform $p \mapsto T_p$ to $T^Q_{\sqcap_{\circ\bullet}}(1) = \sum_i e_i \otimes Qe_i$.
This describes the representation theory of $O^+(Q)$ and $U^+(Q)$.
OPEN: How to deform $p \mapsto T_p$ for larger size blocks? $\exists S^+(Q)$?

Examples of combinatorial descriptions: ④ $SU_q(2)$ and $SU_q(N)$

Definition [Woronowicz 1980s]

Let $N \geq 2$, $q \in (0,1]$. Define $SU_q(N)$ via

$$C(SU_q(N)) := C^*\Big(u_{ij}, 1 \,\Big|\, u \text{ unitary}, \sum_{j_1,\ldots,j_N} (-q)^{I(j_1,\ldots,j_N)} u_{i_1 j_1} \cdots u_{i_N j_N} = (-q)^{I(i_1,\ldots,i_N)} 1\Big)$$

Here, $I(j_1,\ldots,j_N) :=$ minimal number of transpositions from $(1,\ldots,N)$.

For $N = 2$ and $Q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}$: $SU_q(2) = O^+(Q)$. So, have full combinatorial description for $SU_q(2)$.

OPEN: Full description for $SU_q(N)$ (only generators known)?

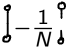
Examples of combinatorial descriptions: (5) "non-easy" QGs

Recall: $S_N \subseteq G \subseteq O_N^+$ "easy", if Mor = category of partitions + $p \mapsto T_p$.

Definition

$S_N \subseteq G \subseteq O_N^+$ *"non-easy"* $:\Longleftrightarrow$ G *not "easy"*

Note: Whenever $S_N \subseteq G$, we have $\mathrm{Mor}_G \subseteq \mathrm{Mor}_{S_N} = \mathrm{span}\{T_p \mid p \in P\}$.
Hence: "non-easy" $\longleftrightarrow$ categories of *linear combinations of* partitions

Functor $\mathcal{P}$: Replace legs in $p \in P(k,l)$ by ; kills singletons
Take $\mathcal{PC}$ as combinatorics (or $\mathcal{VC}$) and $p \mapsto T_p$ as a fiber functor.
Obtain many non-easy QGs, e.g. G isomorphic to the irred. part of S_N^+.
[Gromada-W. 2019]

Skew categories of partitions + $p \mapsto \hat{T}_p$ by subtracting smaller partitions
[Maaßen 2018]

OPEN: Classification of all $S_N \subseteq G \subseteq O_N^+$? Combinatorial description?

Now, consider $G \subseteq S_N^+$ rather than $S_N \subseteq G \subseteq U_N^+$.
$\Gamma = (V, E)$ fin. graph, $|V| = N$, no multiple edges, no self-loops

$$\mathrm{Aut}(\Gamma) = \{\sigma \in S_N \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_N \qquad (\varepsilon \text{ adj. matrix})$$

Definition [Banica 2005]

Define the quantum automorphism group $G_{\mathrm{aut}}^+(\Gamma)$ of Γ via:

$$A_S(N)/\langle u\varepsilon = \varepsilon u\rangle$$

Then, $\mathrm{Aut}(\Gamma) \subseteq G_{\mathrm{aut}}^+(\Gamma) \subseteq S_N^+$.
Γ has quantum symmetries $:\Longleftrightarrow$ $\mathrm{Aut}(\Gamma) \neq G_{\mathrm{aut}}^+(\Gamma)$
Ex.: (a) $G_{\mathrm{aut}}^+(\text{complete graph}) = S_N^+$ has quantum symmetries
(b) $G_{\mathrm{aut}}^+(\text{Petersen graph}) = \mathrm{Aut}(\text{Petersen graph}) = S_5$ no qu. symm. [Schmidt 2018]
Note: $\mathrm{QSym}(C^*(\Gamma)) = G_{\mathrm{aut}}^+(\Gamma)$ [Schmidt-W. 2018]

Examples of combinatorial descriptions:
6 Quantum autom. groups of fin. graphs

Theorem [Mancinska-Roberson et al 2017, 2019]

(AlgC) $\Gamma_1 \cong_q \Gamma_2 \iff \forall K$ *planar:* $|\{\text{hom. } K \to \Gamma_1\}| = |\{\text{hom. } K \to \Gamma_2\}|$

(QIT) $\Gamma_1 \cong_q \Gamma_2 \iff$ *win graph isom. game using qu. strategy with prob. 1*

(QG) $\mathrm{Mor}_{G^+_{aut}(\Gamma)}(k,l) = \mathrm{span}\{T^{K\to\Gamma} \mid K \in \mathcal{P}(k,l)\}$, $\mathcal{P}$ *planar bi-lab. graphs*

[Lovasz 1960s]

$\Gamma_1 \cong_q \Gamma_2 :\iff \exists \pi : A_S(N) \to M_M(H), \pi(u)\varepsilon_1 = \varepsilon_2\pi(u)$. ($M = 1$: $\Gamma_1 \cong \Gamma_2$)

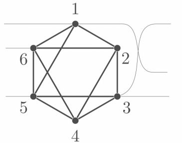
@(AlgC): $\Gamma_1 \cong \Gamma_2 \iff \forall K$ graph: $|\{\text{hom. } K \to \Gamma_1\}| = |\{\text{hom. } K \to \Gamma_2\}|$

@(QIT): Given $v_A, v_B \in V_1 \sqcup V_2$; reply $w_A, w_B \in V_1 \sqcup V_2$; win "$\in E_1 \Leftrightarrow \in E_2$"

@(QG): More generally, consider graph categories

Graph categ.: Bi-labeled graphs K, tensor prod., ...

Fiber functor: $T^{K\to\Gamma} := \#$ hom. $K \to \Gamma$ fixing labels



OPEN: Mor_G of further $G \subseteq O_N^+$, i.e. more examples of graph categories?

e.g. $G = G^*_{aut}(\Gamma) \subseteq G^+_{aut}(\Gamma)$ [Bichon 2003] or $G =$ "easy"$/\langle u\varepsilon = \varepsilon u\rangle$ [Speicher-W. 2019]

Examples of combinatorial descriptions: (7) Vaes-Valveken's example of Property (T)

Triangle presentation: $F = \{1, \ldots, N\}$; $T \subseteq F \times F \times F$ invar. under cyclic permutations + some rules ("of order q") on "predecessors/successors". There is an associated group Γ_T.

Definition [Vaes-Valvekens 2019]

Quantum version of Γ_T defined via:

$$C^*(u_{ij}, 1 \mid u \text{ unitary}, \sum_{a,b,c} \delta_{(a,b,c)\in T} u_{ia} u_{jb} u_{kc} = \delta_{(i,j,k)\in T})$$

The discrete dual of this CMQG has property (T) under good conditions.

Comb. description: Represent T by partitions with blocks of size 1 or 3.
Fiber functor similar to $p \mapsto T_p$ with weights from T-labeling counts
OPEN: T_1, T_2 of same order $\Longrightarrow$ assoc. qu. groups monoidally equiv.?

Examples of combinatorial descriptions:
? Did I forget one?

Sure, many...

- $\{G \subseteq GL_N(\mathbb{C})$ compact group$\} \subseteq$ CMQG $\subseteq$ CQG $\subseteq$ LCQG
- CMQG = Hopf *-algebra + Haar integration
- CMQG may arise as $M_N(\mathbb{C})$-valued solutions of matrix entry relations
- Schur-Weyl/Tannaka-Krein opens the door for rep. theory: combinatorial description = combinatorial category + fiber functor
- Meta conjecture: Every CMQG possesses such a comb. description

	comb. category	fiber functor
"easy" QGs	partitions of sets	$T_p(e_{\mathbf{i}}) := \sum_{\mathbf{j}} \delta_p(\mathbf{i},\mathbf{j}) e_{\mathbf{j}}$
variants of "easy"	+ colors; + 3D	T_p + weights $\{-1, 0, +1\}$
$O^+(Q), U^+(Q)$	part., block size 2	$T^Q \quad (1) = \sum_i e_i \otimes Qe_i$
$SU_q(2), SU_q(N)$	? (generators known)	?
"non-easy" QG	lin. comb.; skew categ.	T_p; ($T_p -$ smaller ones)
$G^+_{\text{aut}}(\Gamma)$	bi-labeled graphs	homomorphism counts
triangle pres. QG	part., block size 1 or 3	T_p + label count weights