LECTURE 1: FREE PROBABILITY

MAIN IDEA:

A unital $\mathbb{C}$-algebra — samples ("random variables")
(& possible further structure)

$\varphi : A \longrightarrow \mathbb{C}$ lin. fctnal — observations ("expectation")
with $\varphi(1) = 1$
(& possible further structure)

$\{\varphi(x^k) \mid k \in \mathbb{N}\}$, $x \in A$ — knowing $x \in A$ ("moments"/"distribution")

Special feature: $xy \neq yx$

Q: How to compute expressions $\varphi(x^{a_1} y^{a_2} x^{a_3} \dots y^{a_n})$? provided we know $\varphi(x^k)$ and $\varphi(y^\ell)$, $k, \ell \in \mathbb{N}$

("What do we know about mixed moments of x and y ?")

$\rightsquigarrow$ INDEPENDENCE CONCEPT FOR NON-COMMUTATIVE SITUATIONS

LECTURE 1: FREE PROBABILITY

MAIN IDEA: $\mathcal{A}$ unital $\mathbb{C}$-algebra

$\varphi : \mathcal{A} \xrightarrow[\text{with } \varphi(1)=1]{} \mathbb{C}$ lin. fctnal

Ex.: a) $\mathcal{A} = L^\infty(\Omega, \mathbb{P}) = \{ X : \Omega \to \mathbb{C} \text{ measurable} \}$ = class. random variables

$\varphi(X) = \mathbb{E}(X) = \int_\Omega X \, d\mathbb{P}$ expectation

Rule for computing mixed moments? INDEPENDENCE: $\mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$

special feature: $xy \neq yx$

b) $\mathcal{A} = M_n(\mathbb{C})$ $n \times n$ matrices, $\varphi = tr$ trace $\boxed{Q: \text{Rule?}}$

c) $\mathcal{A} = M_n(L^\infty(\Omega, \mathbb{P}))$ random matrices, $\varphi = tr \otimes \mathbb{E}$ $\boxed{Q: \text{Rule?}}$

d) $\mathcal{A} = \mathcal{B}(H)$ operators on a Hilbert space, $\varphi(x) = \langle x\xi, \xi \rangle$ $\boxed{Q: \text{Rule?}}$
$\xi \in H$, $\|\xi\| = 1$

Further reading: Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017 - Chapter Basics in free probability

HISTORY OF FREE PROBABILITY THEORY:

Pre:

- Murray & von Neumann 1930's/1940's : von Neumann algebras, inspired from quantum mechanics: $M \subseteq B(H)$ closed by multipl., add., adj., ptws. limits
  generic class of examples : $L(G) := \overline{\mathbb{C}G}^{ptws} \subseteq B(\ell^2 G)$, G discrete group

Then:

"small observations can lead to big discoveries"

- Voiculescu 1980's : wants to solve $\boxed{L(F_n) \not\cong L(F_m),\ n \neq m\ ?}$
  fails, but observes for $x \in L(F_n)$, $y \in L(F_m)$ $\subseteq A := L(F_{n+m})$, $\varphi(x) = \langle x\xi_0, \xi_0 \rangle$:
  $\varphi((x+y)^m)$ depends only on $\varphi(x^k)$, $\varphi(y^\ell)$ $\rightsquigarrow$ Rule: FREE INDEPENDENCE
- Voiculescu 1991: Large N limits of random matrices behave freely indep.
- Speicher early 1990's: Combinatorial approach to free prob., set partitions
- Biane mid 1990's : free increments (stoch. processes), asymptotics of rep. of S_n
- Haagerup mid 1990's : transforms by Fock space oper., use of links RM $\leftrightarrow$ Op. alg

very incomplete list!

- Dykema, Nica, Bercovici, Shlyakhtenko, Mingo, Belinschi, Collins, Thorbjørnsen, Anshelevich, Arizmendi, Kemp, Lehner, Novak, Skoufranis, Bożejko, Capitaine, ...

Further reading: Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017 - Chapter Background and outlook

CONCRETE DEFINITIONS:

a) $(\mathcal{A},\varphi)$ "non-commutative probability space" $:\Leftrightarrow$ $\mathcal{A}$ unital $\mathbb{C}$-algebra
$\varphi: \mathcal{A} \to \mathbb{C}$ linear, $\varphi(1)=1$

POSSIBLE UPGRADES (additional assumptions):

- $(\mathcal{A},\varphi)$ $*$-ncps, if there is an involution $*: \mathcal{A} \to \mathcal{A}$ and φ is positive, i.e. $\varphi(x^*x) \geq 0 \quad \forall x \in \mathcal{A}$
- $(\mathcal{A},\varphi)$ C^*-ncps, if $\mathcal{A}$ is a unital C^*-algebra and φ is positive (i.e. a state)
- $(\mathcal{A},\varphi)$ W^*-ncps, if $\mathcal{A}$ is a von Neumann algebra and φ is a normal state
- $(\mathcal{A},\varphi)$ tracial W^*-ncps, if in addition $\varphi(xy)=\varphi(yx)$ $\forall x,y \in \mathcal{A}$

CONCRETE DEFINITIONS:

a) $(\mathcal{A},\varphi)$ "non-commutative probability space" $:\Leftrightarrow$ $\mathcal{A}$ unital $\mathbb{C}$-algebra
$\varphi: \mathcal{A} \to \mathbb{C}$ linear, $\varphi(1)=1$

Elements $x \in \mathcal{A}$ are called "random variables".

b) $(\mathcal{A},\varphi)$ ncps, $\mathcal{A}_i \subseteq \mathcal{A}$, $i \in I$ subalgebras, $1 \in \mathcal{A}_i$.

(i) $(\mathcal{A}_i)_{i\in I}$ "classical/tensor independent", if $xy=yx$ $\forall x \in \mathcal{A}_i, y \in \mathcal{A}_j, i \neq j$
and $\varphi(x_1 \cdot \ldots \cdot x_n) = \prod_{j=1}^{n} \varphi(x_j)$ whenever $x_j \in \mathcal{A}_{i_j}$ and all i_j are mutually different

(ii) $(\mathcal{A}_i)_{i\in I}$ "freely independent/free", if

$\varphi(x_1 \cdot \ldots \cdot x_n) = 0$ whenever $x_j \in \mathcal{A}_{i_j}$ with $i_1 \neq i_2 \neq \ldots \neq i_n$ and $\varphi(x_j)=0$ $\forall j$

(iii) Elements $x_i \in \mathcal{A}$, $i \in I$ are called free, if $\mathcal{A}_i := \mathrm{alg}(x_i, 1) \subseteq \mathcal{A}$ are free
*-free, if $\mathcal{A}_i := \mathrm{alg}(x_i, x_i^*, 1) \subseteq \mathcal{A}$ are free

CONCRETE DEFINITIONS:

$(\mathcal{A},\varphi)$ "non-commutative probability space" $:\Leftrightarrow$ $\mathcal{A}$ unital $\mathbb{C}$-algebra
$\varphi : \mathcal{A} \to \mathbb{C}$ linear, $\varphi(1)=1$

$(\mathcal{A}_i)_{i\in I}$ "freely independent/free", if
$\varphi(x_1 \cdot \ldots \cdot x_n) = 0$ whenever $x_j \in \mathcal{A}_{i_j}$ with $i_1 \neq i_2 \neq \ldots \neq i_n$ and $\varphi(x_j)=0 \;\forall j$

The rule for computing mixed moments:

x,y classically indep.
$\Rightarrow \varphi(xyxy) = \varphi(x^2)\varphi(y^2)$

$(\mathcal{A},\varphi)$ ncps, $x,y \in \mathcal{A}$ free. $\varphi(xyxy) = ?$

$x,y \in \mathcal{A}$ free $\Rightarrow$ $alg(x,1)$, $alg(y,1)$ free $\Rightarrow$ $x-\varphi(x)1$, $y-\varphi(y)1$ free

Since $\varphi(x-\varphi(x)1)=0$, $\varphi(y-\varphi(y)1)=0$ ("centering trick"),

we have $\varphi\left[(x-\varphi(x)1)(y-\varphi(y)1)(x-\varphi(x)1)(y-\varphi(y)1)\right]=0$

$\Rightarrow \varphi(xyxy) = \varphi(x^2)\varphi(y)^2 + \varphi(x)^2\varphi(y^2) - \varphi(x)^2\varphi(y)^2$

Prop.: $x,y \in (\mathcal{A},\varphi)$ free $\Rightarrow$ All mixed moments may be computed from $\varphi(x^k)$, $\varphi(y^l)$.

CONCRETE DEFINITIONS:

(A,φ) "non-commutative probability space" $:\Longleftrightarrow$ A unital $\mathbb{C}$-algebra
$\varphi: A \to \mathbb{C}$ linear, $\varphi(1)=1$

$(A_i)_{i\in I}$ "freely independent/free", if

$\varphi(x_1\cdot\ldots\cdot x_n)=0$ whenever $x_j\in A_{i_j}$ with $i_1\neq i_2\neq\ldots\neq i_n$ and $\varphi(x_j)=0\ \forall j$

Ex.: a) $A=L^\infty(X,\mathbb{P})$ classical random var., $\varphi=\mathbb{E}=\int_\Omega d\mathbb{P}$.

x,y independent in the usual sense $\Rightarrow$ x,y class./tensor indep.

b) The groups $G_1:=\mathbb{F}_n$, $G_2:=\mathbb{F}_m$ $\subseteq G:=\mathbb{F}_{n+m}$ are free in the sense:

$\boxed{L(\mathbb{F}_n)\not\cong L(\mathbb{F}_m),\ n\neq m\ ?}$

$x_j\in G_{i_j}$, $i_1\neq\ldots\neq i_n$, $x_j\neq e$ $\Rightarrow$ $x_1\cdot\ldots\cdot x_n\neq e$

$A_1:=\mathbb{C}\mathbb{F}_n$, $A_2:=\mathbb{C}\mathbb{F}_m\subseteq A:=\mathbb{C}\mathbb{F}_{n+m}=\left\{\sum^{\text{fin}}_{g\in\mathbb{F}_{n+m}}\alpha_g g\right\}$, $\varphi(\sum\alpha_g g)=\alpha_e$

$x_j\in A_{i_j}$, $i_1\neq\ldots\neq i_n$, $\underbrace{x_j=\sum\alpha^j_g g \text{ with } \alpha^j_e=0}_{\varphi(x_j)=0}$ $\Rightarrow$ $\underbrace{x_1\cdot\ldots\cdot x_n=\sum\beta_g g,\ \beta_e=0}_{\varphi(x_1\cdot\ldots\cdot x_n)=0}$

CONCRETE DEFINITIONS:

(A, φ) "non-commutative probability space" $:\Leftrightarrow$ A unital $\mathbb{C}$-algebra
$\varphi : A \to \mathbb{C}$ linear, $\varphi(1) = 1$

$(A_i)_{i \in I}$ "freely independent/free", if
$\varphi(x_1 \cdot \ldots \cdot x_n) = 0$ whenever $x_j \in A_{i_j}$ with $i_1 \neq i_2 \neq \ldots \neq i_n$ and $\varphi(x_j) = 0 \ \forall j$

Ex.: c) H Hilbert space, Ω vector, $\|\Omega\| = 1$, "vacuum vector".

$$\mathcal{F}(H) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n} \quad \text{Fock space}$$

$A := \mathcal{B}(\mathcal{F}(H))$, $\varphi(x) := \langle x\Omega, \Omega \rangle$, $x \in A$.

$\xi \in H$. $\ell(\xi) \in A$ via $\begin{cases} \ell(\xi)\Omega := \xi \\ \ell(\xi)(\eta_1 \otimes \ldots \otimes \eta_n) := \xi \otimes \eta_1 \otimes \ldots \otimes \eta_n \end{cases}$

"left creation operator"

$\xi_1, \ldots \xi_n \in H$ orthonormal $\Rightarrow$ $\ell(\xi_1), \ldots \ell(\xi_n) \in A$ $*$-free

Further reading: Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017 - Chapter Basics in free probability

ASPECT 1: NON-COMMUTATIVE DISTRIBUTIONS

Def: (A,φ) $*$-ncps, $x_1,\dots x_n \in A$. $\{\varphi(x_{i_1}^{\varepsilon_1}\cdot\ldots\cdot x_{i_k}^{\varepsilon_k}) \mid 1\le i_j\le n,\ \varepsilon_j\in\{1,*\}\}$ "joint $*$-distr." of $x_1,\dots x_n$

Prop.: a) (A,φ), (B,ψ) $*$-ncps, φ,ψ faithful (i.e. $\varphi(x^*x)=0 \Rightarrow x=0$), $x_1,\dots x_n\in A$, $y_1,\dots y_n\in B$.

$$\varphi(x_{i_1}^{\varepsilon_1}\cdot\ldots\cdot x_{i_k}^{\varepsilon_k}) = \psi(y_{i_1}^{\varepsilon_1}\ldots y_{i_k}^{\varepsilon_k}) \quad \forall i,\varepsilon,k \Rightarrow {}^*\text{-alg}(x_1,\dots x_n)\cong {}^*\text{-alg}(y_1,\dots y_n)$$

(i.e. $\mathrm{dist}(x_1,\dots x_n)=\mathrm{dist}(y_1,\dots y_n)$)

$C^*(x_1,\dots x_n)\cong C^*(y_1,\dots y_n)$

$W^*(x_1,\dots x_n)\cong W^*(y_1,\dots y_n)$

b) (A,φ) C^*-ncps, $x\in A$ normal $(x^*x=xx^*)$ $\Rightarrow \exists\mu_x$ measure on $\mathbb{C}$: $\int z^k\bar{z}^\ell\, d\mu_x(z)=\varphi(x^k x^{*\ell})$ $\forall k,\ell\in\mathbb{N}_0$

... $x=x^*$ ... ... on $\mathbb{R}$ ...

Ex.: a) $A=L^\infty(\Omega,\mathbb{P})$, $x\in A \Rightarrow \mu_x=$ distribution of x (classical sense)

b) $A=M_n(\mathbb{C})$, $\varphi=\mathrm{tr}$, $x\in A$ normal, $\lambda_1,\dots\lambda_n\in\mathbb{C}$ Eig.values, $\mu_x=\frac{1}{n}\sum_{k=1}^n\delta_{\lambda_k}$

c) (A,φ) $*$-ncps, $u\in A$, $u^*u=uu^*=1$. "Haar unitary", if $\varphi(u^k)=0$, $k\in\mathbb{Z}\setminus\{0\}$, $\mu_u=\lambda_{S^1}$

d) (A,φ) $*$-ncps, $s\in A$, $s=s^*$, $\sigma\in\mathbb{R}$. "Semicircle", if $\varphi(s^{2m})=\sigma^{2m}C_m$, $\varphi(s^{2m+1})=0$

Further reading: Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017 - Chapter Basics in free probability

ASPECT 2: FREE PROB. & COMBINATORICS

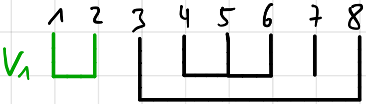
Def: $p \in P(n)$ "partition", if $p = \{V_1, \dots V_r\}$, $V_i \cap V_j = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{r} V_i = \{1, \dots n\}$, $V_i \neq \emptyset$ "blocks"

$p \in NC(n)$ "noncrossing", if $s_1 < t_1 < s_2 < t_2$, $s_1, s_2 \in V_i$, $t_1, t_2 \in V_j$, $i \neq j$ does NOT occur

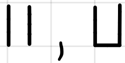
Def: (A, φ) ncps, $\kappa_n : A \times \dots \times A \to \mathbb{C}$ inductively defined by "moment-cumulant formula"

$$\boxed{\varphi(x_1 \cdot \ldots \cdot x_n) = \sum_{p \in NC(n)} \kappa_p(x_1, \dots, x_n)} \quad \text{with} \quad \kappa_p(x_1, \dots x_n) := \prod_{i=1}^{r} \kappa_{|V_i|}\left((x_j)_{j \in V_i}\right) \text{ "cumulants"}$$

Ex.: NC:



$n=1$: $\quad \varphi(x_1) = \kappa_1(x_1) \Rightarrow \kappa_1$ ✓

$n=2$: 

$$\varphi(x_1 x_2) = \kappa_{||}(x_1, x_2) + \kappa_{\sqcup}(x_1, x_2)$$
$$= \kappa_1(x_1)\kappa_1(x_2) + \kappa_2(x_1, x_2) \Rightarrow \kappa_2 \checkmark$$

$n=3$: 

$$\varphi(x_1 x_2 x_3) = \kappa_{|||} + \kappa_{|\sqcup} + \kappa_{\sqcup|} + \kappa_{\sqcup\!\!|} + \kappa_{\sqcup\!\sqcup}$$
$$= \kappa_1(x_1)\kappa_1(x_2)\kappa_1(x_3) + \dots + \kappa_3 \Rightarrow \kappa_3 \checkmark$$

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$p \in NC(n)$ "noncrossing", if $s_1 < t_1 < s_2 < t_2$, $s_1, s_2 \in V_i$, $t_1, t_2 \in V_j$, $i \neq j$ does NOT occur

Def: (A, φ) ncps, $\kappa_n : A \times \dots \times A \to \mathbb{C}$ inductively defined by "moment - cumulant formula"

$$\boxed{\varphi(x_1 \cdot \ldots \cdot x_n) = \sum_{p \in NC(n)} \kappa_p(x_1, \dots, x_n)}$$

with $\kappa_p(x_1, \dots x_n) := \prod_{i=1}^{r} \kappa_{|V_i|}\left((x_j)_{j \in V_i}\right)$ "cumulants"

Ex.: a) $u \in (A, \varphi)$ Haar unitary $\Leftrightarrow$ $u^* u = u u^* = 1$ & $\kappa_{2m}(u, u^*, \dots, u, u^*) = \kappa_{2m}(u^*, u, \dots u^*, u) = (-1)^{m-1} C_{m-1}$, $\kappa = 0$ else

b) $s \in (A, \varphi)$ semicircle $\Leftrightarrow$ $s = s^*$ & $\kappa_2(s, s) = \sigma^2$, $\kappa = 0$ else

$$\boxed{\text{Thm: } A_1, \dots A_m \subseteq (A, \varphi) \text{ free } \Leftrightarrow \forall n: \kappa_n(x_1, \dots x_k, \dots x_\ell, \dots x_n) = 0, \text{ if } \exists\, x_k \in A_{i_k}, x_\ell \in A_{i_\ell}, i_k \neq i_\ell}$$

"vanishing of mixed cumulants"

class.: $\varphi(x_1 \cdot \ldots \cdot x_n) = \sum_{p \in P(n)} c_p(x_1, \dots, x_n)$

$A_1, \dots A_m \subseteq (A, \varphi)$ tensor indep. $\Leftrightarrow$ $\forall n: c_n(x_1, \dots x_k, \dots x_\ell, \dots x_n) = 0$, if $\exists\, x_k \in A_{i_k}, x_\ell \in A_{i_\ell}, i_k \neq i_\ell$

Further reading: Nica, Speicher, Lectures on the combinatorics of free probability, 2016

ASPECT 3: FREE CONVOLUTION OF MEASURES

Let μ, ν be probability measures on $\mathbb{R}$, determined by moments.

Let $(\mathcal{A}, \varphi)$ be a ncps, $x, y \in \mathcal{A}$ free, $\mu_x = \mu$, $\mu_y = \nu$, i.e. $\int_{\mathbb{R}} t^k \, d\mu(t) = \varphi(x^k)$

Def.: a) $\mu \boxplus \nu := \mu_{x+y}$, i.e. $\int_{\mathbb{R}} t^k \, d\mu \boxplus \nu(t) = \varphi((x+y)^k)$

b) $\mu \boxtimes \nu := \mu_{y^{\frac{1}{2}} x y^{\frac{1}{2}}}$ "$= \mu_{xy}$ with xy turned self-adjoint"

Explicit way to compute $\mu \boxplus \nu$:

1.) Compute the "Cauchy/Stieltjes transforms" $G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t} \, d\mu(t)$ & G_ν

2.) Compute the "R-transforms" uniquely determined by $G_\mu\left(\frac{1}{z} + R_\mu(z)\right) = z$ & R_ν

3.) It holds $R_{\mu \boxplus \nu} = R_\mu + R_\nu$ "additivity of the R-transforms"

4.) Use 2.) to obtain $G_{\mu \boxplus \nu}$ and then use "Stieltjes inversion"

$$d\mu \boxplus \nu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} G_{\mu \boxplus \nu}(t + i\varepsilon)$$

Further reading: Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017 - Chapter Random matrices and combinatorics

ASPECT 4: FREE PROB. & THE SEMICIRCLE

Thm (free CLT): $(x_n)_{n\in\mathbb{N}} \subseteq (A,\varphi)$ free, identically distributed, $\varphi(x_n)=0$, $\varphi(x_n^2)=1$.

Then $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i \xrightarrow{\text{distr}} s \in (A,\varphi)$, where s is a semicircle (with $\sigma^2=1$).

Def: $a_n \in (A_n,\varphi_n)$, $n\in\mathbb{N}$, $a\in(A,\varphi)$.

$(a_n)_{n\in\mathbb{N}} \longrightarrow a$ "in distribution"

if $\varphi_n(a_n^k) \xrightarrow[n\to\infty]{} \varphi(a^k) \quad \forall k\in\mathbb{N}$

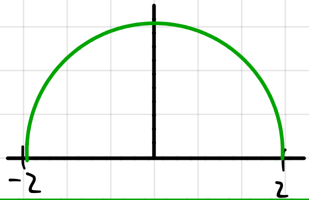
Recall: (A,φ) $*$-ncps, $s\in A$, $s=s^*$, $\sigma^2=1$

$\varphi(s^{2m}) = C_m = \frac{1}{m+1}\binom{2m}{m}$, $\varphi(s^{2m+1})=0$

$\kappa_2(s,s)=1$, $\kappa_n(s,\dots,s)=0$, $n\neq 2$

measure μ_s with $\int_{\mathbb{R}} t^k \, d\mu_s(t) = \varphi(s^k)$

has density $d\mu_s(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,dt$



ASPECT 4: FREE PROB. & THE SEMICIRCLE

Thm (free CLT): $(x_n)_{n\in\mathbb{N}} \subseteq (A,\varphi)$ free, identically distributed, $\varphi(x_n)=0$, $\varphi(x_n^2)=1$.

Then $\displaystyle \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i \xrightarrow{\text{distr}} s \in (A,\varphi)$, where s is a semicircle (with $\sigma^2=1$).

Proof:

$$\varphi\Big(\Big(n^{-\frac12}\sum_{i=1}^{n} x_i\Big)^k\Big) = n^{-\frac{k}{2}} \sum_{i_1,\dots,i_k=1}^{n} \varphi(x_{i_1}\cdots x_{i_k})$$

$$= n^{-\frac{k}{2}} \sum_{i_1,\dots,i_k} \sum_{p\in NC(k)} \kappa_p(x_{i_1},\dots,x_{i_k})$$

$\varphi(x_1\cdots x_n) = \sum_{p\in NC(n)} \kappa_p(x_1,\dots,x_n)$

"moment-cumulant formula"

$$\kappa_p(x_{i_1},\dots,x_{i_k}) = \kappa_p(x_{j_1},\dots,x_{j_k})$$

for $i_1\, i_2\, i_3\, i_4 \dots i_k = j_1\, j_2\, j_3\, j_4 \dots j_k$

freeness (mixed κ vanish) & id. distr.

$$= n^{-\frac{k}{2}} \sum_{p\in NC(k)} \kappa_p \cdot \#\{(i_1,\dots,i_k) \mid i_1\, i_2\, i_3\, i_4 \dots i_k = p\}$$

$$= n^{-\frac{k}{2}} \sum_{p\in NC(k)} \kappa_p \cdot n(n-1)\cdot \dots \cdot (n - \#\text{blocks in } p + 1))$$

$$\approx \sum_{p\in NC(k)} \kappa_p \cdot n^{\#b(p)-\frac{k}{2}} \approx \sum_{p\in NC_2(k)} 1\cdot n^0 = \varphi(s^k)$$

Further reading: Nica, Speicher, Lectures on the combinatorics of free probability, 2016

ASPECT 5: FREE PROB. & RANDOM MATRICES

THE SEMICIRCLE COINCIDENCE

Def: $(\Omega, \mathbb{P})$ class. prob. space, $a_{ij}: \Omega \to \mathbb{C}$ measurable, $a_N := (a_{ij})_{i,j=1,\dots,N}$ "random matrix"

$\left(A := M_N(L^\infty(\Omega, \mathbb{P})), \ \varphi := tr \otimes \mathbb{E} \text{ corresponding ncps} \right)$

a_N "self-adj. Gaussian random matrix" / "Gaussian Unitary Ensemble (GUE)", if

$a_N = a_N^*$ (i.e. $a_{ij} = \overline{a_{ji}}$), a_{ij} indep. complex Gaussian random var., $1 \le i < j \le n$,

$\mathbb{E}[a_{ij}] = 0$, $\mathbb{E}[a_{ij}^2] = 0$, $i \neq j$, $\mathbb{E}[a_{ij}\overline{a_{ij}}] = \frac{1}{N}$

Thm: (a_N) GUE $\Rightarrow$ $a_N \xrightarrow{distr} s$ semicircle, i.e. $\varphi(a_N^k) \to \varphi(s^k)$, $N \to \infty$ $\forall k$

Proof: $\varphi(a_N^k) = \mathbb{E}(tr(a_N^k)) = \frac{1}{N} \sum_{i_1,\dots,i_k=1}^{N} \mathbb{E}[a_{i_1 i_2} a_{i_2 i_3} \cdot \ldots \cdot a_{i_k i_1}]$

$$= \frac{1}{N} \sum_{i_1,\dots,i_k} \sum_{p \in P_2(k)} N^{-\frac{k}{2}} \prod_{j=1}^{r} \delta_{i_j i_{p(r)+1}} = \sum_{p \in P_2(k)} N^{\#\gamma p - 1 - \frac{k}{2}} \xrightarrow[N \to \infty]{} \begin{cases} \# NC_2(k) & \text{even} \\ 0 & \text{odd} \end{cases}$$

a_N "Haar unitary r.m.", if $a_N(\omega)$ unitary $\forall \omega \in \Omega$, Haar distr. $\Rightarrow$ $a_N \xrightarrow{distr} u$ Haar unitary

Further reading: Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017 - Chapter Random matrices and combinatorics

ASPECT 5: FREE PROB. & RANDOM MATRICES

GOOD FOR RANDOM MATRICES: ASYMPTOTIC FREENESS

Def: $(a_N), (b_N)$ random matrices "asymptotically free", if $a_N, b_N \xrightarrow{\text{distr}} a, b \in (A, \varphi)$ free

Thm: a) $(a_N^1), \dots (a_N^n)$ independent GUE matrices $\Rightarrow$ asymptot. free & semicircles

& $(d_N^1), \dots (d_N^n)$ deterministic with $d_N^1, \dots d_N^n \to d_1, \dots d_n$ $\Rightarrow$ & free from $\{d_1, \dots d_n\}$

b) $(a_N), (b_N)$ deterministic, $a_N \to a$, $b_N \to b$, (u_N) Haar unitary r.m. $\Rightarrow$ $(a_N), (u_N b_N u_N^*)$ asymptot. free

Further reading: Mingo, Speicher, Free probability and random matrices, 2017

GOOD FOR OPERATOR ALGEBRAS / FREE GROUP FACTORS: RM MODELS

Thm: a) $L\mathbb{F}_n * R \cong L\mathbb{F}_{n+1}$ $\quad R$ hyperfinite factor $\quad \boxed{L\mathbb{F}_n = W^*(u_1, \dots u_n \text{ free Haar unitaries})}$

b) $L\mathbb{F}_n \cong M_k(L\mathbb{F}_{1+k^2(n-1)})$ $\quad \mathcal{F}(M) = \{t \in \mathbb{R} \mid t > 0,\ M \cong p(M_n(M))p,\ \tau \otimes Tr(p) = t\}$

c) Dichotomy: Either $L\mathbb{F}_n \cong L\mathbb{F}_m \ \forall n \neq m$ or $L\mathbb{F}_n \not\cong L\mathbb{F}_m \ \forall n \neq m$

d) $L\mathbb{F}_n$ has no Cartan subalgebra, is prime, strongly solid, ...

Further reading: Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017 - Chapter Free group factors

ASPECT 6: OPERATOR VALUED FREE PROB.

Def: $(\mathcal{A}, \mathcal{B}, E)$ "operator-valued ncps", if
- $\mathcal{A}$ unital $\mathbb{C}$-algebra
- $\mathcal{B} \subseteq \mathcal{A}$ subalgebra, $1 \in \mathcal{B}$
- $E: \mathcal{A} \to \mathcal{B}$ linear, $E(\underbrace{b_1 a b_2}_{\in \mathcal{BAB}}) = b_1 E(a) b_2$

$\mathcal{B} = \mathbb{C} \rightsquigarrow (\mathcal{A}, \mathcal{B}, E) = (\mathcal{A}, \varphi)$ as before

Ex: $(\mathcal{A}_0, \varphi)$ ncps, $\mathcal{A} := M_2(\mathcal{A})$, $\mathcal{B} := M_2(\mathbb{C})$, $E\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix}$

Def: $\mathcal{A}_i \subseteq \mathcal{A}$, $i \in I$, $\mathcal{B} \subseteq \mathcal{A}_i$ "free with amalgamation over $\mathcal{B}$" / "free wrt E",

if $E(a_1 \cdot \ldots \cdot a_n) = 0$ whenever $a_j \in \mathcal{A}_{i_j}$, $i_1 \neq i_2 \neq \ldots \neq i_n$ and $E(a_j) = 0 \ \forall j$

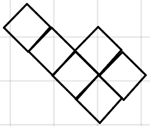
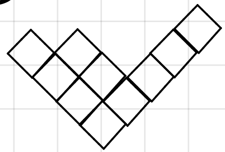
ADVANTAGE: $(\mathcal{A}_i)_{i \in I}$ might not be free (over $\mathbb{C}$) but over $M_2(\mathbb{C})$ ("rough freeness")

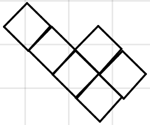
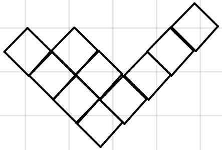
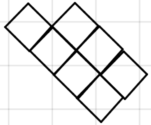
$\rightsquigarrow$ Lecture by Roland Speicher at IHP, September

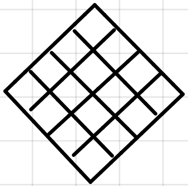
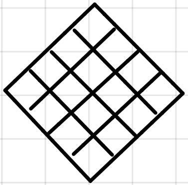
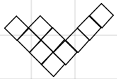
Further reading: Mingo, Speicher, Free probability and random matrices, 2017

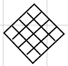
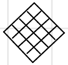
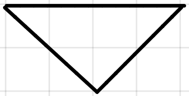
Aspect 7: Free prob. & large rep.'s of S_n

Representations of symmetric group S_n:

Young diagrams  ,  , ... $\rightsquigarrow$ irred. representations

decomposition of irr. rep.:  $\times$  $=$  $+ \dots$

for  $\times$  $= \dots$ $\rightsquigarrow$ generic shape of  ?

Compute μ  $\boxplus$ μ  $\rightsquigarrow$ obtain  as limiting shape

Further reading: Speicher, Free probability and its avatars ..., Jahresber Dtsch Math-Ver, 119:3-30, 2017

ASPECT 8: FURTHER NON-COMM. INDEPENDENCES

- classical / tensor $\quad \varphi(x_1 \cdot \ldots \cdot x_n) = \prod_{j=1}^{n} \varphi(x_j)$ if i_j mutually different, x_i comm.
- free $\quad \varphi(x_1 \cdot \ldots \cdot x_n) = 0$ if $i_1 \neq i_2 \neq \ldots \neq i_n$ and $\varphi(x_j) = 0$
- Boolean $\quad \varphi(x^{a_1} y^{b_1} x^{a_2} y^{b_2} \ldots x^{a_k} y^{b_k}) = \varphi(x^{a_1}) \varphi(y^{b_1}) \cdot \ldots \cdot \varphi(x^{a_k}) \varphi(y^{b_k})$

  Further reading: Bozejko, Speicher, Woroudi,...
- monotone & antimonotone

  Further reading: Naofumi Muraki's work
- traffic freeness

  Further reading: Camille Male's work
- type B freeness

  Further reading: Biane, Goodman, Nica, Belinschi, Shlyakhtenko,...
- second order freeness

  Further reading: Mingo and Speicher's work
- finite free probability

  Further reading: Adam Marcus's work
- bi-free probability

  Further reading: Voiculescu's very recent work
- ...

Thm: (A_1, φ_1), (A_2, φ_2) ncps. How many "natural" rules are there on $A_1 * A_2$ to compute mixed moments from those of A_1 and A_2?

a) A_i unital

b) A_i non-unital

c) without "$\varphi_1 * \varphi_2 = \varphi_2 * \varphi_1$"

OPEN PROBLEMS IN FREE PROB.:

- $L\mathbb{F}_n \cong L\mathbb{F}_m$?
- free entropy (dimension): different approaches coincide? limsup = lim ? vN invariant ?
- regularity of nc distributions: : no atoms, "density", free stochastic dif. eq. ...
- Brown measures
- ...

LITERATURE ON FREE PROB.:

- Voiculescu, Dykema, Nica: Free random variables, 1992.
- Voiculescu: Free probability theory, 1997.
- Hiai, Petz: The semicircle law, free random variables and entropy, 2000.
- Nica, Speicher: Lectures on the combinatorics of free probability, 2006.
- Voiculescu, Stammeier, Weber: Free probability and operator algebras, 2016.
- Mingo, Speicher: Free probability and random matrices, 2017.