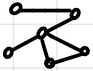
LECTURE 2: QUANTUM GROUPS

MAIN IDEA:

topology/geometry	$\rightarrow$	noncomm. topology/quantum space
X compact space		A possibly noncomm. C^*-algebra
$S^{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$		$C^*(x_1, \dots x_n, 1 \mid x_i = x_i^*, \sum x_i^2 = 1)$
$\Gamma = (V, E) =$ 		$C^*(\Gamma)$ graph C^*-algebra
$\downarrow$		$\downarrow$
symmetry	$\rightarrow$	quantum symmetry
$Iso(S^{n-1}) = O_n$		$QISO(S^{n-1,+}) = O_n^+$
$Aut(\Gamma) \subseteq S_n$, $\lvert V \rvert = n$		$QAut(\Gamma) \subseteq S_n^+$

special feature: $xy \neq yx$

$\rightsquigarrow$ SYMMETRY CONCEPT FOR NON-COMMUTATIVE SITUATIONS

HISTORY OF (COMPACT) QUANTUM GROUPS (CQG):

Pre:

- G loc. comp. abelian group. $\hat{G} := \{\varphi : G \to \mathbb{C} \text{ character}\}$ again loc. comp. abelian grp.
  Pontrjagin duality: $\hat{\hat{G}} \cong G$. But for non-abelian groups? Kac 1960's
- Hopf algebras 1940's ; Yang-Baxter equations 1970's
- deformations of Lie algebras ⇝ rep. th. of groups Drinfeld, Jimbo 1980's

Then:

- Woronowicz 1980's: Def. of CQGs & CMQGs, based on C^*-alg.; Haar integration, quantum Schur-Weyl, example $SU_q(2)$

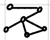
Glorious 1980's:
Connes non-comm. geom.
Jones subfactors
Woronowicz CQG
Voiculescu free prob.

- van Daele 1990's: Haar integration 2.0, duality concepts
- Sh. Wang 1990's: Examples O_n^+, S_n^+, U_n^+ ; Baaj, Skandalis 1990's: multipl. unitaries
- Banica 1990's: rep. th. O_n^+, S_n^+, U_n^+, ...
- Kustermans, Vaes 2000: loc. comp. QG

Further reading: Timmermann, An invitation to quantum groups and duality, 2008, Introduction and CH 4

- Bichon, Collins, Vergnioux, Neshveyev, Skalski, Sołtan, Pusz, Kasprzak, Speicher, Curran, Raum, Freslon, Goswami, Bhowmick, Brannan, ...

[INTERLUDE: C^*-ALGEBRAS]

topology / geometry	$\rightarrow$	noncomm. topology / quantum space ?
X compact space		A possibly noncomm. C^*-algebra
$S^{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$		$C^*(x_1, \dots x_n, 1 \mid x_i = x_i^*, \sum x_i^2 = 1)$
$\Gamma = (V,E) =$ 		$C^*(\Gamma)$ graph C^*-algebra

Def (Gelfand, Naimark 1940's): A is a C^*-algebra $:\Longleftrightarrow$

A is a unital $\mathbb{C}$-algebra with involution $^*: A \to A$, a nice norm $\|\cdot\|$, complete wrt. $\|\cdot\|$

(* antilin., $x^{**} = x$, $(xy)^* = y^* x^*$) ($\|xy\| \leq \|x\| \|y\|$, $\|x^*x\| = \|x\|^2$ i.e. $[x^*x = 0 \Rightarrow x = 0]$)

Fundamental Thm. I: A commutative $\Longleftrightarrow$ $\exists X$ comp. topol. space: $A \cong C(X) := \{f: X \to \mathbb{C} \text{ cont.}\}$

(ptws. oper., $f^*(x) := \overline{f(x)}$, $\|\cdot\|_\infty$)

$\rightsquigarrow$ noncomm. C^*-algebras $\hat{=}$ noncomm. topology / quantum space

Fundamental Thm. II: $\exists \pi: A \hookrightarrow B(H)$ $\rightsquigarrow$ abstract C^*-alg.'s are actually concrete

($A = B(H) = M_n(\mathbb{C})$, $\dim H < \infty$: matrix multipl., $(a_{ij})^* = (\overline{a_{ji}})$, matrix norm $\|\cdot\|$)

universal C^*-algebras (simplified):

$$C^*(x_1, \dots x_n \mid p_i(x_1, \dots x_n) = 0, i \in I) := \overline{\{\text{non-comm. polynomials in } x_1, \dots x_n \text{ and } x_1^*, \dots x_n^*\} / \langle p_i \rangle}^{\|\cdot\|_{max}}$$

Further reading: Blackadar, Operator algebras, 2006

CONCRETE DEFINITIONS:

Def.: $G=(A,\Delta)$ CQG $:\Leftrightarrow$
- A unital C^*-algebra
- $\Delta: A \to A \otimes_{min} A$ unital *-hom. ("co-multiplication")
- $(\Delta \otimes id)\circ\Delta = (id\otimes\Delta)\circ\Delta$
- span $\Delta(A)(1\otimes A)$, span $\Delta(A)(A\otimes 1) \subseteq A\otimes_{min} A$ dense

CQG: Compact Quantum Group

write $A = C(G)$

Rem: a) a CQG is not a group!

b) $\{$compact groups$\} \subsetneq$ CQG: G compact group $\Rightarrow (C(G),\Delta)$ CQG
- $C(G) := \{f: G \to \mathbb{C}$ continous$\}$ unital C^*-algebra
- $\Delta: C(G) \to C(G)\otimes C(G) \cong C(G\times G)$ with $\Delta(f)(s,t) := f(st)$ for $f\in C(G)$, $s,t\in G$

c) (A,Δ) CQG, A commutative $\Rightarrow \exists G$ compact group: $(A,\Delta)\cong(C(G),\Delta)$ as in b)

CONCRETE DEFINITIONS:

Def.: $G = (A, \Delta)$ CQG $:\Longleftrightarrow$
- A unital C^*-algebra
- $\Delta : A \to A \otimes_{min} A$ unital *-hom. ("co-multiplication")
- $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$
- $\operatorname{span} \Delta(A)(1 \otimes A)$, $\operatorname{span} \Delta(A)(A \otimes 1) \subseteq A \otimes_{min} A$ dense

Compact Quantum Group

write $A = C(G)$

Ex.: G discrete group, $\mathbb{C}G := \{ \sum_{g \in G}^{fin} \alpha_g g \mid \alpha_g \in \mathbb{C} \}$

$A := C^*_{(max)}(G) := \overline{\mathbb{C}G}^{\|\cdot\|_{max}}$, $\Delta : C^*(G) \to C^*(G) \otimes C^*(G)$, $g \mapsto g \otimes g$

Further reading: Neshveyev, Tuset, Compact quantum groups and their representation categories, 2013

ASPECT 1: HAAR INTEGRATION ON CQG's:

class. situation: G compact group $\overset{\text{Riesz}}{\Longrightarrow}$ $\exists!\ \mu_G$ "Haar measure":

$$\int_G f(st)\,d\mu_G(s) = \int_G f(s)\,d\mu_G(s) \qquad \forall f \in C(G),\ t \in G$$

Thm: (A,Δ) CQG $\Rightarrow$ $\exists!\ h: A \to \mathbb{C}$ "Haar state":

$$(id \otimes h)\circ\Delta = (h \otimes id)\circ\Delta = h \cdot 1_A$$

Proof: $\omega: A \to \mathbb{C}$ any state $\Rightarrow$ $\frac{1}{n}\sum_{k=1}^{n} \omega^{*k} \to \rho$ (accumul. pt), $\omega_1 * \omega_2 = (\omega_1 \otimes \omega_2)\circ\Delta$

$\Rightarrow K_\omega := \{\rho \mid \rho * \omega = \omega * \rho = \omega(1)\rho\} \neq \emptyset$, $\quad h \in \bigcap_\omega K_\omega \neq \emptyset$ (Cantor's intersect.)

For $a \in A$, $b := h(a) - (id \otimes h)(\Delta(a))$ have $\omega(b) = 0$ $\forall \omega$ state $\Rightarrow b = 0$.

Ex.: G compact group, $h: C(G) \to \mathbb{C}$, $f \mapsto \int_G f(s)\,d\mu_G(s)$ Haar state:

$$(h \otimes id)\circ\Delta(f)(t) = \int_G \Delta(f)(s,t)\,d\mu_G(s) = \int_G f(st)\,d\mu_G(s) = h(f)\,1_{C(G)}(t)$$

Further reading: Weber, Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups, 2017/2018 (?) OR Timmermann OR Neshveyev-Tuset

ASPECT 2: COMPACT MATRIX QUANTUM GROUPS:

Def.: $G=(A,u)$ CMQG $:\Leftrightarrow$ • $A=C^*(u_{ij},1,\ i,j=1,\dots n)$ unital C^*-alg.

Comp. Matrix Qu. Grp. • $u=(u_{ij})$, $\bar{u}=(u_{ij}^*)$ invertible in $M_n(A)$

write $A=C(G)$ • $\Delta: A\to A\otimes_{\min} A$, $u_{ij}\mapsto\sum_k u_{ik}\otimes u_{kj}$ unital $*$-hom

Rem.: a) CMQG $\subsetneq$ CQG

Def.: $G=(A,\Delta)$ CQG $:\Leftrightarrow$ • A unital C^*-algebra

Compact Quantum Group • $\Delta: A\to A\otimes_{\min} A$ unital $*$-hom. ("co-multiplication")

• $(\Delta\otimes \mathrm{id})\circ\Delta=(\mathrm{id}\otimes\Delta)\circ\Delta$

write $A=C(G)$ • span $\Delta(A)(1\otimes A)$, span $\Delta(A)(A\otimes 1)\subseteq A\otimes_{\min} A$ dense

b) $G\subseteq GL_n(\mathbb{C})\subseteq M_n(\mathbb{C})$ comp. group

• $C(G)=C^*(\tilde{u}_{ij},1)$ by Stone-Weierst., $\tilde{u}_{ij}: G\to\mathbb{C}$, $(a_{kl})\mapsto a_{ij}$

• $\Delta(\tilde{u}_{ij})(g,h)=\tilde{u}_{ij}(gh)=\sum_k g_{ik}h_{kj}=\sum_k \tilde{u}_{ik}\otimes\tilde{u}_{kj}(g,h)$

ASPECT 2: COMPACT MATRIX QUANTUM GROUPS:

Def.: $G = (A, u)$ CMQG $:\Leftrightarrow$ • $A = C^*(u_{ij}, 1, i,j = 1, \dots n)$ unital C^*-alg., some $n \in \mathbb{N}$

Comp. Matrix Qu. Grp.

• $u = (u_{ij})$, $\bar{u} = (u_{ij}^*)$ invertible in $M_n(A)$

write $A = C(G)$

• $\Delta: A \to A \otimes_{min} A$, $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ unital $*$-hom

Ex.: a) $A_o(n) := C(O_n^+) := C^*(1, u_{ij}, i,j = 1, \dots n \mid u_{ij} = u_{ij}^*, \sum_k u_{ik} u_{jk} = \sum_k u_{ki} u_{kj} = \delta_{ij})$

$(A_o(n), u)$ CMQG "free orthogonal q.grp." with $A_o(n) / \langle u_{ij} \text{ comm.} \rangle = C(O_n)$

indeed: $\sum_k u_{ik} u_{jk} = \delta_{ij} \Leftrightarrow u u^t = 1$. We have $O_n \subseteq O_n^+$.

b) Def. of U_n^+ with $U_n \subseteq U_n^+$

c) $C(SU_q(2)) := C^*\left(\alpha, \gamma \mid \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ unitary}\right)$, $q \in [-1,1] \setminus \{0\}$

$q = 1$: $C(SU_q(2)) = C(SU(2))$

Further reading: Weber, Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups, 2017/2018 (?) OR Timmermann OR Neshveyev-Tuset

ASPECT 3: SCHUR-WEYL / TANNAKA-KREIN FOR CMQG:

REPRESENTATIONS

classical situation: G compact group, $U: G \to M_n(\mathbb{C})$ rep. of G, if U cont.,

i.e. $U \in C(G, M_n(\mathbb{C})) \cong C(G) \otimes M_n(\mathbb{C})$, $U(g) = \sum_{ij} u_{ij}(g) \otimes e_{ij}$, $u_{ij} \in C(G)$

and $\sum_{ij} \Delta(u_{ij})(g,h) \otimes e_{ij} = U(gh) = U(g)U(h) = \sum_{ij} \left(\sum_k u_{ik} \otimes u_{kj}\right)(g,h) \otimes e_{ij}$

Def.: a) (A, Δ) CQG. $u = (u_{ij}) \in M_n(A)$ "(fin.dim.) unitary rep.", if $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$

& u unitary

b) $u \otimes v := \sum_{ijkl} u_{ij} v_{kl} \otimes e_{ij} \otimes e_{kl} \in M_{n_u}(\mathbb{C}) \otimes M_{n_v}(\mathbb{C}) \otimes A = M_{n_u \cdot n_v}(A)$

c) u irreducible $:\Leftrightarrow$ $\left[T \in M_{n_u}(\mathbb{C}),\ Tu = uT \Rightarrow T = \lambda \cdot 1, \text{ some } \lambda \in \mathbb{C} \right]$

i.e. $u \neq \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$

ASPECT 3: SCHUR-WEYL / TANNAKA-KREIN FOR CMQG:

TANNAKA-KREIN FOR CMQG:

Thm: a) (A,u) CMQG $\Rightarrow$ $\mathrm{Rep}(A,u) := \{$fin. dim. unitary rep.$\}$ $\overline{W}$-tensor categ.

b) $\mathfrak{R}$ W-tensor categ. $\Rightarrow$ $\exists!$ (A,u) CMQG : $\mathfrak{R} = \overline{\mathrm{Rep}}(A,u)$

Hence: Duality $\{W$-tensor categ.$\} \longleftrightarrow \{$CMQG$\}$

Main ingredient of the proof:

interpret morphisms $T \in \mathfrak{R}$ as "intertwiners" $Tu = vT$, $u, v \in \mathfrak{R}$

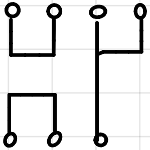
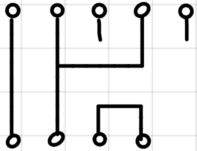
(i.e. $T: H_u \longrightarrow H_v$ linear maps)

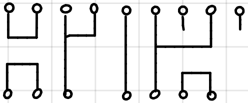
W-tensor category involves $T_1, T_2 \in \mathfrak{R} \Rightarrow T_1 \otimes T_2,\ T_1 T_2,\ T_1^* \in \mathfrak{R}$

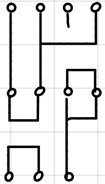
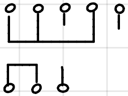
Further reading: Weber, Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups, 2017/2018 (?) OR Neshveyev-Tuset

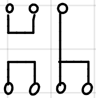
ASPECT 4: BANICA-SPEICHER QUANTUM GROUPS:

Recall:

set partitions $p =$  $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}_0$ $\in P(4,3)$, $q =$  $\in P(5,4)$

operations $p \otimes q =$ 

$pq =$  $=$ 

$p^* =$ 

Ex.: a) $\mathcal{C} = \{\text{all partitions}\}$

b) $\mathcal{C} = \{\text{all blocks size } 2\}$

c) $\mathcal{C} = \{\text{noncross. part.}\}$

d) $\mathcal{C} = (b) \cap (c)$

Def.: $\mathcal{C} = (\mathcal{C}(k,\ell))_{k,\ell \in \mathbb{N}_0} \subseteq \bigcup_{k,\ell \in \mathbb{N}_0} P(k,\ell)$ category of partitions, if

$p, q \in \mathcal{C} \implies p \otimes q,\ pq,\ p^* \in \mathcal{C}$ & ,  $\in \mathcal{C}$

ASPECT 4: BANICA-SPEICHER QUANTUM GROUPS:

Machine: Given a category of partitions $\mathcal{C}$,

define natural linear maps $T_p : (\mathbb{C}^n)^{\otimes k} \longrightarrow (\mathbb{C}^n)^{\otimes \ell}$, $p \in \mathcal{C}(k,\ell)$

$\Longrightarrow$ $\mathcal{R} := \operatorname{span}\{T_p \mid p \in \mathcal{C}\}$ W-tensor category

(since $T_p \otimes T_q = T_{p \otimes q}$ etc.)

$\overset{TK}{\Longrightarrow}$ obtain a Banica-Speicher QG (aka "easy QG")

Ex.:
a) $S_n \leftrightarrow \{$all partitions$\}$
b) $O_n \leftrightarrow \{$all blocks size 2$\}$
c) $S_n^+ \leftrightarrow \{$noncross. part.$\}$
d) $O_n^+ \leftrightarrow$ (b) $\cap$ (c)

Philosophy: BSQG's are completely determined by combinatorics of partitions

Further reading: Weber, Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups, 2017/2018 (?)
OR Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017, CH Easy quantum groups

Def.: $\mathcal{C} = (\mathcal{C}(k,\ell))_{k,\ell \in \mathbb{N}_0} \subseteq \bigcup_{k,\ell \in \mathbb{N}_0} P(k,\ell)$ category of partitions, if

$p, q \in \mathcal{C} \Longrightarrow p \otimes q,\ pq,\ p^* \in \mathcal{C}$ & $\sqcap\ ,\ | \in \mathcal{C}$

ASPECT 5: QUANTUM SYMMETRIES:

ACTIONS

classical situation: G compact group, X compact space

$\mathring{\alpha}: G \times X \to X$ action $\rightsquigarrow$ $\alpha: \mathcal{C}(X) \to \mathcal{C}(G \times X) \cong \mathcal{C}(G) \otimes \mathcal{C}(X)$

$f \longmapsto f \circ \mathring{\alpha}$

Def: (A, Δ) CQG, B unital C^*-algebra. An action is

$\alpha: B \to A \otimes_{\min} B$ unital $*$-hom. s.t.

$(id \otimes \alpha) \circ \alpha = (\Delta \otimes id) \circ \alpha$ & $\operatorname{span} \alpha(B)(A \otimes 1) \subseteq A \otimes_{\min} B$ dense

We say that (A, Δ) is the quantum symmetry of B, if it is "the maximal" quantum group acting on B.

ASPECT 5: QUANTUM SYMMETRIES:

Q. SYMMETRIES

topology/geometry	$\rightarrow$	noncomm. topology / quantum space
X compact space		A possibly noncomm. C^*-algebra
$S^{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$		$C^*(x_1, \dots x_n, 1 \mid x_i = x_i^*, \sum x_i^2 = 1)$
$\Gamma = (V,E) =$ 		$C^*(\Gamma)$ graph C^*-algebra
$\downarrow$		$\downarrow$
symmetry	$\rightarrow$	quantum symmetry
$Iso(S^{n-1}) = O_n$		$QISO(S^{n-1,+}) = O_n^+$
$Aut(\Gamma) \subseteq S_n$, $\lvert V \rvert = n$		$QAut(\Gamma) \subseteq S_n^+$

If (A,u) is a CMQG and $B = C^*(x_1, \dots x_n, 1)$, actions boil down to

$$\alpha : B \longrightarrow A \otimes B$$
$$x_i \longmapsto \sum_k u_{ik} \otimes x_k$$

Ex.: a) $C(S^{n-1,+}) := C^*(x_1, \dots x_n, 1 \mid x_i = x_i^*, \sum_i x_i^2 = 1)$ "free sphere", $C(S^{n-1,+}) \big/ \langle \text{comm.} \rangle \cong C(S^{n-1})$

(A,u) some CMQG acting on $S^{n-1,+}$. Then:

$$\sum_{ij} \delta_{ij} \otimes x_i x_j = 1 \otimes 1 = \alpha(1) = \sum_k \alpha(x_k^2) = \sum_{kij} u_{ki} u_{kj} \otimes x_i x_j = \sum_{ij} \Big(\sum_k u_{ki} u_{kj} \Big) \otimes x_i x_j$$

$\Longrightarrow$ $u \in M_n(A)$ orthogonal $\rightsquigarrow$ $QSym(S^{n-1,+}) = O_n^+$

Goswami & Bhowmick: quantum symmetries of Connes's noncommutative manifolds $\rightsquigarrow$ quantum (Riemannian) isometries

$\rightsquigarrow$ $QISO(S^{n-1,+}) = O_n^+$

Further reading: Goswami, Bhowmick, Quantum isometry groups, 2016

Further reading: Weber, Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups, 2017/2018 (?) OR arxiv:1603.09192

ASPECT 5: QUANTUM SYMMETRIES:

Q. SYMMETRIES

If (A,u) is a CMQG and $B = C^*(x_1, \dots x_n, 1)$,

actions boil down to $\alpha: B \longrightarrow A \otimes B$

$$x_i \longmapsto \sum_k u_{ik} \otimes x_k$$

topology/geometry	→	noncomm. topology/quantum space
X compact space		A possibly noncomm. C^*-algebra
$S^{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$		$C^*(x_1, \dots x_n, 1 \mid x_i = x_i^*, \sum x_i^2 = 1)$
$\Gamma = (V,E) =$ 		$C^*(\Gamma)$ graph C^*-algebra
↓		↓
symmetry	→	quantum symmetry
$Iso(S^{n-1}) = O_n$		$QISO(S^{n-1,+}) = O_n^+$
$Aut(\Gamma) \subseteq S_n$, $\lvert V \rvert = n$		$QAut(\Gamma) \subseteq S_n^+$

Ex.: b) $X = \{t_1, \dots t_n\}$ n points, $Aut(X) = S_n$ symm. group

$$\mathcal{C}(X) = C^*(x_1, \dots x_n, 1 \mid x_i = x_i^* = x_i^2, \sum_i x_i = 1) \text{ (commutative)}$$

(A,u) some CMQG acting on X. Then:

$$\sum_i 1 \otimes x_i = 1 \otimes 1 = \alpha(1) = \sum_k \alpha(x_k) = \sum_{ki} u_{ki} \otimes x_i = \sum_i \Big(\sum_k u_{ki}\Big) \otimes x_i$$

$$A_s(n) := C(S_n^+) := C^*(1, u_{ij}, i,j = 1, \dots n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{kj} = \sum_k u_{ik} = 1 \ \forall i,j)$$

$(A_s(n), u)$ "free symmetric q.grp." with $A_s(n) / \langle u_{ij} \text{ comm.} \rangle = C(S_n)$, $S_n \subseteq S_n^+$

$\rightsquigarrow$ $QSym(n \text{ points}) = S_n^+$ $\rightsquigarrow$ Def. of $QAut(\Gamma) \subseteq S_n^+$

Further reading:
arxiv:1706.08833

ASPECT 6: REPRESENTATION THEORY REV. – FUSION RULES:

Thm: (A,Δ) CQG, u fin.dim. unitary rep. $\Rightarrow$ $u = \bigoplus u_r$ irred. rep.

Hence:
- find all irred. representations
- explain how $u \otimes v = \bigoplus u_r$ decomposes

Ex.: a) irred. representations of O_n^+ indexed by $\mathbb{N}$: $(v^k)_{k\in\mathbb{N}}$

$$v^k \otimes v^\ell = v^{|k-\ell|} \oplus v^{|k-\ell|+2} \oplus v^{|k-\ell|+4} \oplus \ldots \oplus v^{k+\ell}$$

b) irred. representations of S_n^+ indexed by $\mathbb{N}$: $(v^k)_{k\in\mathbb{N}}$

$$v^k \otimes v^\ell = v^{|k-\ell|} \oplus v^{|k-\ell|+1} \oplus v^{|k-\ell|+2} \oplus \ldots \oplus v^{k+\ell}$$

c) for Banica-Speicher QG: can express everything in terms of partitions

Further reading: Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2017, CH Easy quantum groups

ASPECT 7: "ALGEBRAIC" QUANTUM GROUPS:

Def.: $G=(A,\Delta)$ CQG $:\Leftrightarrow$
- A unital C^*-algebra
- $\Delta : A \to A \otimes_{min} A$ unital *-hom. ("co-multiplication")
- $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$
- span $\Delta(A)(1 \otimes A)$, span $\Delta(A)(A \otimes 1) \subseteq A \otimes_{min} A$ dense

Compact Quantum Group

write $A = C(G)$

Our definition of a quantum group:

What about "non-compact"?

Essential quantization step: $(G, \circ : G \times G \to G)$ group $\rightsquigarrow$ $(alg(G), \dots)$

which structure? your choice!

for topologists: C^*-algebras (since "A comm. C^*-alg. $\Rightarrow A \cong C(X)$")

topology

for algebraists: (pointed) Hopf algebras

Def.: (A,Δ) CQG. $u = (u_{ij}) \in M_n(A)$ "(fin.dim.) unitary rep.", if $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ & u unitary

Thm: (A,Δ) CQG, $A_0 := \text{span}\{ u_{ij} \mid u \text{ fin.dim. unitary rep. of } (A,\Delta)\} \subseteq A$.

Then $A_0 \subseteq A$ dense & $(A_0, \Delta|_{A_0} : A_0 \to A_0 \otimes A_0, S, \varepsilon)$ Hopf algebra

dual to multiplication   dual to inverse   dual to neutral element

Further reading: Timmermann, An invitation to quantum groups and duality, 2008, CH 5.4

LITERATURE ON COMPACT QUANTUM GROUPS:

- Timmermann, An invitation to quantum groups and duality, 2008.
- Neshveyev, Tuset, Compact quantum groups and their rep. categ., 2013.
- Goswami, Bhowmick, Quantum isometry groups, 2016.
- Voiculescu, Stammeier, Weber: Free probability and operator algebras, 2016.
- Franz, Quantum symmetries, 2017. feat. Aubrun
- Weber, Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups, 2017/2018?