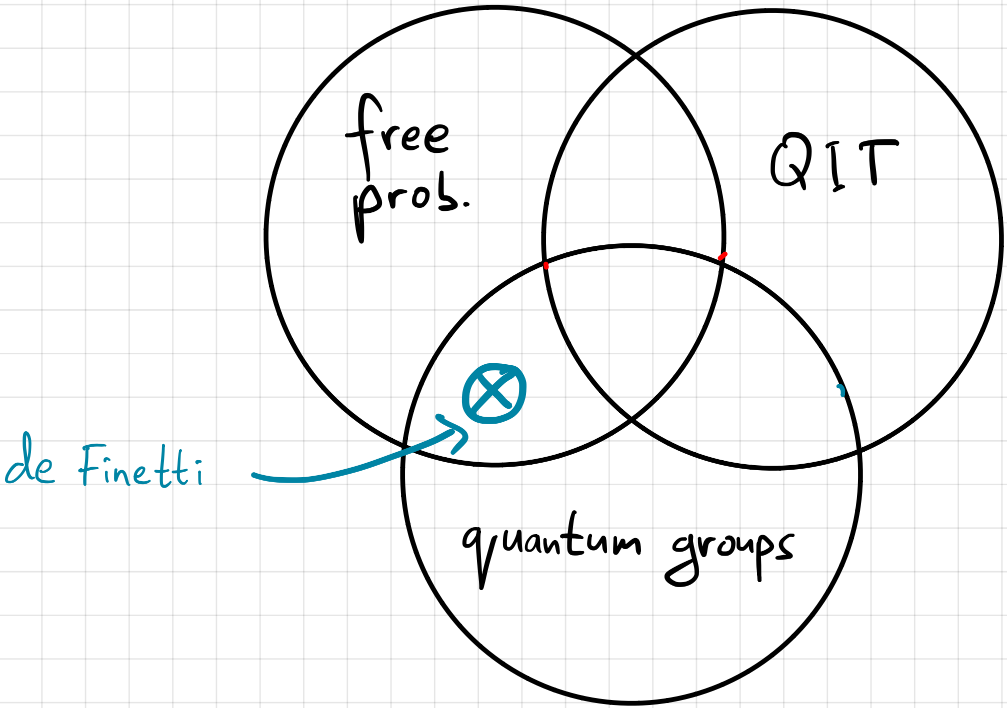
free prob.
QIT
quantum groups
de Finetti

LINK FP-QG: FREE DE FINETTI

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DE FINETTI THEOREMS FOR EASY QUANTUM GROUPS

BY TEODOR BANICA[1], STEPHEN CURRAN AND ROLAND SPEICHER[2]

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A Noncommutative de Finetti Theorem: Invariance under Quantum Permutations is Equivalent to Freeness with Amalgamation

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Abstract: We show that the classical de Finetti theorem has a canonical noncommutative counterpart if we strengthen "exchangeability" (i.e., invariance of the joint distribution of the random variables under the action of the permutation group) to invariance under the action of the quantum permutation group. More precisely, for an infinite sequence of noncommutative random variables $(x_i)_{i\in\mathbb{N}}$, we prove that invariance of the joint distribution of the x_i's under quantum permutations is equivalent to the fact that the x_i's are identically distributed and free with respect to the conditional expectation onto the tail algebra of the x_i's.

1. Introduction

The de Finetti theorem states that an infinite family of random variables whose distribution is invariant under finite permutations (such a family is called *exchangeable*) is independent and identically distributed with respect to the conditional expectation onto the tail algebra of the random variables. Since the implication in the other direction is fairly elementary one has the equivalence between exchangeability and conditional independence. See, e.g., [Kal] for an exposition on the classical de Finetti theorem.

In a noncommutative context classical random variables are replaced by, typically noncommuting, operators on Hilbert spaces. The expectation with respect to a probability measure is then replaced by a state on the algebra generated by these operators. Of course, the notion of invariance of mixed moments still makes sense. Thus one can ask what exchangeability means in such a context. It turns out that in the noncommutative world there are actually many quite different possibilities for exchangeable random variables. It was shown in [Koe1] that they all possess some kind of factorization property; but, as one sees from the variety of examples, one cannot expect that exchangeability implies some fixed kind of independence. Indeed, both independence and freeness

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With the action $\alpha: W^*(x_1,\dots,x_n) \longrightarrow C(S_n^+)\otimes W^*(x_1,\dots,x_n)$

$x_i \longmapsto \sum_j u_{ij}\otimes x_j$

the distribution of $(x_1,\dots,x_n)$ is S_n^+ invariant, iff

$$\varphi(x_{i(1)}\cdots x_{i(m)}) = \sum_{j(1),\dots,j(m)} u_{i(1)j(1)}\cdots u_{i(m)j(m)}\,\varphi(x_{j(1)}\cdots x_{j(m)})$$

Note: for $u_{ij}=\tilde{u}_{ij}: C(S_n)\longrightarrow\mathbb{C},\ (a_{kl})\longmapsto a_{ij}$, obtain class. exch.:

$$\varphi(x_{i(1)}\cdots x_{i(m)}) = \sum_{j(1),\dots,j(m)} \delta_{\sigma(i(1))j(1)}\cdots\delta_{\sigma(i(m))j(m)}\,\varphi(x_{j(1)}\cdots x_{j(m)})$$

$$= \varphi(x_{\sigma(i(1))}\cdots x_{\sigma(i(m))}) \qquad \text{for } \sigma\in S_n$$

$(\mathcal{A},\varphi)$ "non-commutative W^*-probability space"

$:\Leftrightarrow$ $\mathcal{A}$ von Neumann algebra

$\varphi:\mathcal{A}\to\mathbb{C}$ linear, $\varphi(1)=1$

& φ is normal

ow in our main theorem this noncom-
again a very special situation - namely
classical exchangeability is equivalent
eability is equivalent to freeness with

is another instance of the general phi-
ve world the same role as independence
ness is not a hidden assumption in our
eplacing the permutation group by its

de Finetti theorem. All relevant notions
will be defined in Sects. 2 and 4.

Theorem 1.1. *Let $(\mathcal{A},\varphi)$ be a W^*-probability space and consider an infinite sequence of selfadjoint elements $(x_i)_{i\in\mathbb{N}}$ in $\mathcal{A}$. Assume that the x_i ($i\in\mathbb{N}$) generate $\mathcal{A}$ as a von Neumann algebra. Then the following two statements are equivalent:*

(a) *The joint distribution of $(x_i)_{i\in\mathbb{N}}$ with respect to φ is invariant under quantum permutations.*

(b) *The sequence $(x_i)_{i\in\mathbb{N}}$ is identically distributed and free with respect to the φ-preserving conditional expectation E onto the tail algebra of the $(x_i)_{i\in\mathbb{N}}$.*

Our paper is organized as follows. In the next section we collect the preliminaries.

$\{\varphi(x_{i_1}^{\varepsilon_1}\cdots x_{i_k}^{\varepsilon_k}) \mid 1\le i_j\le\infty,\ \varepsilon_j\in\{1,*\}\}$

$A_i\subseteq A$, $i\in I$, $B\subseteq A$: "free wrt E",

if $E(a_1\cdots a_n)=0$

whenever $a_j\in A_{i_j}$, $i_1\neq i_2\neq\dots\neq i_n$

and $E(a_j)=0\ \forall j$

amalgamation implies invariance under quantum
elementary as in the classical case (where it follow
dence is a rule for expressing mixed moments in te
variables) and we will have to use some of the basi
Sect. 4, we will define the tail algebra of our sequ
some basic properties of the corresponding conditional expectation. Section 5 will finally give the proof of the other implication of our de Finetti theorem, Theorem 1.1. The paper closes with an example which shows that, as in the classical case (see [DF]), one needs infinitely many random variables in our de Finetti theorem: quantum exchangeability of *finitely* many random variables does not necessarily imply freeness with amalgamation. We would like to mention that a recent preprint of Curran [Cur], which was inspired

3. Operator-Valued Free Random Variables are Invariant

the distribution of $(x_1,\dots,x_n)$ is S_n^+ invariant, iff

$$\varphi(x_{i(1)}\cdots x_{i(m)}) = \sum_{j(1),\dots,j(m)} u_{i(1)j(1)}\cdots u_{i(m)j(m)}\,\varphi(x_{j(1)}\cdots x_{j(m)})$$

Proposition 3.1. *Let $(\mathcal{A},\varphi)$ be a noncommutative probability space, $\mathcal{B}\subset\mathcal{A}$ a unital subalgebra, and $E:\mathcal{A}\to\mathcal{B}$ a conditional expectation such that $\varphi=\varphi\circ E$. Consider a sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathcal{A}$ which is identically distributed and free with respect to E. Then the joint distribution of the sequence $(x_i)_{i\in\mathbb{N}}$ with respect to φ is invariant under quantum permutations.*

Proof. Fix n, k and $\mathbf{i}=(i(1),\dots,i(n))$ with $1\le i(1),\dots,i(n)\le k$. We have

$$\sum_{j(1),\dots,j(n)=1}^{k} u_{i(1)j(1)}\cdots u_{i(n)j(n)}\cdot\varphi\left(x_{j(1)}\cdots x_{j(n)}\right)$$

$$=\sum_{j(1),\dots,j(n)=1}^{k} u_{i(1)j(1)}\cdots u_{i(n)j(n)}\cdot\varphi\left(E[x_{j(1)}\cdots x_{j(n)}]\right)$$

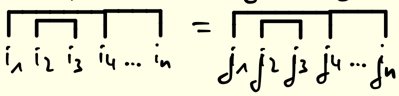
moment cumulant formula

$$=\sum_{j(1),\dots,j(n)=1}^{k} u_{i(1)j(1)}\cdots u_{i(n)j(n)}\cdot\varphi\left(\sum_{\pi\in NC(n)}\kappa_\pi^E[x_{j(1)},\dots,x_{j(n)}]\right)$$

$$=\sum_{\pi\in NC(n)}\sum_{j(1),\dots,j(n)=1}^{k} u_{i(1)j(1)}\cdots u_{i(n)j(n)}\cdot\varphi\left(\kappa_\pi^E[x_{j(1)},\dots,x_{j(n)}]\right).$$

As in the proof of the free CLT:

$\kappa_\pi^E := \kappa_\pi^E(x_{i_1},\dots,x_{i_n}) = \kappa_\pi^E(x_{j_1}\dots x_{j_n})$

for $\pi =$ 

freeness (mixed κ vanish) & id. distr.

$$=\sum_{\pi\in NC(n)}\varphi\left(\kappa_\pi^E\right)\sum_{\substack{j(1),\dots,j(n)=1,\dots,k\\ \ker\mathbf{j}\ge\pi}} u_{i(1)j(1)}\cdots u_{i(n)j(n)}.$$

Now, in $C(S_n^+)$, the following relation holds:

$$\sum_{\substack{j(1),\dots,j(n)=1,\dots,k \\ \ker \mathbf{j} \geq \pi}} u_{i(1)j(1)} \cdots u_{i(n)j(n)} = \begin{cases} 1, & \ker \mathbf{i} \geq \pi \\ 0, & \text{otherwise} \end{cases}.$$

Thus, by recalling that κ_π^E is equal to $\kappa_\pi^E[x_{i(1)}, \dots, x_{i(n)}]$ for any i with $\ker \mathbf{i} \geq \pi$, we have

$$\begin{aligned}
\sum_{j(1),\dots,j(n)=1}^{k} u_{i(1)j(1)} \cdots u_{i(n)j(n)} \cdot \varphi(x_{j(1)} \cdots x_{j(n)}) &= \sum_{\substack{\pi \in NC(n) \\ \ker \mathbf{i} \geq \pi}} \varphi\left(\kappa_\pi^E\right) \\
&= \varphi\left(\sum_{\substack{\pi \in NC(n) \\ \ker \mathbf{i} \geq \pi}} \kappa_\pi^E\right) \\
&= \varphi\left(\sum_{\substack{\pi \in NC(n) \\ \ker \mathbf{i} \geq \pi}} \kappa_\pi^E[x_{i(1)}, \dots, x_{i(n)}]\right) \\
&= \varphi\left(E[x_{i(1)} \cdots x_{i(n)}]\right) \\
&= \varphi\left(x_{i(1)} \cdots x_{i(n)}\right).
\end{aligned}$$

□

For the converse direction (harder):

Proposition 5.1. *Let $(\mathcal{A}, \varphi)$ be a W^*-probability space and consider a sequence of selfadjoint elements $(x_i)_{i\in\mathbb{N}}$ in $\mathcal{A}$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{N}}$ with respect to φ is invariant under classical permutations. Then the sequence $(x_i)_{i\in\mathbb{N}}$ is identically distributed with respect to the conditional expectation E onto the tail algebra of $(x_i)_{i\in\mathbb{N}}$.*

Extended de Finetti theorems:

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Let $(x_i)_{i\in\mathbb{N}} \subseteq (M,\varphi)$ be G invariant

(i.e. $(x_1,\dots x_n)$ is G_n invariant $\forall n$), and let $x_i = x_i^*$.

(1) *Free case:*

(a) *If* $G_n = S_n^+$, *then* $(x_i)_{i\in\mathbb{N}}$ *are freely independent and identically distributed with amalgamation over* $\mathcal{B}$.

(b) *If* $G_n = H_n^+$, *then* $(x_i)_{i\in\mathbb{N}}$ *are freely independent, and have even and identical distributions, with amalgamation over* $\mathcal{B}$.

(c) *If* $G_n = O_n^+$, *then* $(x_i)_{i\in\mathbb{N}}$ *form a* $\mathcal{B}$*-valued free semicircular family with mean zero and common variance.*

(d) *If* $G_n = B_n^+$, *then* $(x_i)_{i\in\mathbb{N}}$ *form a* $\mathcal{B}$*-valued free semicircular family with common mean and variance.*

LAWS OF CHARACTERS

Since a CMQG (A,u) comes with a Haar state h, we have a natural C^*-ncps (A,h).

Which noncommutative distribution does $\chi := \sum_{i=1}^{n} u_{ii} \in A$ have?

Recall:

(A,φ) C^*-ncps, $x = x^* \Rightarrow \exists \mu_x$ prob. measure on $\mathbb{R}$ st. $\int z^k \, d\mu_x(z) = \varphi(x^k)$ $\forall k \in \mathbb{N}_0$

For O_n^+: $\chi = \chi^* \in C(O_n^+)$ has semicircle distribution wrt h

For O_n: $\chi = \chi^* \in C(O_n)$ has Gaussian distribution wrt h

For S_n^+: $\chi = \chi^* \in C(S_n^+)$ has Marchenko-Pastur distribution wrt h

For S_n: $\chi = \chi^* \in C(S_n)$ has Poisson distribution wrt h