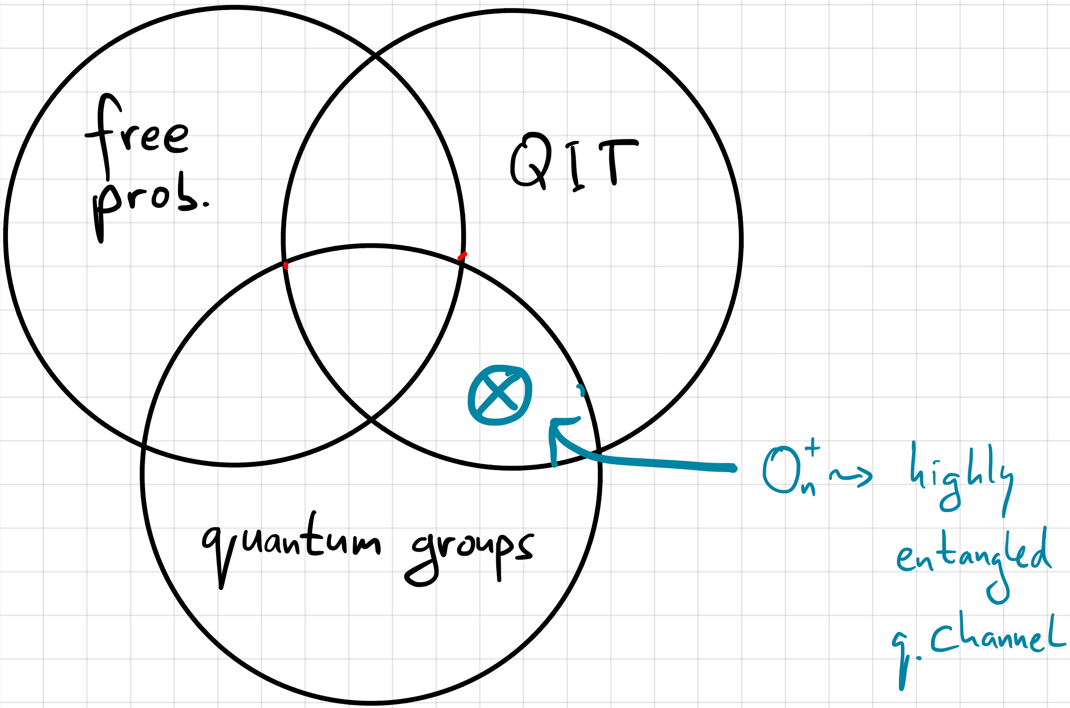
free prob.
QIT
quantum groups
$O_n^+ \rightsquigarrow$ highly entangled q. channel

QGQIT15 Conference on QUANTUM GROUPS & QUANTUM INFORMATION THEORY

JULY 13 - 17, 2015 at Herstmonceux Castle, UK

THIS PAST SUMMER Herstmonceux Castle was the site of a Fields Conference on Quantum Groups and Quantum Information Theory. Herstmonceux is a beautiful medieval castle located southeast of London, and is one of England's most important historical treasures; the use of the Castle was made possible through Queen's University (Kingston, Canada).

The event, held from July 13 to July 17, 2015, was organized by Benoit Collins (University of Ottawa/Kyoto University), James Mingo (Queen's University), Ashley Montanaro (University of Bristol), Maria Grazia Viola (Lakehead University), and Moritz Weber (Saarland University).

This conference had a pioneering role in several regards. Firstly, it marked the first time that a Fields event has been held in Europe. The vision is that Herstmonceux plays a continuing role in the Fields agenda as an alternate site. This way, Fields can establish closer working relationships with European institutions and thereby extend its reach both scientifically and financially.

Secondly, the conference opened the door to new interactions in mathematics. While Quantum Groups and Quantum Information Theory are well established and very active areas of research, the links between the two fields are just beginning to be recognized and developed. Instrumental in this regard has been the work of M. Neufang, J. Crann, M. Junge, B. Collins, M. Brannan, and others, who have used quantum groups to provide new counterexamples and channels in quantum information theory.

The conference at Herstmonceux Castle was an opportunity for researchers in Quantum Groups and in Quantum Information Theory to explore these links, learn from each other, and start new collaborations. Forty seven participants from Canada, the United States, Europe, Australia, Korea and Japan gathered at Herstmonceux Castle for the five day workshop. They included many of the leading experts in each of the two areas, several postdoctoral fellows and graduate students.

Moreover, the London Mathematical Society selected the event to be part of the activities commemorating its 150th anniversary. To celebrate the occasion two special talks were given by Stanislaw Woronowicz (University of Warsaw) and Aram Harrow (MIT) on the first afternoon on the conference, providing an introduction to each of the fields of research. Apart the excellent talks given by twenty-three speakers, two very interesting open problems sessions were held during the workshop.

For five days every room in the castle and every corner of the beautiful courtyard was vibrating with mathematical ideas and discussions. Similar to the workshops held at the Banff International Research Station, participation at the conference was by invitation, and the attendees were accommodated at the castle during their stay. All the participants truly enjoyed not only the workshop, but also the spectacular surroundings.

Sponsorship of the event included the generous support of the Fields Institute, and funds provided by the London Mathematical Society, the European Research Council Advanced Grant of

(continued next page...)

General Scientific Activities

(continued from page 15)

Roland Speicher, the European Research Council Starting Grant of Matthias Christandl, and Lakehead University.

Planning is already underway for a second “test” conference at Herstmonceux Castle, which will take place from June 20 to June 26, 2016. It will be a Retrospective Workshop for the 2013 Thematic Program on Calabi-Yau Varieties: Arithmetic, Geometry and Physics, organized by Mark Gross (University of Cambridge), Radu Laza (Brook University), Matthias Schütt (Leibniz University Hannover), and Noriko Yui (Queen’s University).

The possibility of organizing additional conferences at Herstmonceaux is now open to the entire Fields community. Members are invited to join this initiative by filing a proposal for a conference at the castle during the summer months of 2017 or possibly 2018. A good fit requires a substantial European participation profile, and requires that the Fields funds be matched by one or more European sources. Further information, or inquiries should be directed to Bob Erdahl at Queen’s University (robert.erdahl@queensu.ca).

Maria Grazia Viola (Lakehead University)

LINK QG-QIT: Brannan, Collins, 2016

HIGHLY ENTANGLED, NON-RANDOM SUBSPACES OF TENSOR PRODUCTS FROM QUANTUM GROUPS

MICHAEL BRANNAN AND BENOÎT COLLINS

ABSTRACT. In this paper we describe a class of highly entangled subspaces of a tensor product of finite dimensional Hilbert spaces arising from the representation theory of free orthogonal quantum groups. We determine their largest singular values and obtain lower bounds for the minimum output entropy of the corresponding quantum channels. An application to the construction of d-positive maps on matrix algebras is also presented.

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1. INTRODUCTION

excellent introduction

Entanglement is one of the most important properties that differentiates quantum phenomena from classical phenomena. This property pertains to bi-partite or multi-partite systems. In a classical context, a multi-partite system is modeled by a Cartesian product of sets (e.g. of state spaces), whereas in the quantum context, where linear structures are required, the Cartesian product is replaced by the tensor product (of Hilbert spaces describing each individual system). For example, if H_A and H_B describe the states of systems A and B, then the bipartite system AB is described by the Hilbert space tensor product $H = H_A \otimes H_B$.

Given a Hilbert space H, a (pure) state $\xi \in H$ is a vector of norm 1, taken up to a phase factor. Equivalently, a pure state ξ can be viewed as the rank one projection $\rho_\xi = |\xi\rangle\langle\xi|$ onto $\mathbb{C}\xi \subseteq H$ in $\mathcal{B}(H)$. The (closed) convex hull of pure states is called the state-space of H, and denoted by $\mathcal{D}(H)$. This is a convex compact set, and its extremal points are the rank one projectors, i.e., pure states. Given a bipartite system modeled by the Hilbert space tensor product $H = H_A \otimes H_B$, a state $\rho \in \mathcal{D}(H)$ is said to *separable* if it belongs to the convex hull of the set of product states $\rho = \rho_A \otimes \rho_B$, where $\rho_A \in \mathcal{D}(H_A)$ and $\rho_B \in \mathcal{D}(H_B)$. A state ρ is called *entangled* if it is not separable. We shall call a Hilbert subspace $H_0 \subset H_A \otimes H_B$ an *entangled subspace* if all of its associated pure states are entangled.

For the sake of simplicity in this introduction (precise definitions will be given later), we say that a Hilbert subspace $H_0 \subset H = H_A \otimes H_B$ is *highly entangled* if the set of pure states on H associated to H_0 are uniformly "far away" from the set of product states $\rho_A \otimes \rho_B \in \mathcal{D}(H)$. A maximally entangled state on H is usually called a *Bell state*: All of the singular values (Schmidt-coefficients) associated to a Bell state are equal. It is easy to see that if the dimensions of H_A and H_B are equal, the only subspace $H_0 \subseteq H$ such that all its associated pure states are maximally entangled is a dimension one space $H_0 = \mathbb{C}\rho_B$ spanned by a Bell state ρ_B.

Naturally, the larger the dimension of subspace $H_0 \subseteq H$, the less likely it will be highly entangled, as per the above definition of entanglement. In recent years it has become a very important problem in Quantum Information Theory (QIT) to: *Find subspaces H_0 of large relative dimension in a tensor product $H = H_A \otimes H_B$ such that all states are highly entangled.*

Aim: Want to find subspaces $H_0 \subseteq H_A \otimes H_B$ such that

$$\mathcal{E}_\mu = \sup_{\xi \in H_0, \|\xi\|=1} \{ H(\rho_\xi) + \mu(\log \dim H_0 - \log \dim H) \} \quad 0<\mu<1$$

is large (and positive)

Sources for highly entangled subspaces: random techniques

but: only know that examples must exist - how to find them?

Brannan-Collins: non-random techniques

Al Nuweiran 2013: irred. rep.'s of $SU(2) \rightsquigarrow$ entangled subspaces,

but not "highly entangled"

$\longrightarrow$ Brannan-Collins: work with O_n^+ instead

(Note: may deform $O_n^+ \rightsquigarrow O_n^+(Q)$. $\exists Q: O_2^+(Q) = SU_q(2)$)

Use Vergnioux's property of rapid decay (RD)) quantum world, property RD was observed by Vergnioux to be intrinsically connected to the geometry of the relative position of a subrepresentation of a tensor product of irreducible representations of a given quantum group. More precisely, Vergnioux [Ver07, Section 4] points out that property RD for a given quantum group $\mathbb{G}$ is related to the following geometric requirement: *Given any pair of irreducible representations H_A, H_B of $\mathbb{G}$, all multiplicity-free irreducible subrepresentations $H_0 \subset H_A \otimes H_B$ must be asymptotically far from the cone of decomposable tensors in $H_A \otimes H_B$.*

An exploration of this premonitory remark turns out to be extremely fruitful for a certain class of compact quantum groups, called the *free orthogonal quantum groups* $(O_N^+)_{N\geq 3}$. This remarkable class of quantum groups, introduced by Wang [Wan95], forms a centerpiece in the theory of C*-algebraic compact quantum groups. O_N^+ arises as a certain universal non-commutative deformation of the function algebra on the classical real orthogonal group O_N, and has been the topic of much study over the past 20 years. See, for example, the survey [Bra16] and the references therein. One remarkable fact for our purposes, discovered by Banica [Ban96], is that the quantum groups O_N^+ have a unitary representation theory that closely parallels that of $SU(2)$. In particular, the unitary irreducible representations of O_N^+ have the same fusion rules as $SU(2)$, and their construction is well understood in terms of the planar calculus of the Temperley-Lieb category [KL94]. This close parallel with $SU(2)$, on the one hand, allows for a highly computable framework (like one has for $SU(2)$). On the other hand, the genuinely quantum features of O_N^+ result in a much higher degree of entanglement in subrepresentations of tensor products, in comparison to what can be obtained for $SU(2)$.

For the free orthogonal quantum groups $(O_N^+)_{N\geq 3}$, we show that one can describe very precisely the largest singular values of states that appear in irreducible subrepresentations of tensor product representations (see Theorem 3.4). As a result we describe very precisely a new non-random class of subspaces of tensor products with the property of being highly entangled and of large relative dimension. In particular we find deterministic examples of entangled subspaces of large relative dimension such that the quantity $\mathcal{E}_\mu$ defined in (1) is strictly positive for any $\mu < 1/2$. We also deduce from our entanglement results some interesting properties for the class of quantum channels associated to these subspaces. We compute explicitly the $\mathcal{S}^1 \to \mathcal{S}^\infty$ norms of these channels, and obtain large lower bounds on their minimum output entropies (see Section 4.1).

It is our hope that this paper will be a first step towards substantiating the claim that quantum groups form a rich well of entangled subspaces and quantum channels with interesting analytic properties. Along the way, we revisit the fundamental geometric inequality associated to the rapid decay property for O_N^+ (Proposition 3.1), and improve our understanding thereof. In particular, we show that entanglement inequality for property RD is essentially optimal for the free orthogonal quantum groups, and establish a higher-rank generalization of it (Theorem 3.4).

The remainder of our paper is organized as follows: After this introduction, we recall in the first part of Section 2 some concepts related to entangled subspaces, quantum channels, and minimum output entropy of quantum channels. The second half of Section 2 introduces the free orthogonal quantum groups and describes aspects of their irreducible unitary representation theory that will be used in the sequel. The main section of the paper is Section 3 where we study the entanglement of irreducible subrepresentations of tensor products of

Ingredients of the construction. Recall:

1) $A_o(n) := C(O_n^+) := C^*(1, u_{ij}, i,j=1,\dots n \mid u_{ij} = u_{ij}^*, \sum_k u_{ik}u_{jk} = \sum_k u_{ki}u_{kj} = \delta_{ij})$

$(A_o(n), u)$ CMQG "free orthogonal q.grp." with $A_o(n) / \langle u_{ij} \text{ comm.} \rangle = C(O_n)$

indeed: $\sum_k u_{ik}u_{jk} = \delta_{ij} \iff uu^t = 1$. We have $O_n \subseteq O_n^+$.

2) Tannaka-Krein for CMQG's:

a) (A,u) CMQG $\Rightarrow$ $Rep(A,u) := \{$fin. dim. unitary rep.$\}$ $\overline{W}$-tensor categ.

b) $\mathcal{R}$ W-tensor categ. $\Rightarrow \exists! \ (A,u)$ CMQG : $\overline{\mathcal{R}} = Rep(A,u)$

Hence: Duality $\{W\text{-tensor categ.}\} \longleftrightarrow \{CMQG\}$

Main ingredient of the proof:

interpret morphisms $T \in \mathcal{R}$ as "intertwiners" $Tu = vT$, $u, v \in \mathcal{R}$

$Hom(u,v) := \{T : H_u \to H_v \text{ linear} \mid Tu = vT\}$ intertwiner space

3) Fusion rules for O_n^+:

irred. representations of O_n^+ indexed by $\mathbb{N}$: $v^k \in C(O_n^+) \otimes \mathcal{B}(H_k)$

$$v^k \otimes v^\ell = v^{|k-\ell|} \oplus v^{|k-\ell|+2} \oplus v^{|k-\ell|+4} \oplus \dots \oplus v^{k+\ell}, \quad k,\ell \in \mathbb{N}$$

Some notation:

$$q := \frac{1}{n}\left(\frac{2}{1+\sqrt{1-\frac{4}{n^2}}}\right) \in (0,1], \quad [k]_q := q^{-(k-1)}\left(\frac{1-q^k}{1-q^2}\right) \approx n^{k-1}, \quad \dim H_k = [k+1]_q$$

$T_1 := \sum e_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^n$ "Jones-Wenzl projection"

$$\exists!\, p_k \in \mathrm{Hom}(u^{\otimes k}, u^{\otimes k}) \subseteq \mathcal{B}((\mathbb{C}^n)^{\otimes k}) \text{ s.t. } (id^{\otimes i-1} \otimes T_1 T_1^* \otimes id^{\otimes k-i-1}) p_k = 0$$

For $(k,\ell,m) \in \mathbb{N}_0^3$ "admissible" (i.e. $k = \ell+m-2r$ for some $0 \leq r \leq \min(\ell,m)$):

$$\langle A_k^{\ell,m} \rangle = \mathrm{Hom}(v^k, v^\ell \otimes v^m) \subseteq \mathcal{B}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes(\ell+m)}) \quad \text{one-dimensional}$$

with $A_k^{\ell,m} := (p_\ell \otimes p_m)(id_{H_{\ell-r}} \otimes T_r \otimes id_{m-r}) p_k$, $\quad T_r := (id_{H_1} \otimes T_1 \otimes id_{H_1}) T_{r-1}$

Then $\alpha_k^{\ell,m} : H_k \to H_\ell \otimes H_m$ isometry, $\quad \alpha_k^{\ell,m} := \frac{1}{\|A_k^{\ell,m}\|} A_k^{\ell,m}$

the previous section. Recall that we set $q = \frac{1}{N}\Big(\frac{2}{1+\sqrt{1-4/N^2}}\Big) \in (0,1)$. Our main interest is to study the entanglement of the $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$, and the following proposition gives a measure of this.

Proposition 3.1. *Fix $N \geq 3$ and let $(k,l,m) \in \mathbb{N}_0^3$ be an admissible triple. Then for any unit vectors $\xi \in H_k, \eta \in H_l, \zeta \in H_m$, we have*

$$|\langle \alpha_k^{l,m}(\xi) | \eta \otimes \zeta \rangle| \leq \Big(\frac{[k+1]_q}{\theta_q(k,l,m)}\Big)^{1/2} \leq C(q) q^{\frac{l+m-k}{4}}, \approx \Big(\frac{\dim H_k}{\dim H_l \dim H_m}\Big)^{\frac{1}{4}}$$

where

$$C(q) = (1-q^2)^{-1/2}\Big(\prod_{s=1}^{\infty} \frac{1}{1-q^{2s}}\Big)^{3/2}$$

Theorem 3.2. *For k,l,m as above, the subspaces $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$ are (highly) entangled provided $k < l+m$. When $k = l+m$, the highest weight subspace $\alpha_{l+m}(H_{l+m}) \subset H_l \otimes H_k$ is a separable subspace.*

Theorem 3.4. *Let $(k,l,m) \in \mathbb{N}_0^3$ be an admissible triple, $N \geq 3$, and $d \leq (N-2)(N-1)^{\frac{l+m-k-2}{2}}$. Then there exists a unit vector $\xi \in H_k$ such that $\alpha_k^{l,m}(\xi)$ has a singular value decomposition $\alpha_k^{l,m}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i$ with $\lambda_1 \geq \lambda_2 \geq \ldots$ satisfying*

$$\lambda_1 = \lambda_2 = \ldots = \lambda_d = \frac{[k+1]_q}{\theta_q(k,l,m)} \geq q^{\frac{l+m-k}{2}}.$$

Corollary 4.2. *Given any admissible triple $(k,l,m) \in \mathbb{N}_0^3$ and $N \geq 3$, we have*

$$H_{\min}(\Phi_k^{\bar{l},m}), H_{\min}(\Phi_k^{l,\bar{m}}) \geq \log\Big(\frac{\theta_q(k,l,m)}{[k+1]_q}\Big) \geq -\Big(\frac{l+m-k}{2}\Big)\log(q) - 2\log(C(q)).$$

Talk: M. Brannan, IHP/Centre Émile Borel, Tue 12 Sept

FURTHER: Crann, M.Sc. thesis, 2012

Quantum Channels Arising From Abstract Harmonic Analysis

by

Jason A. Crann

from the introduction:

cation, we show that every normalized positive definite functional on a locally compact quantum group yields a quantum channel via the Junge–Neufang–Ruan representation theorem [41]. Furthermore, we introduce a systematic method of constructing the resulting channel in terms of its unitary co-representation. We finish the chapter

See also: Crann, Neufang, Quantum channels arising from abstract harmonic analysis, 2013