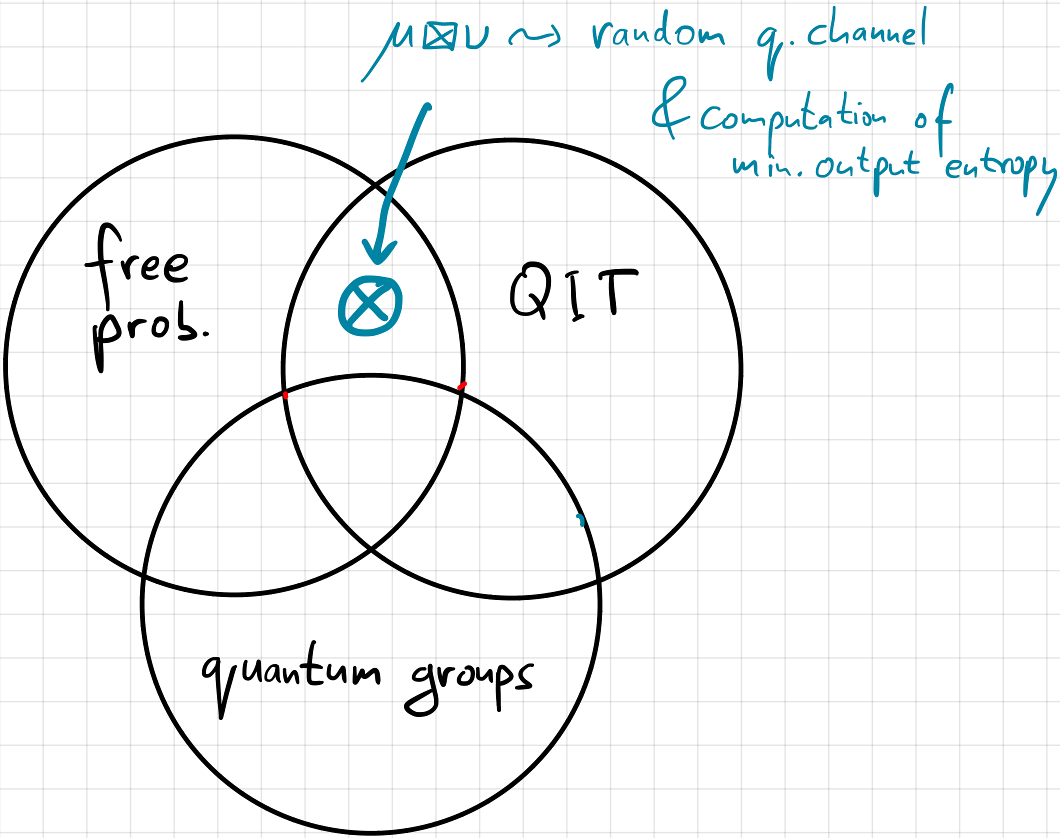
free prob.
QIT
quantum groups
$\mu \boxtimes \nu \rightsquigarrow$ random q. channel
& computation of
min. output entropy

LINK FP-QIT: Belinschi, Collins, Nechita, 2012 & 2016

Invent math (2012) 190:647–697
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Eigenvectors and eigenvalues in a random subspace of a tensor product

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Commun. Math. Phys. 341, 885–909 (2016)
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Communications in Mathematical Physics

CrossMark

Almost One Bit Violation for the Additivity of the Minimum Output Entropy

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Eigenvectors and eigenvalues in a random subspace of a tensor product

Serban Belinschi · Benoît Collins · Ion Nechita

Received: 21 April 2011 / Accepted: 4 February 2012 / Published online: 22 February 2012

Abstract Given two positive integers n and k and a parameter $t \in (0, 1)$, we choose at random a vector subspace $V_n \subset \mathbb{C}^k \otimes \mathbb{C}^n$ of dimension $N \sim tnk$. We show that the set of k-tuples of singular values of all unit vectors in V_n fills asymptotically (as n tends to infinity) a deterministic convex set $K_{k,t}$ that we describe using a new norm in $\mathbb{R}^k$.

Our proof relies on free probability, random matrix theory, complex analysis and matrix analysis techniques. The main result comes together with a law of large numbers for the singular value decomposition of the eigenvectors corresponding to large eigenvalues of a random truncation of a matrix with high eigenvalue degeneracy.

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Mathematics Subject Classification (2000) Primary 15A52 · Secondary 52A22 · 46L54

1 Introduction

In [15], it was observed that if one takes at random a vector subspace V_n of $\mathbb{C}^k \otimes \mathbb{C}^n$ of relative dimension t for large n and fixed k, with very high probability, some sequences of numbers in $\mathbb{R}_+^k$ never occur as singular values of elements in V_n as n becomes large. This result was used to provide a systematic understanding of some non-additivity theorems for entropies in Quantum Information Theory. We refer to the bibliography of [15] for more information on this topic.

Our aim in this paper is to provide a definitive answer to the question of which sequences of numbers in $\mathbb{R}_+^k$ occur or not as singular values of elements in V_n. Our main result can be sketched as follows—for the statement with complete definitions, we refer to Theorem 5.2:

Theorem 1.1 *Let $t \in (0, 1)$ be a parameter and for any n, V_n a vector subspace of $\mathbb{C}^k \otimes \mathbb{C}^n$ of dimension $N \sim tnk$ chosen at random. Then, there exists a compact set $K_{k,t} \subset \mathbb{R}_+^k$ such that any k-tuple λ in the interior of $K_{k,t}$ occurs with high probability as the singular value vector of some norm one vector $x \in V_n$. Moreover, the probability that some vector $v \notin K_{k,t}$ occurs as the singular value vector of some element $y \in V_n$ is vanishing when $n \to \infty$.*

We now introduce the convex body $K_{k,t} \subset \Delta_k$ as follows:

$$K_{k,t} := \{\lambda \in \Delta_k \mid \forall a \in \Delta_k, \langle \lambda, a \rangle \leqslant \|a\|_{(t)}\}, \tag{18}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product in $\mathbb{R}^k$. We shall show in Theorem 6.4 that this set is intimately related to the (t)-norm: $K_{k,t}$ is the intersection of the dual ball of the (t)-norm with the probability simplex Δ_k. Since it

Definition 3.2 For a positive integer k, embed $\mathbb{R}^k$ as a self-adjoint real subalgebra $\mathcal{R}$ of a II_1 factor $\mathcal{A}$ endowed with trace φ, so that $\varphi((x_1, \ldots, x_k)) = (x_1 + \cdots + x_k)/k$. Let p_t be a projection of rank $t \in (0, 1]$ in $\mathcal{A}$, free from $\mathcal{R}$. On the real vector space $\mathbb{R}^k$, we introduce the following norm, called the *(t)-norm*:

$$\|x\|_{(t)} := \|p_t x p_t\|_\infty, \tag{8}$$

$= \mu_x \boxtimes \mu_{p_t} \leftrightarrow \mu_x^{\boxplus \frac{1}{t}}$

where the vector $x \in \mathbb{R}^k$ is identified with its image in $\mathcal{R}$.

because of the initial quantum-information theoretical motivation.

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Communications in **Mathematical Physics**

Almost One Bit Violation for the Additivity of the Minimum Output Entropy

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Received: 16 September 2014 / Accepted: 17 November 2015
Published online: 11 January 2016 –

Abstract: In a previous paper, we proved that, in the appropriate asymptotic regime, the limit of the collection of possible eigenvalues of output states of a random quantum channel is a deterministic, compact set $K_{k,t}$. We also showed that the set $K_{k,t}$ is obtained, up to an intersection, as the unit ball of the dual of a free compression norm. In this paper, we identify the maximum of ℓ^p norms on the set $K_{k,t}$ and prove that the maximum is attained on a vector of shape $(a, b, \ldots, b)$ where $a > b$. In particular, we compute the precise limit value of the minimum output entropy of a single random quantum channel. As a corollary, we show that for any $\varepsilon > 0$, it is possible to obtain a violation for the additivity of the minimum output entropy for an output dimension as low as 183, and that for appropriate choice of parameters, the violation can be as large as $\log 2 - \varepsilon$. Conversely, our result implies that, with probability one in the limit, one does not obtain a violation of additivity using conjugate random quantum channels and the Bell state, in dimension 182 and less.

1. Introduction

The question of determining the optimal rate at which classical information can be transmitted through a noisy quantum channel is central in the theory of quantum information. A conjectured one letter formula for the classical capacity of quantum channels, introduced by Holevo [24] was shown to be a strict lower bound for the optimal rate in the celebrated work of Hastings [23], who showed, using randomly constructed examples, that several quantities of interest in quantum information theory were non-additive [33].

Hastings' counter-example, as well as several follow-up constructions, make use of the same central idea: random quantum channels, constructed from random isometrical embeddings of large Hilbert spaces inside tensor products, violate the additivity of the *minimum output entropy* (MOE). The non-additivity proofs consist of two bounds: an upper bound for the MOE of a tensor product of two channels and a lower bound for the MOE of single channels. In our past work [5], we showed that the lower bound for result for the minimum output p-entropies of random quantum channels.

Theorem 5.2. *Let p be a real number in $[1, \infty]$ and $\Phi_n : M_{d_n}(\mathbb{C}) \to M_k(\mathbb{C})$ a sequence of random quantum channels with constant output space of dimension k, environment of size $n \to \infty$ and input space of dimension $d_n \sim tkn$. Then, almost surely as $n \to \infty$,*

$$\lim_{n\to\infty} H_p^{\min}(\Phi_n) = H_p(x_t^*), \tag{53}$$

with x_t^* defined in Eq. (6).

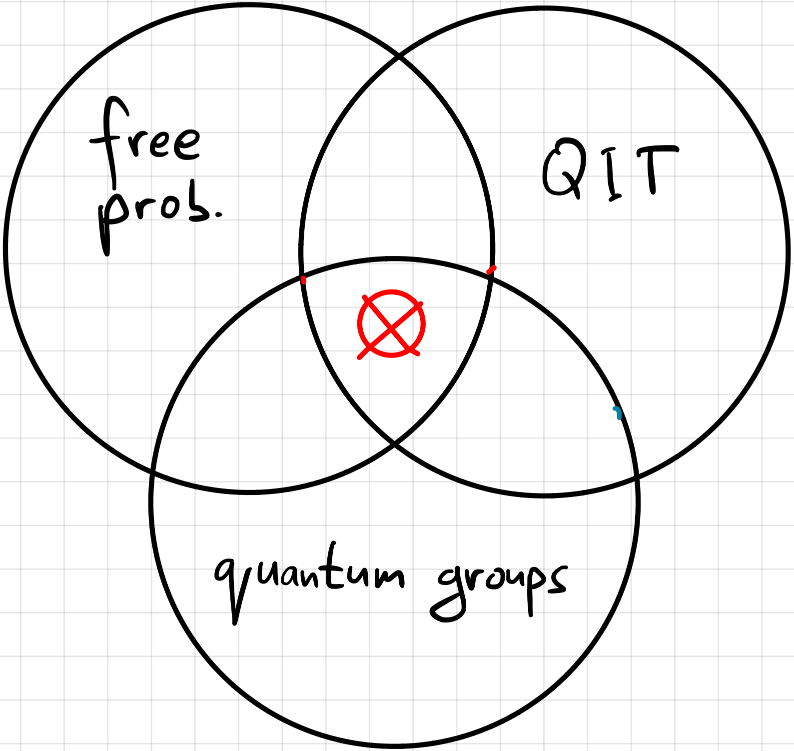
Recall that $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^k$ and let

$$x_t^* = \left(\|e_1\|_{(t)}, \underbrace{\frac{1 - \|e_1\|_{(t)}}{k-1}, \ldots, \frac{1 - \|e_1\|_{(t)}}{k-1}}_{k-1 \text{ times}} \right). \tag{6}$$

With this notation we are able to state our main result

Theorem 2.4. *For any $p > 1$, the maximum of the ℓ^p norm on $K_{k,t}$ is reached at the point x_t^*.*

LINK FP-QG-QIT: YOUR JOB!



APPENDIX: A NONCOMMUTATIVE DICTIONARY

As soon as noncommutative (algebraic) structures are involved,

i.e. $xy \neq yx$: Think noncommutative!

Further reading: Franz, Skalski, Noncommutative mathematics for quantum systems, 2017.

classical	noncommutative	
topology	C^*-algebras [Gelfand, Naimark 1940's]	Further reading: Blackadar, Operator algebras, 2006; Davidson, C*-algebras by example, 1996; Murphy, C*-algebras and operator theory, 1990.
measure theory	von Neumann algebras [Murray, von Neumann 1930's/40's]	Further reading: Blackadar, Operator algebras, 2006; Lecture notes Vaughan Jones, 2009; Lecture notes Jesse Petersen, 2013.
probability / independence	free probability / free independence [Voiculescu 1980's]	Further reading: Nica, Speicher, Lectures on the combinatorics of free probability, 2006; Voiculescu, Stammeier, Weber, Free probability and operator algebras, 2016; Mingo, Speicher, Free probability and random matrices, 2017.
symmetry / (compact) groups	(compact) quantum groups [Woronowicz 1980's]	Further reading: Timmermann, An invitation to quantum groups and duality, 2008; Neshveyev, Tuset, Compact quantum groups and their representation categories, 2013; Franz, Quantum symmetries, 2017; Weber, Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups, 2017.
information	quantum information [Feynmann, Deutsch, ... 1980's]	Further reading: Nielsen, Chuang, Quantum computation and quantum information, 2000; Watrous, The theory of quantum information, 2017; Kitaev, Shen, Vyalyi, Classical and quantum computation, 2002; Aaronson, Quantum computing since Democritus, 2013.
complex analysis	free analysis [J.L. Taylor 1970's]	Further reading: Taylor, Functions of several noncommuting variables, 1973; Voiculescu, Aspects of free analysis, 2008, Kaliuzhnyi-Verbovetskyi, Vinnikov, Foundations of free noncommutative function theory, 2014.
differential geometry	noncommutative geometry [Connes 1980's]	Further reading: Connes, Noncommutative geometry, 1994; Landi, An introduction to Noncommutative spaces and their geometries, 1997; Gracia-Bondia, Figueroa, Varilly, Elements of noncommutative geometry, 2000; Madore, Noncommutative geometry for pedestrians, 1999.
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