Universität des Saarlandes
FR 6.1 Mathematik
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10. Exercise sheet for Geometric Group Theory

Exercise 1.

Let G be a group, $S \subset G$ be a finite generating set and $\Sigma_{G,S}(z)$ be the corresponding growth series.

a) Show that we have for the generating set $B_2(S) = \{g \in G \mid |g|_S \leq 2\} = \{\mathbb{1}\} \cup S \cup S^2$ the growth series

$$\Sigma_{G,B_2(S)}(z) = \frac{1}{2}\big((1+\sqrt{z})\Sigma_{G,S}(\sqrt{z}) + (1-\sqrt{z})\Sigma_{G,S}(-\sqrt{z})\big).$$

b) Conclude that if $\Sigma_{G,S}(z)$ is rational, then $\Sigma_{G,B_2(S)}(z)$ is rational, too.
Hint: Write it as $A(z) + \sqrt{z}B(z)$ *and consider* $\mathrm{Gal}(\mathbb{C}(\sqrt{z})/\mathbb{C}(z))$.

c) Can you generalize this to $B_m(S)$?

Exercise 2.

For $N, d \in \mathbb{N}$ we consider the generating set $S_{N,d} := \{1, N, N^2, \ldots, N^d\} \subset \mathbb{Z}$. Compute $\Sigma_{\mathbb{Z},S_{N,d}}(z)$ and compare it with $\Sigma_{\mathbb{Z}^d,\mathrm{basis}}(z)$ for large N.

Exercise 3.

Let $G_1 = \langle S_1 \rangle$ and $G_2 = \langle S_2 \rangle$ be two finitely generated groups. Consider their direct product $G := G_1 \times G_2$ with generating set $S := B_1(S_1) \times B_1(S_2) = \{(s_1, s_2) \mid s_i \in S_i \cup \{\mathbb{1}\}\}$. For two formal power series

$$F(z) = \sum_{i=0}^{\infty} f_i z^i \quad G(z) = \sum_{i=0}^{\infty} g_i z^i$$

we define their Hadamard product as

$$(F \circ G)(z) := \sum_{i=0}^{\infty} (f_i g_i) z^i.$$

a) Prove that we have $\Sigma_{G,S}(z) = (1-z) \cdot (\frac{\Sigma_{G_1,S_1}}{1-z} \circ \frac{\Sigma_{G_2,S_2}}{1-z})(z)$.

b) Prove that if $F(z)$ and $G(z)$ are rational, then $(F \circ G)(z)$ is also rational.
Hint: Prove it for $F(z), G(z)$ *of the form* $\sum_{i\geq 0}(\lambda^i i^d)z^i$ *and apply bilinearity.*

You can hand in the exercise sheet until Monday the 27. 06. 2022 at 12 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a (∗) might be a little bit trickier.

Exercise 4.

Consider the three hyperplanes

$$A := \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle, \quad B := \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle, \quad C := \langle \begin{pmatrix} \sin(\pi/3) \\ \cos(\pi/3) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

in $\mathbb{R}^3$. We denote with $S := \{\sigma_A, \sigma_B, \sigma_C\}$ the set of the orthogonal reflections at A, B and C ,respectively. Let $G := \langle S \rangle \leq \mathrm{GL}_3(\mathbb{R})$ be the group that they generate.

a) Show that G is a Coxeter group and write down the corresponding presentation.

b) What is the growth series $\Sigma_{G,S}$?

c) The images of the hyperplanes A, B, C under G cut $\mathbb{R}^3$ into disjoint cones. Fix your favourite cone and count the number of different cones you can reach by passing over at most $n \in \mathbb{N}$ hyperplanes (you are only allowed to walk over a single hyperplane at once). Have you seen these numbers before?

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