Universität des Saarlandes
FR 6.1 Mathematik
Prof. Dr. L. Bartholdi
Prof. Dr. G. Weitze-Schmithüsen
Dr. C. Steinhart

11. Exercise sheet for Geometric Group Theory

Exercise 1.

a) Show that the concatenation of two quasi-isometries is again a quasi-isometry.

b) Let X and Y be two metric spaces and $\varphi : X \to Y$ be a map. Show that φ is a quasi-isometry if and only if there exists a function $\psi : Y \to X$ and constants $\lambda, C, D \in \mathbb{R}_{>0}$ such that the following holds for all $x, x' \in X, y, y' \in Y$:

- $d_Y(\varphi(x), \varphi(x')) \leq \lambda d_X(x, x') + C$ and $d_X(\psi(y), \psi(y')) \leq \lambda d_Y(y, y') + C$.
- $d_Y(\varphi(\psi(y)), y) \leq D$ and $d_X(\psi(\varphi(x)), x) \leq D$.

c) Conclude that $\sim_{\mathrm{QI}}$ is an equivalence relation on a set of metric spaces.

Exercise 2.

Let G and H be two finitely generated groups. Prove the following statements:

a) Let $N \trianglelefteq$ be a finite, normal subgroup, then the quotient map $\pi : G \to G/N$ is a quasi-isometry.

b) Let $\phi : G \to H$ be a group homomorphism. Then ϕ is a quasi-isometry (with the corresponding word metrics) if and only if $\mathrm{Kern}(\phi)$ and the index $[H : \mathrm{Im}(\phi)]$ are finite.

Exercise 3.

We define the logarithmic spiral as the path

$$\gamma : \mathbb{R}_{>0} \to \mathbb{R}^2, t \mapsto \big(t \cdot \cos(\ln(t)), t \cdot \sin(\ln(t))\big).$$

a) Show that γ is a quasi-geodesic, that is a quasi-isometric embedding from $\mathbb{R}_{>0}$ to $\mathbb{R}^2$ (with the euclidean metric).

b) Show that γ intersects all geodesics in $\mathbb{R}^2$ and has infinite distance[1] to all geodesics.

[1] We use here the Hausdorff distance $d_H(X, Y) = \max\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(X, y)$ for subsets $X, Y \subset \mathbb{R}^2$.

You can hand in the exercise sheet until Monday the 04. 07. 2022 at 12 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a $(*)$ might be a little bit trickier.

Exercise 4.

a) Let $X := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-plane. For which pairs of the following metrics is the identity a quasi-isometry?

   (i) The euclidean metric $d_{eukl}(w, z) := |w - z|$

   (ii) The Manhattan metric $d_{Man}(w, z) := |\mathrm{Re}(w - z)| + |\mathrm{Im}(w - z)|$.

   (iii) The SNCF metric with Paris at 0, i.e. $d_{SNCF}(w, z) := \begin{cases} d_{eukl}(w, z) & , \text{ if } w = \lambda z \text{ for some } \lambda \in \mathbb{R} \\ |w| + |z| & , \text{ otherwise} \end{cases}$.

   (iv) The hyperbolic metric d_{hyp}.

b) Show that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not quasi-isometric for $n \neq m$.

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