Universität des Saarlandes
FR 6.1 Mathematik
Prof. Dr. L. Bartholdi
Prof. Dr. G. Weitze-Schmithüsen
Dr. C. Steinhart

12. Exercise sheet for Geometric Group Theory

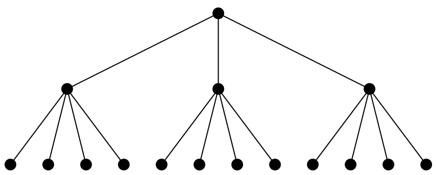
Exercise 1.

Let G, H be two groups and X be a set on which H. The set $G^X := \mathrm{Abb}_0(X, G)$ of functions $f : X \to G$ with finite support is together with pointwise operation $f \star g : X \to G, x \mapsto f(x) \star g(x)$ is itself a group on which H acts on the left via

$$\phi : H \times G^X \to G^X, (h, f) \mapsto (f \circ h^{-1} : x \mapsto f(h^{-1}x)).$$

The wreath product[1] is the semidirect product $G^X \rtimes_\phi H$ and is written $G \wr_X H$. Typically we have $X = H$ with the usual left-multiplication.

a) Write the automorphism group of the graph



as a wreath product of S_3 and S_4.

b) The *lamplighter group* is the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ (with the usual action of $\mathbb{Z}$ on itself). Show that the lamplighter group is finitely generated and has exponential growth rate.
Remark: The group can be illustrated as a lamplighter on a street with infinitely many lampposts. The lamplighter can now walk along the lampposts and turn them on=1 and off=0.

Exercise 2.

We consider the upper triangular group and unitriangular group

$$U := \left\{ \begin{pmatrix} \star & \star & \dots & \star \\ 0 & \star & \ddots & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 0 & \star \end{pmatrix} \right\}, \quad V := \left\{ \begin{pmatrix} 1 & \star & \star & \dots & \star \\ 0 & 1 & \star & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \star \\ 0 & \dots & 0 & & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_n(K)$$

for some field K.

[1] Sometimes this is also called the restricted wreath product and if one drops the finite support it is called the unrestricted wreath product.

You can hand in the exercise sheet until Monday the 11. 07. 2022 at 2 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a $(*)$ might be a little bit trickier.

a) Show that the unitriangular group V is nilpotent.
Hint: Calculate the commutator for two addition matrices.

b) What can you say about the growth rate of the upper triangular matrices over $\mathbb{Z}$?

c) Show that the upper triangular group U over $\mathbb{R}$ is solvable but not nilpotent.

Exercise 3.

a) Consider the three points

$$A := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C := \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

and the subgroup $G := \langle \sigma_{AB}, \sigma_{BC}, \sigma_{AC} \rangle \leq \mathrm{Isom}(\mathbb{R}^2)$ generated by the reflections at the geodesics through them. Show that G has quadratic growth.

b) Can you generalize a) for all triangles $\overline{ABC} \subset \mathbb{R}^2$ with angles of the form $\frac{\pi}{q}$ and $q \in \mathbb{N}$? What happens for irrational angles?

Exercise 4.

Let G be a group and $G := G_0 \geq G_1 \geq \cdots \geq G_s := \{e_G\}$ be a series of subgroups with $[G, G_i] \leq G_{i+1}$ for all $i < s$.

a) Show that all G_i are normal subgroups of G and all G_i/G_{i+1} are abelian.

b) Let $N \trianglelefteq G$ be a normal subgroup and G, N and G/N be finitely generated. Show that we can bound the growth rate of G by the growth rates of N and G/N as follows:

$$\gamma_G(n) \precsim \gamma_N(n^2) \cdot \gamma_{G/N}(n).$$

c) Let G be a finitely generated, nilpotent group. Conclude that G has polynomial growth rate.

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