Universität des Saarlandes
FR 6.1 Mathematik
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13. Exercise sheet for Geometric Group Theory

Exercise 1.

Let $n \in \mathbb{N}$ be a natural number and $X := \{0,1\}^n = (\mathbb{Z}/2\mathbb{Z})^n$.

a) Write down a finite state automaton which describes multiplication with 3 in $\mathbb{Z}_2$. What is the inverse of 3 in $\mathbb{Z}_2$?

b) Draw a finite state automaton which realises division by 3 in $\mathbb{Z}_2$. Let $x \in \mathbb{Z}_2$ be an element which is periodic at some point, conclude that $x/3 \in \mathbb{Z}_2$ also is periodic at some point.

c) Let $G := \mathrm{Aff}_n(\mathbb{Z}[\frac{1}{3}]) := \{x \mapsto Ax + t \mid A \in \mathrm{GL}_n(\mathbb{Z}[\frac{1}{3}]), t \in \mathbb{Z}[\frac{1}{3}]\}$ be the group of affine transformations over $\mathbb{Z}[\frac{1}{3}]$. Show that the action of G on $\Omega \cong (\mathbb{Z}_2)^n = X^\omega$ is self-similar. Note that the action of the subgroup $\mathrm{GL}_n(\mathbb{Z}[\frac{1}{3}]) \leq G$ is not self-similar.
Remark: If you are having trouble with $\mathbb{Z}[\frac{1}{3}]$ you are allowed to work with $\mathbb{Z}$ instead. The proof is the same.

d) For $q \in \mathbb{N}$ and $p, k \in \mathbb{Z}$ we define the affine transformation

$$f : \mathbb{Z}_2 \to \mathbb{Z}_2, \quad x \mapsto 3^k x + \frac{p}{3^q}.$$

Show that f can be described with a finite-state automaton.

e) The Baumslag-Solitar group[1] $BS(1,3)$ can be defined as the set of all those affine transformations

$$BS(1,3) := \{f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 3^k x + \frac{p}{3^q} \mid p, q \in \mathbb{N}, k \in \mathbb{Z}\}.$$

Write down generators of $BS(1,3)$ and describe an automaton whose states generate $BS(1,3)$.

[1] More generally for $n, m \in \mathbb{Z}$ one defines $BS(n,m) := \langle a, b \mid ba^n b^{-1} = a^m \rangle$.

You can hand in the exercise sheet until Monday the 18. 07. 2022 at 12 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a (∗) might be a little bit trickier.

Exercise 2.

We consider the Grigorchuk group $G = \langle a, b, c, d \rangle$ from the lecture generated by the following automaton:

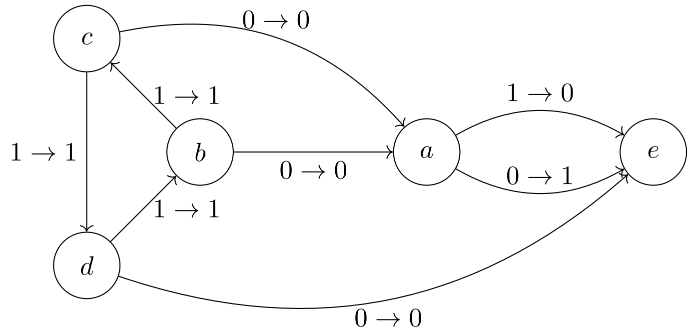

Figure 1: The automaton generating the Grigorchuk group.

a) Show that the map

$$\sigma : G \to G, \quad \begin{aligned} a &\mapsto aca \\ b &\mapsto d \\ c &\mapsto b \\ d &\mapsto c \end{aligned}$$

is a group homomorphism.

b) Show that for all $n \in \mathbb{N}$ we have $\sigma^n((ad)^4) = \sigma^n((ababad)^4) = 1$.

c) ($*$) Show that G has the presentation

$$G = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd \text{ and } \sigma^n((ad)^4), \sigma^n((ababad)^4) \text{ for all } n \in \mathbb{N} \rangle.$$

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