Universität des Saarlandes
FR 6.1 Mathematik
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2. Exercise sheet for Geometric Group Theory

Exercise 1.

Let X be a set and $\{X_i\}_{i\in I}$ be a family of topological spaces together with functions $f_i : X_i \to X$. The final topology on X is defined as

$$U \subseteq X \text{ is open} \iff \forall i \in I : f_i^{-1}(U) \subseteq X_i \text{ is open.}$$

a) Show that the following are equivalent:

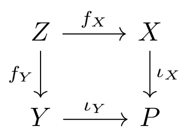
(i) X has the final topology.

(ii) Any function $g : X \to Y$ to some topological space Y is continuous if and only if also the concatenations $g \circ f_i : X_i \to Y$ are continuous for all $i \in I$.

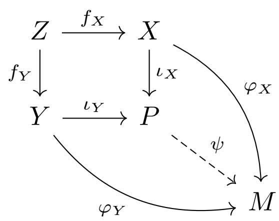
Can you write the quotient topology and the topological sum as final topologies?

b) Let X, Y, Z be objects in a category and $f_X : Z \to X$, $f_Y : Z \to Y$ be two morphisms. We say f_X and f_Y have a *pushout*, if there exists an object P and morphisms $\iota_X : X \to P$, $\iota_Y : Y \to P$ with the following properties:

(i) The following diagram commutes



(ii) (P, ι_X, ι_Y) is universal in the sense, that for any other object M and morphisms $\varphi_X : X \to M$, $\varphi_Y : Y \to M$ which fulfil (i), there exists a unique morphism $\psi : P \to M$ such that the following diagram commutes:



Show that in the category *Set* of sets the pushout of two maps f_X and f_Y is $X \sqcup_Z Y := X \sqcup Y/\sim$ with the obivous maps ι_X, ι_Y and „$\sim$“ the equivalence relation induced by

$$x \sim y \iff \exists z \in Z : f_X(z) = x \text{ and } f_Y(z) = y.$$

Can you find a topology on $X \sqcup_Z Y$ to make it a pushout in the category *Top* of topological spaces?

You can hand in the exercise sheet until Monday the 02. 05. 2022 at 2 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a (∗) might be a little bit trickier.

Exercise 2.

Let Γ be a (not necessarily finite) connected graph.

a) Show that Γ contains a subgraph T which is a tree and contains all vertices of Γ, i.e. we have $V(\Gamma) = V(T)$. Such a subgraph is called a *spanning tree*.
Hint: A famous Lemma from Zorn might help you.

b) Let (Γ, d) be a graph with a metric. Show that Γ is a tree if and only if for any two points $A, B \in (\Gamma, d)$ there exists a unique shortest path from A to B.

Exercise 3.

Let Γ be a finite, connected graph. We call the number $\chi(\Gamma) := \#V(\Gamma) - \#E(\Gamma)$ the *Euler characteristic* of Γ[1].

a) Show that we have $\chi(\Gamma) \leq 1$. Furthermore show that equality $\chi(\Gamma) = 1$ holds if and only if Γ is a non-empty tree.

b) For some $k \in \mathbb{N}$ let additionally Γ be 3-regular and $\chi(\Gamma) = -k$. How many edges does Γ have? Can you draw all such Γ for $k = 1$?

c) Is there a finite 6-regular graph with odd Euler characteristic?

Exercise 4.

Let $(H, \circ)$ and $(N, \bullet)$ be two groups and $\varphi : H \to \mathrm{Aut}(N)$ be a group homomorphism. We define on the set $G := N \times H$ the operation

$$(n_1, h_1) \star (n_2, h_2) := (n_1 \bullet \varphi(h_1)(n_2), h_1 \circ h_2).$$

a) Show, that $(G, \star)$ is a group, which contains $N \times \{1_H\}$ as a normal subgroup and $\{1_N\} \times H$ as a subgroup. $(G, \star)$ is called *semidirect product* of N and H and is denoted by $N \rtimes_\varphi H$.

b) The Diedergroup D_5 is the automorphism group of the cycle graph $C_5 = \Gamma(\mathbb{Z}/5\mathbb{Z}, \{\overline{1}\})$. Write D_5 as a semidirect product of two non-trivial subgroups.

c) Show that $\mathbb{Z}$ can not be written as a non-trivial semidirect product.

[1]Beware that the term Euler characteristic is in the context of planar graphs slightly different, namely one also adds the number of „areas“.

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