Universität des Saarlandes
FR 6.1 Mathematik
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# 3. Exercise sheet for Geometric Group Theory

**Exercise 1.**

Recall from last exercise sheet the topological pushout. We consider the following two special cases of pushouts for topological graphs:

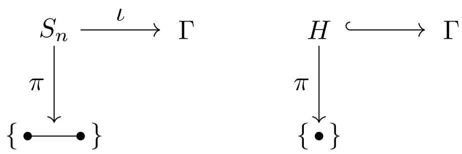


Figure 1: Pushout diagrams for folding and collapsing of a graph

where $\Gamma$ is a graph, $S_n$ is the star graph and $H \subseteq \Gamma$ a subgraph, respectively their topological realisations. More precisely:

- Let $n \geq 2$ and $S_n$ be a the star graph, that is the unique tree with an $n$-valent vertex and $n$ leaves. *Folding* $\iota(S_n)$ is the pushout of a (typically injective) graph morphism $\iota : S_n \to \Gamma$ and the graph morphism $\pi : S_n \to \{\bullet\!\!-\!\!\!-\!\!\bullet\}$ which sends each leave of $S_n$ to a single edge.
- For a connected[1] subgraph $H \subset \Gamma$ a *collapsing of* $H$ is the pushout of the embedding $H \hookrightarrow \Gamma$ and the map which sends $H^{top}$ to a single point.

a) Show that the pushout of folding and collapsing yield again a (topological) graph. Which of the resulting morphisms are graph morphisms?

b) Consider the Petersen-graph $\Gamma$ in Figure 2. What is the resulting graph after folding the red star?

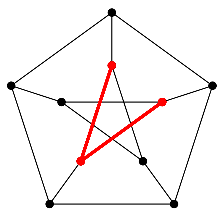

Figure 2: The Petersen-graph

c) Find a spanning tree of the Petersen-graph and collapse it. What is the resulting graph?

[1] You can also collapse not connected subgraphs. In that case each component of $H$ is sent to a different point.

You can hand in the exercise sheet until Monday the 09. 05. 2022 at 2 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a $(*)$ might be a little bit trickier.

**Exercise 2.**

Let $G$ be a group, $S \subset G$ be a generating set and $\Gamma = \Gamma(G, S)$ its Cayley graph. Furthermore, let $N \trianglelefteq G$ be a normal subgroup which acts on $\Gamma$ by left-multiplication. Show that $\Gamma/N$ is a Cayley graph of $G/N$.

**Exercise 3.**

Consider the infinte tree $C :=$  .

a) What are the possible $\mathbb{Z}$-actions on $C$?

b) Let $\Gamma$ be a tree together with a free $\mathbb{Z}$-action. Show that there exists a $\mathbb{Z}$-equivariant graph-morphism from $C$ to $\Gamma$.
*Hint: Start with a vertex $v$ and its edge-path to $1 \bullet v$ and to $2 \bullet v$. When you cut something off, remember that each edge and vertex in a tree is separating.*

**Exercise 4.**

Let $G$ be a finite group which acts on a tree $\Gamma$ (which is not necessarily finite). Show that this action has a global fixpoint in $\Gamma$, that is there exists a vertex or geometric edge in $\Gamma$ which is fixed by all elements of $G$.

**Exercise 5.**

Let $G$ be a group and $S \subseteq G$ be a generating set of $G$ without elements of order $\leq 2$. Let furthermore $\Gamma = \Gamma(G, S)$ be the corresponding Cayley graph together with the usual left multiplication $\rho : G \to \mathrm{Aut}(\Gamma)$.

a) For a vertex $v \in V(\Gamma)$ let $\mathrm{Stab}(v) := \{\phi \in \mathrm{Aut}(\Gamma) \mid \phi(v) = v\}$ be the stabiliser of $\Gamma$. Show that every element in $\psi \in \mathrm{Aut}(\Gamma)$ can be written as $\psi = \rho(g)\phi$ for some $g \in G$ and $\phi \in \mathrm{Stab}(1_G)$.

b) Let $E^+(\Gamma) := \{(g, s) \in E(\Gamma) \mid g \in G, s \in S\}$ together with the labelling $w : E^+(\Gamma) \to S, (g, s) \mapsto s$. Furthermore let

$$\begin{aligned}\mathrm{Aut}(\Gamma, w) := \{\phi \in \mathrm{Aut}(\Gamma) \mid \phi(E^+(\Gamma)) = E^+(\Gamma) \text{ and} \\ \forall e_1, e_2 \in E^+(\Gamma) : \big(w(e_1) = w(e_2) \Rightarrow w(\phi(e_1)) = w(\phi(e_2))\big) \}\end{aligned}$$

be the set of automorphisms which preserve the orientation of the edges and respect the labelling. Show that $\rho(G)$ is a normal subgroup of $\mathrm{Aut}(\Gamma, w)$, but not necessarily normal in $\mathrm{Aut}(\Gamma)$. Deduce that $\mathrm{Aut}(\Gamma, w)$ is a semi-direct product of $G$ and $\mathrm{Stab}(1_G) \cap \mathrm{Aut}(\Gamma, w)$.

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