Universität des Saarlandes
FR 6.1 Mathematik
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4. Exercise sheet for Geometric Group Theory

Exercise 1.

Let Γ and Γ' be two finite[1] graphs and $\phi : \Gamma \to \Gamma'$ be a graph homomorphism.

a) For an edge $e \in E(\Gamma)$ with initial vertex $v := o(e)$ let

$$S_{v,e} := \{f \in E(\Gamma) \mid v = o(f) \text{ and } \phi(e) = \phi(f)\}$$

be the star of v in Γ consisting of all edges with the same image under ϕ. We denote by $\Gamma/_{\sim_{v,e}}$ the folding of Γ along $S_{v,e}$ (see last exercise sheet).

Show that ϕ factors through the folding, that is there exists a graph homomorphism $\tilde{\phi} : \Gamma/_{\sim_{v,e}} \to \Gamma'$ such that $\phi = \tilde{\phi} \circ \pi$ holds where $\pi : \Gamma \to \Gamma/_{\sim_{v,e}}$ is the usual folding morphism.

b) Show that ϕ can be written as a concatenation of finitely many folds and an immersion. An immersion is a locally injective map ι, that means for each point p there exists an open neighbourhood $U \ni p$ such that its restriction $\iota|_U$ is injective.
Remark: Graph morphisms are always locally injective along edges, hence you have to only check near vertices.

Exercise 2.

Let X and Y be two topological spaces and $C(X, Y)$ be the set of continuous maps from X to Y.

a) Show that "being homotopic to each other" is an equivalence relation on $C(X, Y)$.

We say X and Y are homotopy equivalent and write $X \simeq Y$, if there exist continuous maps $f : X \to Y$ and $\tilde{f} : Y \to X$ such that $f \circ \tilde{f} \simeq \mathrm{id}_Y$ and $\tilde{f} \circ f \simeq \mathrm{id}_X$ holds.[2]

b) Show that homotopy equivalence is indeed an equivalence relation on any set of topological spaces.

c) Show that X is contractible if and only if X is homotopy equivalent to the space $\{\star\}$ with only one point.

d) Consider the cirlce, the disk and the punctured disks

$$\mathbb{S}^1 := \{x \in \mathbb{R}^2 \mid |x| = 1\}, \quad \mathbb{D} := \{x \in \mathbb{R}^2 \mid |x| < 1\} \quad \text{and } \mathbb{D}^* := \mathbb{D} \setminus \{0\}$$

with the usual subset topology of $\mathbb{R}^2$. Show that $\mathbb{S}^1$ is homotopy equivalent to $\mathbb{D}^*$ but not to $\mathbb{D}$.

[1] We only need that Γ is finite in b). The rest also works for infinite graphs.
[2] We then say, that $\tilde{f}$ is the homotopy inverse of f. Be aware that homotopy inverses are typically not unique.

You can hand in the exercise sheet until Monday the 16. 05. 2022 at 2 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a (∗) might be a little bit trickier.

Exercise 3.

a) We consider $\mathbb{R}^2$ with the Karlsruhe and the Manhattan metric, that is:

$$d_{KA}\left(\begin{pmatrix} x_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) := \begin{cases} |y_1 - y_2| & \text{, if } x_1 = x_2 \\ |x_1 - x_2| + |y_1| + |y_2| & \text{, if } x_1 \neq x_2 \end{cases}$$

$$d_{Man}\left(\begin{pmatrix} x_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) := |x_1 - x_2| + |y_1 - y_2|$$

For which of the above metrics is $\mathbb{R}^2$ an $\mathbb{R}$-tree?

b) The star S_3 is also called a tripod. Show that a metric space (X, d) is an $\mathbb{R}$-tree if and only if each geodesic triangle in (X, d) is isometric to a metric tripod.
Hint: You have already seen one implication in the lecture.

Exercise 4.

a) Show that a $CAT(0)$ space is uniquely geodesic, that is between any two points there exists a unique geodesic.

b) We consider the sphere

$$\mathbb{S}^2 := \{v \in \mathbb{R}^3 \mid \|v\| = 1\}$$

with the two induced metrics

$$d_{sub}(v, w) := \|v - w\|$$
$$d_{len}(v, w) := \inf\{l(\gamma) \mid \gamma \text{ is a path in } \mathbb{S}^2 \text{ joining } v \text{ and } w\}.$$

For which of these is the sphere $\mathbb{S}^2$ a $CAT(0)$ space? You can use here, that the length metric d_{len} is the same as the angle between the two points measured at the center and its geodesics are segments of great circles, also called orthodromes.

c) Is the following octant of the sphere

$$Q := \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{S}^2 \mid 0 \leq x, y, z \leq 1\}$$

a $CAT(0)$ space for one of the above metrics?

Exercise 5. $(*)$

For $n \in \mathbb{N}$ let X_n be the space consisting of n unit-squares cyclically glued together at adjacent sides, that is we identify one corner of all squares and two neighbour squares share a side.

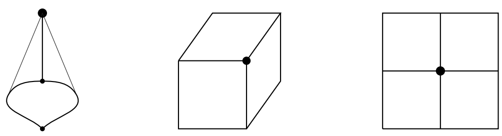

Figure 1: X_1, X_3 and X_4

We endow X_n with the length metric, that is the distance between two points is the length of the shortest path, in terms of the typical metric on each square, connecting them. For which n is X_n a $CAT(0)$-space.
Warning: This exercise is starred, as we do not have yet a feasible proof for $n \geq 5$ with your current tools.

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