Universität des Saarlandes
FR 6.1 Mathematik
Prof. Dr. L. Bartholdi
Prof. Dr. G. Weitze-Schmithüsen
Dr. C. Steinhart

5. Exercise sheet for Geometric Group Theory

Exercise 1.

Let Γ be a finite graph and $H \subset \Gamma$ be a connected subgraph. Recall from exercise sheet 3 the collapsing of a subgraph and let Γ/H denote the resulting graph from collapsing H in Γ.

a) Show that if H is a tree, then the collapsing map $\pi : \Gamma \to \Gamma/H$ is a homotopy equivalence.

b) Recall the Euler characteristic from sheet 2. Show that if two finite, connected graphs have the same Euler characteristic, then they are homotopy equivalent.

c) Consider a star $\iota : S_n \to \Gamma$ in Γ (see sheet 3). Show that if ι is injective, then folding along ι is a homotopy equivalence.

Remark: We will see in the next exercise sheet, that the reverse statements also hold.

Exercise 2.

Let G be a group, $S \subset G$ be a generating set and $\Gamma = \Gamma(G, S)$ its Cayley graph. Show that Γ is a tree if and only if G is isomorphic to the free group $F(S)$.
Hint: A word in G corresponds to a path in Γ.

Exercise 3.

Let X, Y be two finite[1] sets and $F(X)$ and $F(Y)$ their corresponding free groups. We want to show that $F(X)$ and $F(Y)$ are isomorphic if and only if X and Y have the same cardinality.[2]

a) Show that $|X| = |Y|$ already implies $F(X) \cong F(Y)$.

b) Consider the subgroup $P(X) := \langle w^2 \mid w \in F(X)\rangle$ which is generated by squares. Show that $P(X)$ is a characteristic subgroup, that is for any $\phi \in \mathrm{Aut}(F(X))$ we have $\phi(P(X)) \subseteq P(X)$. In particular $P(X)$ is a normal subgroup.

c) Show that $F(X)/P(X) \cong (\mathbb{Z}/2\mathbb{Z})^{|X|}$.

d) Deduce that if $F(X) \cong F(Y)$ holds, we already have $|X| = |Y|$.

[1]This works as well with infinite sets, but for simplicity you may assume they are finite.
[2]Hence one typically writes $F(X) = F_n$ for $n = |X| \in \mathbb{N}$ and calls it the free group of rank n. Do not confuse this notation with the finite fields $\mathbb{F}_q$!

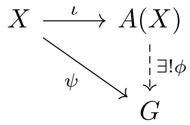
You can hand in the exercise sheet until Monday the 23. 05. 2022 at 12 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a $(*)$ might be a little bit trickier.

Exercise 4.

Similarly to free groups there exists a notion of free abelian groups. Let X be a set, then a (or the) *free abelian group* $A(X)$ generated by X is an abelian group with the following properties:

(i) There exists an injective map $\iota : X \to A(X)$.

(ii) For every abelian group G and map $\psi : X \to G$ there exists a unique group homomorphism $\phi : A(X) \to G$, such that the following diagramm commutes:



Show the following properties:

a) Free abelian groups generated by a set X are unique up to isomorphism.

b) The free abelian group $A(X)$ is the abelianisation of $F(X)$.

c) $\mathbb{Z}^n$ is a free abelian group.

Exercise 5.

a) Give a finite presentation of the Diedergroup D_5.
Hint: From exercise sheet 2 we already have $D_5 = \mathbb{Z}/5\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$.

b) Let $G := N \rtimes_\varphi H$ be a semi-direct product. Can you find a presentation of G with the help of presentations of N and H?

c) Which two groups have the following presentations

$$G := \langle x, y \mid x^2 = y^3, xyx = yxy \rangle$$
$$H := \langle x, y \mid (xy)^2 = (yx)^3, y^2 = x^4, x^2y^4 = \mathbb{1}_H \rangle?$$

Exercise 6. (Just for fun!)

The phonetic group is presented by $\langle ABC \mid \text{sound} \rangle$, i.e. it is the group generated by the 26 letters in the alphabet and two words are equal, if they sound the same, e.g. we have the relation $flour = flower$. Try to show, that the phonetic group is trivial for a fixed language of your choice.

You can hand in the exercise sheet until Monday the 23. 05. 2022 at 12 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a (∗) might be a little bit trickier.