Universität des Saarlandes
FR 6.1 Mathematik
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6. Exercise sheet for Geometric Group Theory

Exercise 1.

a) Let (X, x) and (Y, y) be two punctered, topological spaces, $f : (X, x) \to (Y, y)$ be a morphism between them and $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ its push-forward. Refute or prove the following statements:

(i) If f is surjective, then f_* is also surjective.

(ii) If f is injective, then f_* is injective.

(iii) If f_* is bijective, then f is bijective.

b) Show converse statements from exercise 1 from last sheet, i.e. let Γ be a finite, connected graph and H be a subgraph, then the following holds:

(i) If collapsing H is a homotopy equivalence, then H is a tree.

(ii) If folding a star is a homotopy equivalence, then we didn't fold a multi-edge, that is two edges with the same endpoints.

Exercise 2.

Let $n \in \mathbb{N}$ and we consider the abelianisation $ab : F_n \to \mathbb{Z}^n$.

a) Show that every automorphism $\phi : F_n \to F_n$ induces a unique automorphism $\tilde{\phi} : \mathbb{Z}^n \to \mathbb{Z}^n$ with $ab \circ \phi = \tilde{\phi} \circ ab$.

b) Show that the set

$$\mathrm{Inn} := \{\kappa_g \in \mathrm{Aut}(F_n) \mid g \in F_n \text{ and } \kappa_g(x) := gxg^{-1}\}$$

of conjugations is a normal subgroup of $\mathrm{Aut}(F_n)$. These are called the *inner automorphisms* of F_n.

c) Show that the map $\tilde{ab} : \mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z}), \phi \mapsto \tilde{\phi}$ from a) is surjective.

d) Show that $\tilde{ab}$ factors through the *outer automorphisms of* F_n $\mathrm{Out}(F_n) := \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$, that is we can write $\tilde{ab}$ as the concatenation

$$\mathrm{Aut}(F_n) \overset{\pi}{\twoheadrightarrow} \mathrm{Out}(F_n) \overset{\psi}{\twoheadrightarrow} \mathrm{GL}_n(\mathbb{Z})$$

for the usual quotient map π and a unique morphism ψ.

e) Show for $n \geq 3$ that the function ψ in d) is not injective.
Hint: Try a "partial conjugation".

You can hand in the exercise sheet until Monday the 30. 05. 2022 at 2 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a $(*)$ might be a little bit trickier.

Exercise 3.

Let $\Pi_g := \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = \mathbb{1} \rangle$ be the surface group of genus g. Show the following properties:

a) The abelianisation of Π_g is $\mathbb{Z}^{2g}$.

b) The map

$$\phi : \{a_1, a_2, b_1, b_2\} \to S_4, \begin{cases} a_1, a_2 & \mapsto (1\ 2\ 3) \\ b_1, b_2 & \mapsto (1\ 3\ 4) \end{cases}$$

can be extended to a group homomorphism from Π_2 to S_4.

c) Π_g is not abelian for $g \geq 2$ and abelian for $g = 1$.

Exercise 4.

a) Let G be a group, $H \leq G$ be a subgroup and $\langle\langle H \rangle\rangle \trianglelefteq G$ be its normal hull in G, i.e. the smallest normal subgroup of G containing H. Show that we can write the quotient group as amalgamization $G/\langle\langle H \rangle\rangle \cong G *_H \{\mathbb{1}\}$, where we use the embedding of H as subgroup of G.

b) Construct a surface with fundamental group $\mathbb{Z}/2\mathbb{Z}$.
Extra: If you manage to bring such a surface to the exercise session, your TA will bake a cake for the next session (you may have to "cheat" a little bit as we live only in three dimensions).

c) Show that every finitely presented group is the fundamental group of a surface.

Exercise 5.

Ferdinand likes to hang his pictures on a rope lying over two nails (see Figure 2). But whenever he

Figure 1: Ferdinand helping out a fellow scientist.

invites his extended family for a visit, someone keeps stealing one of the two nails of each picture – he has always suspected that his great-grandaunt had had an affair with a magpie. To find out and surprise the culprit, Ferdinand uses his knowledge about groups and in particular commutators and hangs his pictures in such a way over two nails, that whichever nail one pulls out, the whole picture comes down. How does he do it? To give the thief even a little bit more temptation, can he also do it with three or more nails, such that if any single nail is pulled, the whole picture falls down?
Remark: Since Ferdinand is a scientific owl, he uses a rope which is basically frictionless, arbitrarily thin and shrinks or extends its length as needed, that is the trick neither lies in the length or friction of the rope.

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