Universität des Saarlandes
FR 6.1 Mathematik
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7. Exercise sheet for Geometric Group Theory

Exercise 1.

Let Γ and H be two finite, connected graphs and $\rho : \Gamma \to H$ be a covering map. Prove or refute the following statements:

a) Γ is regular if and only if H is regular.

b) If $e \in E(\Gamma)$ is separating, then $\rho(e) \in E(H)$ is separating.

c) Let $e \in E(\Gamma)$ be an edge and its image $\rho(e) \in E(H)$ is a separating edge, then e is also a separating edge.

d) Γ is a tree if and only if H is a tree.

e) The map $\rho_* : \pi_1(\Gamma, x) \to \pi_1(H, \rho(x))$ is injective.

f) The map $\rho_* : \pi_1(\Gamma, x) \to \pi_1(H, \rho(x))$ is surjective.

Exercise 2.

a) We consider the wedge of shrinking circles[1] as a metric graph Γ with one vertex and with

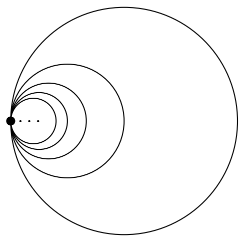

Figure 1: Wedge of shrinking circles

edges $\{e_n \mid n \in \mathbb{N} \text{ and } l(e_n) = \frac{1}{n}\}$. Show that Γ does not have a universal covering.

b) Can you find a simply connected space, which is not locally connected?
Hint: Start with something locally connected and add a contracting point.

c) What are the possible coverings of the circle $\mathbb{S}^1$?

d) The projective plane is defined as the set

$$\mathbf{P}\mathbb{R}^2 := \{v \in \mathbb{R}^3 \setminus \{0\}\}/\sim$$
$$\text{with } v \sim w :\iff \exists \lambda \in \mathbb{R} : \lambda v = w$$

together with the quotient topology, i.e. the finest topology such that the quotient map $\pi : \mathbb{R}^3 \to \mathbf{P}\mathbb{R}^2$ is continuous. Show that the sphere is the universal covering of $\mathbf{P}\mathbb{R}^2$.

[1] In the literature this is sometimes also the Hawaiian earring.

You can hand in the exercise sheet until Monday the 06. 06. 2022 at 2 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a (∗) might be a little bit trickier.

Exercise 3.

An origami is a finite covering $\rho : \mathcal{O} \to \mathbb{T}^*$ of the punctered torus $\mathbb{T}^*$. Typically one describes the punctured torus as the unit-square without the corners where we identify opposing sides. In this context an origami is written as a union of unit squares glued together at their sides and each square is sent pointwise to the single square representing the punctered torus. Consider the origami in Figure 2, where we already glued some edges and identify the rest of the edges with the same label:

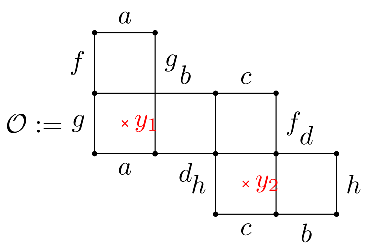

Figure 2: An origami with two points y_1 and y_2 in the same fiber.

a) Draw the lifts of the path $\gamma \in \pi_1(\mathbb{T}^*, y)$ depicted in Figure 3 starting at y_1 and y_2 in $\mathcal{O}$.

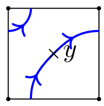

Figure 3: A path in the once punctered torus.

b) Is the covering $\mathcal{O} \to \mathbb{T}^*$ from Figure 2 normal? How many deck transformations does it have?

Exercise 4.

Let $p : Y \to X$ be a covering and $x \in X$. For any $\gamma \in \pi_1(X, x)$ and $y \in p^{-1}(x)$ let $\tilde{\gamma}_y : [0, 1] \to Y$ be the lift of γ starting with $\tilde{\gamma}_y(0) = y$.

a) Show that $\pi_1(X, x)$ acts on the fibers $p^{-1}(x)$ on the right by $y \cdot \gamma := \tilde{\gamma}_y(1)$. This is called the *monodromy action.*

b) Show that the monodromy action is compatible with the deck group, that is for $\phi \in \mathrm{Deck}(p)$ we have $\phi(y) \cdot \gamma = \phi(y \cdot \gamma)$.

Exercise 5.

a) Prove Proposition 3.3, that is: Let X be a Hausdorff space together with a group action $\rho : G \to \mathrm{Homeo(X)}$. Show that the quotient map $p : X \to X/G$ is a covering if and only if ρ is free and properly discontinuously.

b) For which of the following actions is the quotient map a covering?

(i) $\mathbb{Z}$ acts on $\mathbb{S}^1$ by rotating by the angle $2\pi/\sqrt{2}$.

(ii) $\mathbb{Z}$ acts on $\mathbb{R}$ by $\rho(1) : x \mapsto x + e^x$.

(iii) $\mathbb{Z}/k\mathbb{Z}$ acts on $\mathbb{D}$ by rotating by the angle $2\pi/k$.

(iv) $\mathbb{Z}/6\mathbb{Z}$ acts on $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ by $\rho(1)(\begin{pmatrix} [x] \\ [y] \end{pmatrix}) := \begin{pmatrix} [x + 1/3] \\ [y + 1/2] \end{pmatrix}$.

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