Universität des Saarlandes
FR 6.1 Mathematik
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8. Exercise sheet for Geometric Group Theory

Exercise 1.

Let F_n be the free group of rank $n \in \mathbb{N}$ and $U \leq F_n$ be a subgroup.

a) Show that U is a free group.
Hint: Consider the Cayley-graph of F_n and the action of U on it.

b) Let U be now of finite index $d := [F_n : U]$ in F_n. Write the rank k of $U \cong F_k$ in terms of d and n.

c) Show that all free groups F_n of finite rank $n \geq 2$ contain each other as subgroups.

Exercise 2.

Recall the origami in Figure 1 from the last sheet:

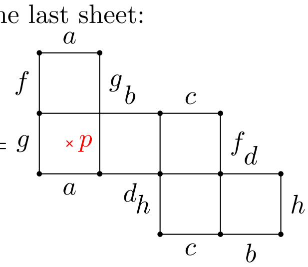
$\mathcal{O} :=$ 

Figure 1: An origami which does not lay eggs.

a) We want to determine the intermediate origamis of $\mathcal{O}$, that is the origamis $\rho' : \mathcal{O}' \to \mathbb{T}^*$ such that $\rho : \mathcal{O} \to \mathbb{T}^*$ can be written as the concatenation

$$\mathcal{O} \xrightarrow{\tilde{\rho}} \mathcal{O}' \xrightarrow{\rho'} \mathbb{T}^*$$

for some covering $r\tilde{h}o : \mathcal{O} \to \mathcal{O}'$.

(i) Show that there is no intermediate origami of degree 3 (over $\mathbb{T}^*$).
Hint: The monodromy action factors through intermediate coverings. Consider the action of x^4 and y^4, what does it tell you about a possible $\mathcal{O}'$ of degree 3?

(ii) What are the intermediate origamis of $\mathcal{O}$?

b) We write $\pi_1(\mathbb{T}^*, \rho(p)) \cong F_2 := \langle x, y \rangle$ as the free group generated by the "horizontal" loop x and "vertical" loop y. We also identify $\pi_1(\mathcal{O}, p)$ via the map $\rho_* : \pi_1(\mathcal{O}, p) \to \pi_1(\mathbb{T}^*, \rho(p))$ as a subgroup U of F_2.

(i) What is the index of U in F_2?

(ii) Can you write down generators of U and representants of the right coset F_2/U?

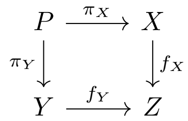
(iii) Which subgroups of F_2 contain U as a subgroup and what is the normal hull of U in F_2?

You can hand in the exercise sheet until Monday the 13. 06. 2022 at 2 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a (*) might be a little bit trickier.

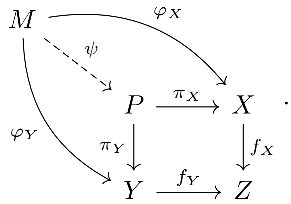
Exercise 3.

Dual to the pushout from sheet 2 there is a notion of pullback in categories[1]. Let X, Y and Z be objects and let $f_X : X \to Z, f_Y : Y \to Z$ be two morphisms. We say f_X and f_Y have a *pullback*, if there exists an object P and morphisms $\pi_X : P \to X, \pi_Y : P \to Y$ with the following properties:

(i) The following diagram commutes



(ii) (P, π_X, π_Y) is universal in the sense, that for any other object M and morphisms $\varphi_X : M \to X$, $\varphi_Y : M \to Y$ which fulfil (i), there exists a unique morphism $\psi : M \to P$ such that the following diagram commutes:



a) Show that in the category of topological spaces (with continuous maps) the pullback is $P := \{(x, y) \in X \times Y \mid f_X(x) = f_Y(y)\}$ together with the product topology and the projections $\pi_X((x, y)) = x$ and $\pi_Y((x, y)) = y$.

b) Let now X, Y and Z be finite, topological graphs and f_X, f_Y be two graph morphisms. Give a brief argument, why their pullback is again a finite, topological graph.

c) We consider the two coverings of the rose R_2 depicted in Figure 2, which send the two vertices

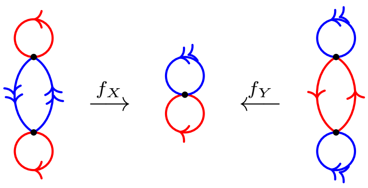


Figure 2: Two coverings of the rose R_2

to the one vertex of the rose and the edges to the same colored edge. What is their pullback?

d) Show that if f_X and f_Y are (finite) coverings, then π_X and π_Y are also (finite) coverings.

e) Let P be the pullback of two coverings and $p \in P$ be a point. Show that we have for the fundamental groups the identity $(f_X \circ \pi_X)_*(\pi_1(P, p)) = (f_X)_*(\pi_1(X, \pi_X(p))) \cap (f_Y)_*(\pi_1(Y, \pi_Y(p)))$. In other words the fundamental group of P is the intersection of the fundamental groups of X and Y as subgroups of the fundamental group in Z.

f) We consider the two subgroups $G := \langle a^2, b, aba \rangle$ and $H := \langle b^2, a, bab \rangle$ of $F_2 = \langle a, b \rangle$. What is the rank of their intersection $G \cap H$? Can you give free generators of this intersection?

g) Conclude that the intersection of two finite index subgroups of F_n is again a finite index subgroup of F_n.

[1] That is to say a pullback is a pushout in the opposite category

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