Universität des Saarlandes
FR 6.1 Mathematik
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9. Exercise sheet for Geometric Group Theory

Exercise 1.

Let $\mathbb{H} := \{z = x + yi \in \mathbb{C} \mid \mathrm{Im}(z) = y > 0\}$ be the upper half-plane. We consider the action of $\mathrm{GL}_2(\mathbb{R})$ on $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by Möbius transformations[1]:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet z := \begin{cases} \frac{az+b}{cz+d} & \text{, if } z \neq \infty \\ \frac{a}{c} & \text{, if } z = \infty \end{cases} \quad \text{and put } \frac{x}{0} := \infty \text{ for } x \neq 0.$$

a) Show that Möbius transformations yield indeed a well defined group action and that the center $Z(\mathrm{GL}_2(\mathbb{R})) = \{\lambda I_2 \mid \lambda \in \mathbb{R} \setminus \{0\}\}$ acts trivially.

b) What is the stabiliser of $\mathbb{H}$?

c) What is the orbit of i and its fixgroup?

Exercise 2.

We consider $\mathbb{H}$ together with the hyperbolic metric. Show that the three Möbius transformations

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad D := \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

for any $\lambda, t > 0$ are isometric.

Exercise 3.

Let X be a locally compact Hausdorff space and $G \curvearrowright X$ be a properly discontinuous action. We call a closed subset $F \subset X$ a *fundamental domain* if it satisfies the following two properties:

(i) $\bigcup_{g \in G} gF = X$

(ii) There exists an open subset $F_O \subset F$ with $\overline{F_O} = F$ and $gF_O \cap F_O = \emptyset$ for all $g \in G \setminus \mathrm{id}$.

a) Let $H \leq G$ be a finite index subgroup of G and F be a fundamental domain of G. Can you describe a fundamental domain of H in terms of F?

b) Let now $X = \mathbb{H}$ be the hyperbolic half-plane. A discrete subgroup $G \leq PSL_2(\mathbb{R})$ is called a *Fuchsian group*. One can show that this is equivalent to the action of G on $\mathbb{H}$ is properly discontinuously (you don't need to show that). Show that the action of a Fuchsian group has a fundamental domain.
Hint: Fix a point $z \in \mathbb{H}$ and consider points which are closer to z than to any other orbit point of z.

[1] This also works just as well with $\mathrm{GL}_2(\mathbb{C})$ with the same definition.

You can hand in the exercise sheet until Monday the 20. 06. 2022 at 12 pm. Either give it directly to Christian Steinhart or throw it into box 47 in the basement in building E 2.5. The exercises marked with a (∗) might be a little bit trickier.

Exercise 4.

We consider the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ by Möbius transformations and denote

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

a) Draw a fundamental domain for each of the two subgroups $\langle S \rangle$ and $\langle T \rangle$. What are the two quotients $\mathbb{H}/\langle S \rangle$ and $\mathbb{H}/\langle T \rangle$ topologically?

We now want to prove that $\mathcal{F} := \{z \in \mathbb{H} \mid |z| \geq 1,\ |\mathrm{Re}(z)| \leq 1/2\}$ is a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$:

b) Show that for any $z \in \mathbb{H}$ and $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $\mathrm{Im}(A \bullet z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$.

c) Follow that the supremum $\sup_{A \in \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(A \cdot z)$ is achieved for some $A \in \mathrm{SL}_2(\mathbb{Z})$. Show furthermore that $\mathrm{Im}(z)$ is maximal in its $\mathrm{SL}_2(\mathbb{Z})$ orbit if and only if $|cz + d| \geq 1$ holds for all coprime $c, d \in \mathbb{Z}$.

d) Show for $z \in \mathcal{F}_O := \{z \in \mathbb{H} \mid |z| > 1,\ |\mathrm{Re}(z)| < 1/2\}$ the following inequalities:

$$|cz + d|^2 > c^2 + cd + d^2 \geq 1$$

for all coprime $c, d \in \mathbb{Z}$.

e) Show that $\mathcal{F}$ is a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 5.

a) Compute the growth function γ and the growth series $\Sigma(z)$ for the group $\mathbb{Z}^2$ with generating set $S := \{(a, b) \mid a, b \in \{-1, 0, 1\}\ \}$.

b) What is the growth function γ and growth series $\Sigma(z)$ for $\mathbb{Z}^2$ as a monoid with generating set $S := \{(0, 1),\ (1, 0),\ (-1, -1)\}$.

c) For the above generating sets let $K(S)$ be the convex hull of S as a polytope in $\mathbb{R}^2$. Verify that we have asymptotically $\gamma(n) \sim \mathrm{vol}(K)n^2$ as $n \to \infty$.

d) Viewing $\gamma(n)$ as a polynomial function, note that $\gamma(-n) = \gamma(n-1)$. (In fact, $\gamma(-n)$ counts the number of points in the interior of nK. The relation above is true as soon as S contains all the points in its convex hull, look at `https://en.wikipedia.org/wiki/Ehrhart_polynomial`.)

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