UNIVERSITÄT DES SAARLANDES

SYSTOLES OF TRANSLATION SURFACES

Bachelorarbeit

Pascal Schumann

Betreuer:
Prof. Dr. Gabriela Weitze-Schmithüsen

Erstkorrektor:
Prof. Dr. Gabriela Weitze-Schmithüsen

Zweitkorrektor:
Prof. Dr. Laurent Bartholdi

August 31, 2022

Sworn declaration

I declare under oath that I have prepared the paper at hand independently and without the help of others and that I have not used any other sources and resources than the ones stated. Parts that have been taken literally or correspondingly from published or unpublished texts or other sources have been labelled as such. This paper has not been presented to any examination board in the same or similar form before.

Saarbrücken, ______________ ______________________

Declaration of Consent

I agree to make both versions of my thesis (with a passing grade) accessible to the public by having them added to the library of the Computer Science Department.

Saarbrücken, ________________ ______________________

Acknowledgments

First and foremost, I want to show my sincere gratitude to Prof. Weitze-Schmithüsen. Without her guidance, advice and support, this thesis would not have been possible. Her dedication is inspiring and she always manages to create a friendly and humorous environment.

I am grateful to Prof. Bartholdi for agreeing to be a second reviewer for this thesis and his insights that inspired the prospect about cyclic covers in the principal strata.

I want to show my appreciation to Alexander Rogovskyy for introducing me to cyclic torus covers and providing me with graphics he created for his thesis.

I wish to express my sincere thanks to my family who continuously and unconditionally supported me my entire life.

Finally, I want to thank my partner Katharina Schackmar for her constant support and reassurance especially in stressful times.

Contents

List of Symbols

Conventions

X, Y	Topological spaces
U, V	Neighborhoods
p, q	Coverings
(X, ω)	Finite translation surface
σ	Singularities of translation surfaces
$[i], [j]$	Singularities of origamis
g	Genus of a surface / Group element
O	Origami
σ_x, σ_y	Origami Permutations
φ, ψ	Charts
G, H	Finite groups
γ, δ	Paths
C_n	Cyclic group of order n
$H_n(X)$	Singular homology groups

Definitions

d_X	Metric on translation surfaces: Lemma I.1.2
Σ	Set of singularities: Definition I.1.3
$H(k_1, \ldots, k_n)$	Stratum: Definition I.2.7
sys	Length of the systole: Definition I.5.1

SR	Systolic ratio: Definition I.5.1
S, T	Generators of $\mathrm{SL}_2(\mathbb{Z})$: Eq. (II.1)
$O_{(G,g,h)}$	Origami from Cayley graph
$G(n,k)$	$\langle r, s \mid r^{2^{k+1}} = s^{2^{n-k-1}} = 1, s^{-1}rs = r^{-1} \rangle$, $0 \leq k \leq n-2$
D_n	Dihedral group: $\langle r, s \mid r^n = s^2 = rsrs = 1 \rangle$
Γ	Graph of saddle connections: Definition III.1.1
$\Gamma_S, \Gamma_v, \Gamma_{S_\varepsilon}$	Subgraphs of the graph of saddle connections: Definition III.1.4
M_g	Genus g surface: Proposition I.3.3
$M \# N$	Connected sum of two manifolds

Introduction

In this thesis, we study a type of surfaces called *finite translation surfaces*. They are two-dimensional manifolds equipped with a *translation structure*, i.e. an atlas such that the transition maps are translations. Alternatively, they can be obtained by identifying opposite edges of polygons in the Euclidean plane. We will see that we can divide the space of translation surfaces by their singularities into *strata* that can be endowed with the structure of an orbifold.

A special type of translation surfaces are *origamis*. They are square-tiled surfaces or equivalently coverings of the once punctured torus. Origamis have multiple interesting properties such as being entirely determined by two permutations describing the horizontal and vertical gluing.

Our attention mostly belongs to *systoles* of translation surfaces. Systoles are in some sense the shortest closed curves of translation surfaces. We will show that for translation surfaces with genus $g \geq 2$ there is always a systole that passes through a *cone point* (i.e. a singularity with angle $2\pi(k+1), k \in \mathbb{N}$). We will then compute the length of systoles for special families of origamis, in particular for *normal origamis* with certain deck transformation groups and origamis induced by cyclic covers of the $(n \times n)$-torus. For this purpose, we will present two algorithms introduced in *Systolic geometry of translation surfaces* [1] that, given an origami, compute the length of the systole. We will also show a potential implementation of these algorithms in GAP (Groups, Algorithms Programming) [3].

Chapter I.

Finite Translation Surfaces

In this chapter, we introduce the notion of finite translation surfaces. We will see that they can be equipped with a metric and we are thus able to talk about their singularities with respect to this metric.

1. Definitions and Notations

We will see that there are multiple equivalent definitions of finite translation surfaces. The first definition is in terms of gluing polygons in the Euclidean plane together [10].

Definition I.1.1: Let $P_1, \ldots, P_n \subset \mathbb{R}^2$ be finitely many disjoint polygons, let P_i^* denote the polygon P_i without its vertices and let $D := \bigcup_{i=1}^n \partial P_i^*$ be the union of the edges of all polygons without their vertices. Moreover, we choose an orientation of $\mathbb{R}^2$ and an involution $T : D \to D$ such that the restriction of T on the interior of an edge is a translation to an edge with opposite orientation. If the surface

$$X := (\bigcup_{i=1}^{n} P_i^*)/T$$

is connected, then (X, ω) is called a *finite translation surface*, where ω is the atlas obtained via the embeddings of the polygons in $\mathbb{R}^2$ and the identification of the edges via T.

We will refer to finite translation surfaces as just *translation surfaces*. Visually speaking one obtains a translation surface by gluing parallel sides of polygons together. Examples of translation surfaces are the torus (see Figure I.1) or the translation surface in Figure I.3.

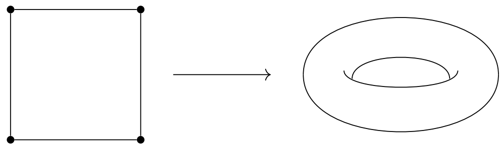

Figure I.1.: The torus is a translation surface obtained by gluing opposite sides of a unit square.

In the following we want to define a metric on our translation surface. We do this by considering the quotient metric of $\bigcup_{i=1}^{n} P_i^*$ induced by the Euclidean metric of $\mathbb{R}^2$. Generally, the quotient space of a metric space does not have to be a metric space itself, but we will see that in our case the space behaves as desired. For this chapter let (X, ω) be a translation surface.

Lemma I.1.2: *The function $d_X : X \times X \to [0, \infty)$ defined by*

$$d_X(x,y) = inf\, \{\sum_{i=1}^{m-1} d_{\mathbb{R}^2}(p_i, p_{i+1})\ |p_1, \ldots, p_m \in \mathbb{R}^2 \ \ s.t\ [p_1] = x, [p_m] = y \ and$$
$$\forall\, 1 \leq i \leq m-1\, \exists\, P_j^* \ \ s.t\ p_i, p_{i+1} \in P_j^*\}$$

defines a metric on X, called the flat metric *on X.*

Proof: We first show the identity of indiscernibles, i.e $d_X(x,y) = 0 \Leftrightarrow x = y$.

Let $x, y \in X$ with $x = y$. Choose an arbitrary point $p \in \pi^{-1}(x) \subset \mathbb{R}^2$, where π is the projection. Then we have $[p] = x = y$. Since $d_{\mathbb{R}^2}$ is a metric we have $d_{\mathbb{R}^2}(p,p) = 0 \implies d_X(x,y) = 0$.

Now let $x \neq y$. Let $p_1, \ldots, p_m \in \mathbb{R}^2$ such that $[p_1] = x$ and $[p_m] = y$. First, assume that neither x nor y are on an edge of a polygon. Let

$$\varepsilon := \inf\,(\{d_{\mathbb{R}^2}(\pi^{-1}(x), p) \mid p \in \bigcup_{j=1}^{n} \partial P_j^*\} \cup \{d_{\mathbb{R}^2}(\pi^{-1}(x), \pi^{-1}(y))\}),$$

where ∂P_j^* are the edges of the polygon P_j without its vertices. Then $0 < \varepsilon \leq \sum_{i=1}^{m-1} d_{\mathbb{R}^2}(p_i, p_{i+1})$, so $d_X(x,y) \geq \varepsilon > 0$.

Now assume that x and y are on some edges of the polygons. If $p \in \mathbb{R}^2$ is a point on an edge of a polygon without vertices, denote by E_p the edge p is on and by E_p^{-1} its opposite edge. Let

$$\delta := \inf\,(\{d_{\mathbb{R}^2}(a,b) \mid [a] = x, [b] = y\}$$
$$\cup\, \{d_{\mathbb{R}^2}(p,q) \mid [p] = x \text{ or } [p] = y, q \in (\bigcup_{j=1}^{n} \partial P_j^*) \setminus (E_p \cup E_p^{-1}\})$$

We have $0 < \delta \leq \sum_{i=1}^{m-1} d_{\mathbb{R}^2}(p_i, p_{i+1}) \implies d_X(x,y) \geq \delta > 0$. Lastly, w.l.o.g. let y be on an edge of a polygon and x not. Let

$$\mu := \inf \{d_{\mathbb{R}^2}(\pi^{-1}(x), p) \mid p \in \bigcup_{j=1}^{n} \partial P_j^*\}.$$

The same argument as in the other two cases shows $d_X(x,y) \geq \mu > 0$. All cases combined imply that $d_X(x,y) \neq 0$ and thus $d_X(x,y) = 0 \Leftrightarrow x = y$.

Furthermore, symmetry follows from the symmetry of $d_{\mathbb{R}^2}$.

Now let $x, y, z \in X$. Since every path from x to z to y is a path from x to y, this path is already considered by the infimum in the definition of $d_X(x,y)$. Hence,

$$d_X(x,z) + d_X(z,y) \geq d_X(x,y).$$

Thus, the triangle inequality holds and d_X is a metric. $\square$

The metric space we obtain does not have to be complete, but every metric space M has a metric completion which can be constructed by considering equivalence classes of Cauchy sequences in M with respect to the equivalence relation of having distance 0. A more thorough construction can be found in [8].

Definition I.1.3: Let $\overline{X}$ be the metric completion of X. We call the points $\Sigma := \overline{X} \setminus X$ the *singularities* of X.

The set Σ of singularities is finite, even discrete. By construction of X, they are exactly the removed vertices of the polygons. In the following section, our goal is to find some sort of classification of these singularities. Henceforth let (X, ω) always denote a translation surface and $\overline{X}$ its metric completion.

2. Classification of Singularities

We first introduce a more general notion of translation surfaces where we allow translation surfaces to be infinite.

Definition I.2.1: We call a set of *translation charts*, i.e., charts whose transition maps are translations, *translation atlas*. We say that two translation atlases are equivalent if for all charts of both atlases there are biholomorphic transition maps. A *translation structure* is then an equivalence class of a translation atlas.

Definition I.2.2: A *(infinite) translation surface* (X, ω) is a connected, two-dimensional manifold X equipped with a translation structure ω.

One can show that we also have a metric for infinite translation surfaces ([10] Proposition 1.8) and singularities with respect to this metric.

Definition I.2.1 inspires the following definition of finite translation surfaces.

Definition I.2.3: A *finite translation surface* (X, ω) is a connected manifold X together with a translation structure ω such that $\overline{X}$ is compact and the set Σ of singularities is finite.

We just need to make sure that the two definitions are talking about the same object. We give a proof which can also be found in [10] Satz 1.

Proposition I.2.4: *The two definitions of finite translation surfaces (Definition I.1.1, Definition I.2.3) are equivalent.*

Proof: Let (X, ω) be a finite translation surface as given in Definition I.2.3. Consider a finite triangulation of $\overline{X}$ such that every singularity is a vertex of a triangle and that for every triangle $\Delta \subset X$ there is a chart (U, φ) such that $\Delta \subset U$ and $\varphi(U)$ is a triangle in $\mathbb{R}^2$.

For all vertices of triangles in $\mathbb{R}^2$ that do not correspond to a singularity glue all the adjacent triangles together. This is possible because the transition maps are translations. This gives polygons in $\mathbb{R}^2$ that can once again be glued together by identifying edges that are shared by triangles in $\overline{X}$ and hence a translation surface according to Definition I.1.1.

Now let (X, ω) be a finite translation surface as given in Definition I.1.1. Let $x \in X$. Then the preimage of the projection π either consists of a single point in the interior of a polygon or of two points on the edges of two not necessarily distinct polygons.

In the first case consider a sufficiently small neighborhood U of x such that π is invertible on U. Then $(U, (\pi_{|U})^{-1})$ is a chart around x.

Similarly, we can find a chart $(V, (\pi_{|V})^{-1})$ in the second case. Since the preimage of V under π consists of two connected components, we need to transform the image of $(\pi_{|V})^{-1}$ on one of the connected components via translations to ensure that the chart is well defined.

Doing this for every $x \in X$ gives a translation atlas. Then X together with the equivalence class of this atlas gives a finite translation surface according to Definition I.2.3. $\square$

We will later give a third definition of finite translation surfaces (see Definition I.4.1) which is also equivalent to the other two definitions.

Lastly, to talk about the classification of singularities we introduce translation coverings.

Definition I.2.5: Let (X, ω) and (Y, ν) be two not necessarily finite translation surfaces and let $p : X \to Y$ be a covering such that p can be extended to a continuous map $\overline{X} \to \overline{Y}$. p is called *translation covering* if for any $x \in X$ there are charts $(U, \varphi) \in \omega, (V, \psi) \in \nu$ with $x \in U$ and $p(U) \subset V$ such that for every $z \in \varphi(U)$ we have

$$(\psi \circ p \circ \varphi^{-1})(z) = z + t \quad \text{for some fixed } t \in \mathbb{R}^2 .$$

We say that p is *k-cyclic* if $\mathrm{Deck}(p) \cong \mathbb{Z} / k\, \mathbb{Z}$ and *infinite cyclic* if $\mathrm{Deck}(p) \cong \mathbb{Z}$.

Informally speaking one can see that we have some sort of angle around the singularities of X. This observation can be made mathematically rigorous.

Definition I.2.6: Let σ be a singularity of (X, ω). σ is called

(i) *flat point* if there is a chart (U, φ) of $X \cup \{\sigma\}$ with $\sigma \in U$ that is compatible with ω.
(ii) *cone angle singularity* or *cone point* if there exist $k > 1, \epsilon > 0$ and a neighborhood U of σ in $\overline{X}$ such that there is a k-cyclic translation covering of $U \setminus \{\sigma\}$ on the once punctured disk $B_\epsilon(0) \setminus \{0\} \subset \mathbb{R}^2$. We call k the *multiplicity* of σ.
(iii) *infinite angle singularity* if there exists an $\epsilon > 0$ and a neighbourhood U of σ in $\overline{X}$ such that there is an infinite cyclic translation covering of $U \setminus \{\sigma\}$ on the once punctured disk $B_\epsilon(0) \setminus \{0\} \subset \mathbb{R}^2$.
(iv) *wild singularity* if none of the other classifications apply.

The translation surface in Figure I.1 has one flat point. It can be shown that on finite translation surfaces only flat points and cone points can occur (see [10] Proposition 2.5). When talking about translation surfaces that are not finite, this does not need to hold.

Definition I.2.7: For $k_1, \ldots, k_n \in \mathbb{N}$ the set

$$\begin{aligned} H(k_1, \ldots, k_n) := \{(X, \omega) \text{ is a finite translation surface} \mid \\ X \text{ has } n \text{ singularities of multiplicities } k_1 + 1, \ldots, k_n + 1\} / \sim \end{aligned}$$

is called *stratum*, where two finite translation surface $(X, \omega), (Y, \nu)$ are isomorphic if there is a homeomorphism $\varphi : X \to Y$ that is locally a translation.

3. Homology of Translation Surfaces

In this section, we want to investigate the homology groups of a translation surface X. This will be useful for establishing a connection between the singularities of X and the genus of X.

We give a brief introduction to singular homology (with coefficients in $\mathbb{Z}$) as given in Allen Hatcher's book about algebraic topology [4].

Let $v_0, \ldots, v_n \in \mathbb{R}^m$ be $n + 1$ affine independent points. Then denote by $[v_0, \ldots, v_n] \subset \mathbb{R}^m$ the smallest convex set containing $v_0, \ldots, v_n$. This set is called *n-simplex* and is the n-dimensional analog of the triangle. Further, denote by

$$\Delta^n := [e_0, \ldots, e_n] \tag{I.1}$$

the *standard n-simplex*, where $\{e_0, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^{n+1}$.

Removing one point v_i of an n-simplex leaves us with an $(n-1)$-simplex. We write $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ for this simplex.

Now let X be an arbitrary topological space. A *singular n-simplex* is a continuous map

$$\sigma : \Delta^n \to X$$

where Δ^n is the standard simplex given in Eq. (I.1).

Define $C_n(X)$ to be the free abelian group with the set of singular n-simplices as its basis. The elements of $C_n(X)$ are called *n-chains* and are of the form $\sum_i n_i \sigma_i$, where $n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X$ and the sum is finite.

The *boundary map* $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined as

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \ldots, \widehat{v_i}, \ldots, v_n]},$$

with the property that $\partial_n \circ \partial_{n+1} = 0$.

We then define the *singular homology groups* as

$$H_n(X) := \ker \partial_n / \mathrm{Im}\ \partial_{n+1}.$$

Equivalently, we could consider the *cellular homology groups*, as they are isomorphic to the singular homology groups.

Definition I.3.1: The *Euler characteristic* of a topological space X is given by

$$\chi(X) := \sum_n (-1)^n \ \mathrm{rank}(H_n(X)),$$

where the H_n are the singular homology groups.

Let M, N be two orientable and connected n-dimensional manifolds, let $D \subset M$ and $D' \subset N$ be two closed n-disks and let $\varphi : \partial D \to \partial D'$ be a homeomorphism. Define the set

$$S = (M \setminus \mathrm{Int}(D)) \cup (N \setminus \mathrm{Int}(D')).$$

The *connected sum* $M \# N$ of M and N is then the quotient space obtained by identifying the boundarys of D and D' via φ. The fact that the connected sum is unique is highly non-trivial and uses the Annulus theorem proven by Kirby in [7].

Less rigorously speaking, the connected sum of two manifold M, N is obtained by removing a ball in each manifold and gluing the two resulting manifolds together along the boundary spheres.

We want to compute the homology groups of connected sums for closed, connected and orientable manifolds of dimension n. The following lemma can be found in [4].

Lemma I.3.2: *Let M, N be two closed, connected and orientable n-dimensional manifolds. Then we have*

$$H_i(M \# N) = \begin{cases} \mathbb{Z}, i = 0 \ \textit{or}\ i = n, \\ H_i(M) \oplus H_i(N), 1 \leq i \leq n-1 \\ 0, i > n \end{cases}$$

In the following, we will show how the Euler characteristic of the genus g surface M_g, defined as the connected sum of g many tori, relates to g. We will then use this information to find a connection between the Euler characteristic of a translation surface and its genus g.

Proposition I.3.3: *Let M_g be the connected sum of g many tori. Then*

$$\chi(X) = 2 - 2g.$$

Proof: Since M_g is path connected, we have $H_0(M_g) = \mathbb{Z}$ (see Lemma I.3.2). The other homology groups

$$H_1(M_g) = H_1(T^2) \oplus \cdots \oplus H_1(T^2) = \mathbb{Z}^2 \oplus \cdots \oplus \mathbb{Z}^2 = \mathbb{Z}^{2g}$$

and

$$H_2(M_g) = \mathbb{Z}\,.$$

and lastly for $n > 2$

$$H_n(M_g) = 0.$$

follow from Lemma I.3.2. Hence, we have

$$\chi(M_g) := \sum_n (-1)^n \operatorname{rank}(H_n(M_g)) = 1 - 2g + 1 = 2 - 2g. \qquad \square$$

Moreover, we use the following classification theorem for closed surfaces from which we can deduce that every translation surface is homeomorphic to M_g for some g. We remind the reader that we restrict ourselves to finite translation surfaces.

Theorem I.3.4 (Classification of Closed Surfaces): *Let X be a closed and connected surface. Then X is homeomorphic to one of the following:*

(i) *The genus g surface, i.e. the connected sum of g many tori. We consider the sphere as the sum of 0 tori.*
(ii) *The connected sum of k real projective planes $k \geq 1$.*

A proof of this theorem can be found for example in Conway's ZIP Proof [2].

Since the transition maps of a translation surface X are translations and the Jacobian matrix of a translation is the identity matrix, it immediately follows that X is orientable. As the real projective plane is not orientable, we get that $X \cong M_g$ for some genus g. Due to the homology groups being topological invariants, we also know the homology groups of the translation surface. Thus we have the following corollary:

Corollary I.3.5: *Let X be a translation surface of genus g. Then*

$$\chi(X) = 2 - 2g.$$

The following basic fact from the theory of translation surfaces (see e.g. [10] Proposition 3.13) establishes a connection between the genus of a translation surface and the multiplicities of its singularities.

Proposition I.3.6: *Let $X \in H(k_1, \ldots, k_n)$ be a translation surface of genus g. Then*

$$2g - 2 = \sum_{i=1}^{n} k_i$$

Proof: Consider a finite triangulation of $\overline{X}$ such that every singularity of X is a vertex of a triangle. Denote by v the number of vertices, by e the number of edges and by f the number of faces of the triangulation. Then we know by Corollary I.3.5

$$2g - 2 = \chi(\overline{X}) = v - e + f.$$

Since every triangle has 3 edges but every edge is shared by two faces, we have that $e = \frac{3f}{2}$.

We compare the sum of all interior angles of the triangles:

$$f \cdot \pi = \sum_{i=1}^{n} (k_i + 1) \cdot 2\pi + (v - n) \cdot 2\pi$$

Dividing by π leaves us with

$$f = \sum_{i=1}^{n} 2(k_i + 1) + 2(v - n)$$

and thus

$$\begin{aligned}
\chi(X) &= v - \frac{3f}{2} + f \\
&= v - \frac{f}{2} \\
&= v - \frac{\sum_{i=1}^{n} 2(k_i + 1) + 2(v - n)}{2} \\
&= v - \sum_{i=1}^{n} (k_i + 1) - (v - n) \\
&= -\sum_{i=1}^{n} (k_i + 1) + \sum_{i=1}^{n} 1 \\
&= -\sum_{i=1}^{n} k_i.
\end{aligned}$$

Multiplying both sides by -1 yields

$$2g - 2 = -(2 - 2g) = -(-\sum_{i=1}^{n} k_i) = \sum_{i=1}^{n} k_i. \qquad \square$$

With this connection, one may easily verify that the following statements are true.

Corollary I.3.7: *Let $H(k_1, \ldots, k_n)$ be a stratum. Then all translation surfaces in $H(k_1, \ldots, k_n)$ share the same genus g.*

Corollary I.3.8: *Let X be a translation surface. Then X has at least one cone point singularity if and only if X has genus $g \geq 2$.*

Corollary I.3.9: *Let n be odd and $k_i = 1$ for $1 \leq i \leq n$. Then the stratum $H(k_1, \ldots, k_n)$ is empty.*

4. Period Coordinates

In the following, we want to equip the set of translation surfaces with a space structure. We first introduce an equivalent definition of translation surfaces using holomorphic 1-forms. For an introduction to differential forms, we refer to [16].

Definition I.4.1: Let X be a compact and connected Riemann surface and let $\omega \neq 0$ be a holomorphic 1-form. Let Σ be the set of zeroes of ω and set $X^* := X \setminus \Sigma$. Then (X^*, ω) is a finite translation surface.

The singularities of X^* are then exactly the zeroes of ω. An atlas can be obtained by integrating over the 1-form ω. For a detailed proof why the two definitions are equivalent, we refer to [10] Satz 1.

The following introduction to *period coordinates* is heavily based on the introduction given by Daniel Massart in [9] and Alex Wright in [19].

We have seen that we can divide the set of equivalence classes of translation surfaces into strata. Let $H(k_1, \ldots, k_s)$ be a stratum consisting of translation surfaces of genus g. Then the stratum can be endowed with the structure of a complex orbifold of dimension $n = 2g + s - 1$ (see [19] Proposition 1.15).

Let (X^*, ω) be a translation surface of genus g, as defined above, and let $\Sigma = \{x_1, \ldots, x_s\}$ be the set of zeroes of the holomorphic differential ω. Choose a basis of the relative homology $H_1(X, \Sigma)$. For example, let $\alpha_1, \dots, \alpha_{2g}$ be simple closed curves based at x_i which generate $H_1(X)$ and for each for each $1 \leq i \leq s-1$ let c_i be a simple arc joining x_i to x_{i+1}. Then the relative homology classes of $\alpha_1, \dots, \alpha_{2g}, c_1, \dots, c_{s-1}$ generate $H_1(X, \Sigma)$. We can then define the period coordinates of (X^*, ω) as follows:

Definition I.4.2: Let (X^*, ω) be a translation surface. The *period coordinates* of (X^*, ω) with respect to the basis $[\alpha_1], \dots, [\alpha_{2g}], [c_1], \dots, [c_{s-1}]$ of $H_1(X, \Sigma)$ are the $n = 2g + s - 1$ complex numbers

$$\int_{\alpha_1} \omega, \dots, \int_{\alpha_{2g}} \omega, \int_{c_1} \omega, \dots, \int_{c_{s-1}} \omega.$$

The period coordinates assign to each stratum an atlas of charts to $\mathbb{C}^n$ with transition functions in $\mathrm{GL}_n(\mathbb{Z})$. These transition functions are change of basis matrices for the first relative homology group $H_1(X, \Sigma)$.

5. Systoles of Translation Surfaces

We define a *curve* as a continuous function $\gamma : I \to \overline{X}$, where $I \subset \mathbb{R}$ is a closed interval. We say that γ is *closed* if the start point and endpoint of γ coincide. In this thesis, we want to talk about curves that are locally shortest paths. Mainly, we are interested in the locally shortest curves that connect two not necessarily distinct singularities in $\overline{X}$.

A *geodesic* is a curve $\gamma : I \to \overline{X}$ such that for every $x \in I \setminus \partial I$ there is a neighborhood U of x such that $\gamma|_U$ is an isometry. A geodesic in $\overline{X}$ which has its starting point and endpoint in two not necessarily distinct singularities and does not contain any more singularities is called *saddle connection*.

One of our main objects of interest are *systoles* of translation surfaces. They are in some sense the shortest closed curves. The following definition makes this more precise:

Definition I.5.1: A *systole* of X is a shortest, simple closed, not null-homotopic geodesic of $\overline{X}$. We denote by $\mathrm{sys}(X)$ the length of the systole. Moreover, denote by

$$\mathrm{SR}(X) := \frac{\mathrm{sys}(X)^2}{\mathrm{area}(X)}$$

the *systolic ratio* of X.

Remark I.5.2: We allow systoles to pass through cone point singularities of X. Therefore, if we talk about systoles of X we also allow them to lie in $\overline{X}$.

While the systole of a translation surface does not have to be unique (not even up to homotopy), it always exists on the surface $\overline{X}$. If the genus of X is ≥ 2, we have that there always exists a systole that passes through a cone point (see [1], Proposition 2.8). We give the proof of this statement in the following.

Theorem I.5.3: *Let X be a translation surface with genus $g \geq 2$, then there exists a systole of X that passes through a cone point.*

Proof: Let X be a translation surface and γ be a simple closed geodesic in X which does not pass through a cone angle singularity. Let $\varepsilon > 0$ be sufficiently small such that the cylinder

$$C_\varepsilon(\gamma) := \{p \in X \mid d_X(p, \gamma) < \varepsilon\}$$

around γ does not contain any cone points. Due to Corollary I.3.8, there exists at least one cone point. Thus, we can expand the cylinder until at width δ the boundary of $\overline{C_\delta(\gamma)}$ contains a cone point p (see Figure I.3) or the upper and the lower segment of the cylinder coincide. Note that the second case implies that X is a torus which contradicts $g \geq 2$.

We can decompose the boundary of the cylinder into two parts, namely the upper and lower segment of the boundary of the cylinder (left/right lines in Figure I.2)

$$\partial\overline{C_\delta(\gamma)} = \partial_1\overline{C_\delta(\gamma)} \cup \partial_2\overline{C_\delta(\gamma)}.$$

Note that $\partial_1\overline{C_\delta(\gamma)}$ and $\partial_2\overline{C_\delta(\gamma)}$ do not need to be disjoint (i.e. the segments may partially lie on identified edges).

Let without loss of generality $\gamma' = \partial_1\overline{C_\delta(\gamma)}$ be the part of the boundary containing p.

We have that $\gamma' \setminus \{p\}$ is either an open segment (see Figure I.2) or consists of at least two connected components (see Figure I.3)

$$\gamma' \setminus \{p\} = \mathring{\gamma}_1 \sqcup \mathring{\gamma}_2 \sqcup \cdots \sqcup \mathring{\gamma}_k.$$

In the first case let $\gamma'' = \gamma'$. In the second case $\mathring{\gamma}_i \cup \{p\}$ are simple closed geodesics and we can choose $\gamma'' = \mathring{\gamma}_1 \cup \{p\}$.

We have the following estimate

$$l(\gamma) = l(\gamma') \geq l(\gamma'').$$

This shows that for every simple closed geodesic γ which does not pass through a cone point there exists a simple closed geodesic of equal or smaller length that intersects a cone point. Thus, the minimum $\operatorname{sys}(X)$ is attained in at least one simple closed geodesic that passes through a cone point. $\square$

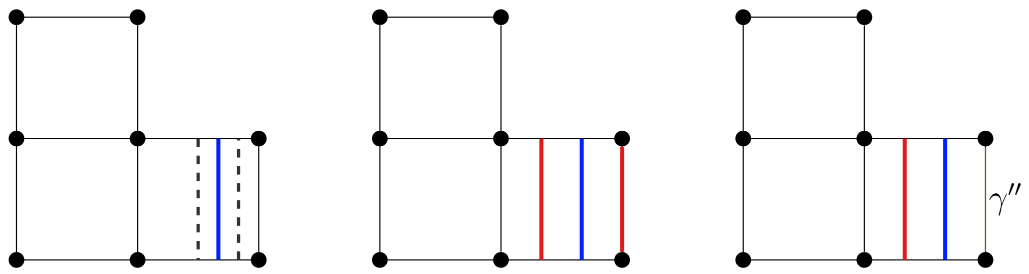

Figure I.2.: Three images of the same translation surface in $H(2)$, where opposite edges are identified. The blue line is a simple closed geodesic γ. The dotted/red lines represent the boundary of the cylinder before and after the expansion, respectively. The green line is a simple closed geodesic γ'' that runs through a cone point. In particular, in this example, γ'' is a systole of the surface.

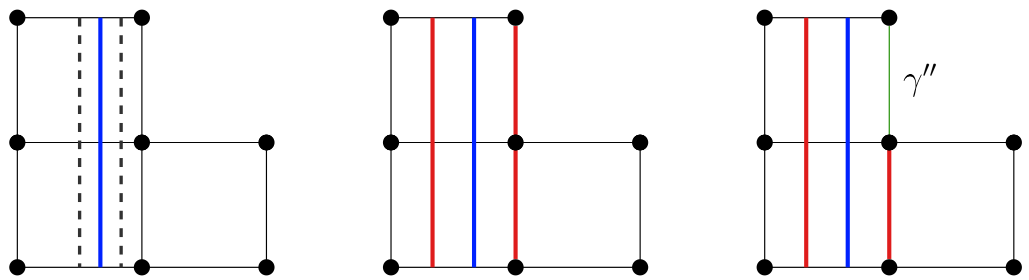

Figure I.3.: The same illustration as in Figure I.2, but this time $\gamma' \setminus \{p\}$ (right red line) consists of two connected components. The green line γ'' is then the choice of such a connected component together with the point p.

Chapter II.

Origamis and the $\mathrm{SL}_2(\mathbb{Z})$-action

The goal of this chapter is to introduce special types of translation surfaces called *origamis*, which lie dense in the strata. They are obtained by gluing finitely many unit squares together. We will see, that $\mathrm{SL}_2(\mathbb{Z})$ defines an action on the set of origamis. Other good references for origamis include [15], [14] and [13].

Definition II.0.1: A finite translation surface (O, ω) is called *origami* if it can be represented as a gluing of finitely many unit squares. The number of squares is called *degree* of O and is denoted by $\deg(O)$.

The simplest example of an origami is the torus as can be seen in Figure I.1. Another example is given in Figure I.3. There we see an origami of degree 3 in $H(2)$.

Remark II.0.2: A convenient property of origamis is that they are entirely determined by two permutations σ_x and σ_y in S_d describing the horizontal and vertical gluing, respectively. We write $O = (\sigma_x, \sigma_y)$. We obtain the translation surface from the permutations as follows:

We label the squares of the origami by $1, \dots, \deg(O)$.

- The right edge of the square labeled with i is glued with the left edge of the square labeled with $\sigma_x(i)$.
- The upper edge of the square labeled with i is glued with the lower edge of the square labeled with $\sigma_y(i)$.

Note that we could use an arbitrary underlying set V with $\#V = d$ and consider $\sigma_x, \sigma_y \in \mathrm{Sym}(V)$.

Equivalently, one may define an origami O as a finite unramified covering of the once punctured torus $p : X^* \to T^*$. The degree d of the origami is the degree of the covering and the two permutations describing O can be obtained by considering the monodromy $\rho : \pi_1(T^*) \to S_d$ of p. In particular, if x, y are horizontal, respectively vertical, closed paths on T^* that do not lie on an edge, then x and y generate $\pi_1(T^*)$ and we set $\sigma_x = \rho(x)$ and $\sigma_y = \rho(y)$.

We say that the origami O is *normal* if the covering $p : X^* \to T^*$ is normal. Figure II.1 shows an example of a normal origami.

Remark II.0.3: Assume that (X, ω) is an origami. Then we can choose the curves $\alpha_1, \dots, \alpha_{2g}, c_1, \dots, c_{s-1}$ in Definition I.4.2 such that they lie on the edges of the squares. Hence the period coordinates lie in $\mathbb{Z}[i]$.

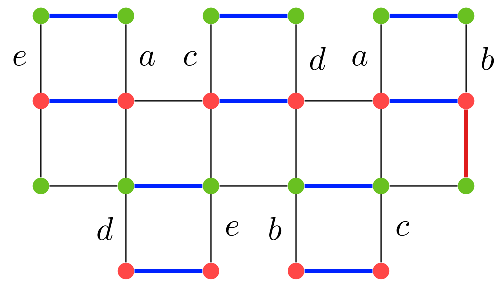

Figure II.1.: A normal origami in the stratum $H(4, 4)$. The blue curves $\alpha_1, \dots, \alpha_{10}$ together with the red curve c_1 form a basis of the $\mathbb{Z}$-module $H_1(X, \Sigma)$.

1. Origamis and Cayley Graphs

In Origamis and permutation groups [20], David Zmiaikou shows how to construct origamis given a 2-generated finite group G together with generators $g, h \in G$ such that G is the deck group of these origamis.

We will need the following definitions.

Definition II.1.1: A *labeled digraph* is a triple (V, E, L) where V is a set whose elements are called *vertices*, L a totally ordered alphabet whose elements are called *labels* and $E \subset V \times V \times L$ a subset whose elements are called *edges*. If there are k labels in L and every label $l_i, \in L$ $(1 \le i \le k)$ appears in an edge, then the digraph is called *k-labeled*.
An *isomorphism between two labeled digraphs* (V, E, L) and (V', E', L') is a pair

of bijections ($f : V \to V', \phi : L \to L'$), such that ϕ respects the orders on L and L' and we have $(u, v, l) \in E$ if and only if $(f(u), f(v), \phi(l)) \in E'$.

Definition II.1.2: Let V be a finite set and let O be an origami corresponding to a pair of permutations $(\sigma_x, \sigma_y) \in \mathrm{Sym}(V) \times \mathrm{Sym}(V)$. Then the *origamal digraph* of O is defined as the 2-labeled digraph (V, E, L), where $L = \{l_1, l_2\}$, $l_1 < l_2$ is an ordered alphabet and

$$E = \{(v, \sigma_x(v), l_1) \mid v \in V\} \cup \{(v, \sigma_y(v), l_2) \mid v \in V\}.$$

The labels l_1 and l_2 are in correspondence with the horizontal and vertical permutations, respectively.

Lastly, we will need the following definition.

Definition II.1.3: Let G be a group and let $\{g, h\}$ be a generating set. Consider G as the set of vertices and let $L = \{g, h\}, g < h$, be the alphabet. Moreover, let there be an edge with label g (resp. h) between two vertices u and v if and only if $v = ug$ (resp. $v = uh$). Then the resulting labeled digraph is called *Caley graph*. We write $C(G; L) = (G, E, L)$.

Normally, the Cayley graph is defined for arbitrary groups and generating sets but in this thesis we restrict ourselves to groups with two generators. Given a presentation $G = \langle X \mid R \rangle$ of G, we write $C(X; R)$ for the Cayley graph of G with respect to the generating set X.

Now let G be a finite group with generators g and h. The group G defines a right group action on itself via right multiplication (note that Zmiaikou uses left multiplication). We get a *permutation representation*

$$p_{\mathrm{reg}} : G \to \mathrm{Sym}(G)$$

of this group action (which is an antihomomorphism).

Then the Cayley graph $C(G, \{g, h\})$ is isomorphic to the origamal digraph defined by $(p_{\mathrm{reg}}(g), p_{\mathrm{reg}}(h)) \in \mathrm{Sym}(G) \times \mathrm{Sym}(G)$. We denote the corresponding normal origami by $O_{(G,g,h)}$.

In general, different generators of the same group give different origamis. However, in some cases, different generators can give rise to the same origami, as Zmiaikou shows in the following lemma ([20], Lemma 4.2).

Lemma II.1.4: *Let (g, h) and (g', h') be two pairs of generators of a finite group G. The origamis $O_{(G,g,h)}$ and $O_{(G,g',h')}$ coincide if and only if there exists an automorphism $\alpha \in Aut(G)$ such that $\alpha(g) = g'$ and $\alpha(h) = h'$.*

In the next section, we will see an example of this construction, namely for so-called p-origamis.

2. p-Origamis

In the dissertation of Thevis [17], a special class of origamis called *p-origamis* was introduced. They are normal origamis whose deck transformation groups are *p-groups*.

Definition II.2.1: Let G be a group. If there is a prime p such that for every $g \in G$ there is an $n \in \mathbb{N}$ such that $\operatorname{ord}(g) = p^n$, then G is called a *p-group*.

For finite groups, it is sufficient for the order of the group to satisfy this condition.

Remark II.2.2: Let G be a finite group. Then G is a p-group if and only if $\operatorname{ord}(G) = p^n$ for some $n \in \mathbb{N}$.

Proof: Let G be a finite p-group. Assume there is a prime $q \in \mathbb{N}$, $q \neq p$, such that $\operatorname{ord}(G)$ is divisible by q. According to the first Sylow theorem, there is a q-Sylow subgroup H of G such that $\operatorname{ord}(H) = q^k$ for some $k \in \mathbb{N}$. Let $h \in H, h \neq 1$. Then $1 \neq \operatorname{ord}(h) \mid q^k$ and thus, since p and q are primes, $\operatorname{ord}(h)$ is not a p-power. This contradicts G being a p-group. Hence, $\operatorname{ord}(G) = p^n$ for some $n \in \mathbb{N}$.

Now let G be a finite group such that $\operatorname{ord}(G) = p^n$ and let $g \in G$. Due to Lagrange's theorem, the order of g divides p^n. Since p is prime, we have that $\operatorname{ord}(g) = p^m$ for a natural number $m \leq n$. Hence, G is a p-group. $\square$

Definition II.2.3: A normal origami O whose deck group is a p-group is called *p-origami*.

The following example shows a family of 2-groups of order 2^n introduced by Thevis in [17] Proposition 3.1.4, obtained as a semidirect product of two cyclic groups. We will later compute the systoles of the 2-origamis obtained by these groups.

Example II.2.4: Let $n, k \in \mathbb{N}$ such that $n > 2$ and $k \leq n - 2$. Moreover, let $C_{2^{k+1}} = \langle r \rangle$ and $C_{2^{n-k-1}} = \langle s \rangle$ be two cyclic groups of order 2^{k+1} and 2^{n-k-1}, respectively. Define the group automorphism

$$\alpha : C_{2^{k+1}} \to C_{2^{k+1}}, r^m \mapsto r^{-m}.$$

Then the map

$$\varphi : C_{2^{n-k-1}} \to \mathrm{Aut}(C_{2^{k+1}}), s^m \mapsto \alpha^m$$

is a group homomorphism. We can then define the semidirect product

$$G_{(n,k)} := C_{2^{k+1}} \rtimes_\varphi C_{2^{n-k-1}} = \langle r, s \mid r^{2^{k+1}} = s^{2^{n-k-1}} = 1, s^{-1}rs = r^{-1} \rangle.$$

Then $G_{(n,k)}$ is a 2-group of order 2^n with generators generators r, s. For a more detailed description of the construction, we refer to Thevis's dissertation. Note that Thevis denotes the same group by $G^2_{(n,k)}$.

Let $n = 3$ and $k = 1$. Then $O_{(G_{(3,1)},r,s)}$ is the following origami:

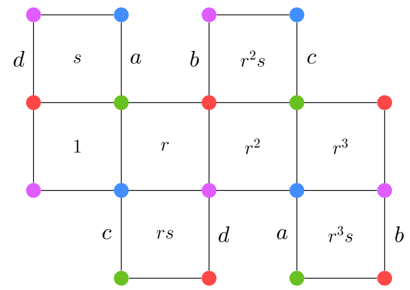

Figure II.2.: The origami $O_{(G_{(3,1)},r,s)}$. Edges without a label are glued with their opposite edge.

The origami in Figure II.2 is a 2-origami. For 2-origamis, Thevis gives the following classification of their strata ([17], Theorem 3.2.3).

Theorem II.2.5: *Let $n \in \mathbb{N}$. For 2-origamis of degree 2^n, the following strata appear:*

(i) $H(0)$
(ii) $H(2^{n-k} \times (2^k - 1))$, *where* $1 \leq k \leq n-2$

Remark II.2.6: For $k = 1$, the origamis $O_{(G_{(n,k)},r,s)}$ defined in Example II.2.4 lie in the stratum

$$H(\underbrace{1, \ldots, 1}_{2^{n-1}\text{-times}}).$$

This is a direct consequence of the proof of Theorem II.2.5.

3. The $\mathrm{SL}_2(\mathbb{Z})$-Action

The group $\mathrm{SL}_2(\mathbb{Z})$ defines an action on the set of origamis (see [18]). For a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ and an origami O we obtain a new origami $A \cdot O$ by applying the affine map $z \mapsto Az$ on every square of O.

In the following, we want to determine the permutations of the origami $A \cdot O$ in terms of the permutations of O. We fix the generators S and T of $\mathrm{SL}_2(\mathbb{Z})$ given by:

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{II.1}$$

The following lemma (as given in [1] Lemma 5.4) yields a way to compute the permutations of $A \cdot O$, where $A \in \{S, T, S^{-1}, T^{-1}\}$, in terms of the permutations of O. Additionally, it establishes a connection between the singularities of O and the singularities of $A \cdot O$.

Lemma II.3.1: *Let O be an origami with permutations (σ_x, σ_y). We have*

(i)
- $S(\sigma_x, \sigma_y) = (\sigma_y^{-1}, \sigma_x)$,
- $T(\sigma_x, \sigma_y) = (\sigma_x, \sigma_y \sigma_x^{-1})$,
- $S^{-1}(\sigma_x, \sigma_y) = (\sigma_y, \sigma_x^{-1})$,
- $T^{-1}(\sigma_x, \sigma_y) = (\sigma_x, \sigma_y \sigma_x)$.

(ii) *Let $[i]$ be the singularity of O associated with the lower left vertex of the square labeled with i and let $[j]$ be the singularity of $A \cdot O$ corresponding to $[i]$. Then the following statements hold:*
- $[j] = [\sigma_y^{-1}(i)]$, *if* $A = S$,
- $[j] = [\sigma_x^{-1}(i)]$, *if* $A = S^{-1}$,
- $[j] = [i]$, *if* $A = T$ *or* $A = T^{-1}$.

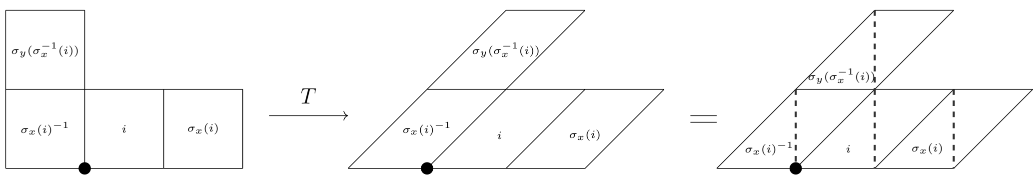

Figure II.3.: Application of T to an origami.

Proof: Figure II.3 shows four squares of O labeled by $i, \sigma_x(i), \sigma_x^{-1}(i), \sigma_y(\sigma_x^{-1}(i))$ on O. Moreover, it shows how the labeling carries over to a labeling of $T \cdot O$. The surface $T \cdot O$ can be tiled by squares, as suggested by Figure II.3, by keeping the horizontal edges of the parallelograms and taking the diagonals of the parallelograms as vertical edges (hinted by the dotted lines). We can label the resulting squares with the numbers of the parallelograms which intersect their right lower part. This yields exactly the stated permutations of $T \cdot O$. Figure II.3 also shows the singularity $[i]$. One can easily see that $[i]$ is mapped to the square labeled also with i. Every other case follows in similar a fashion as shown in Figure II.4 and Figure II.5.

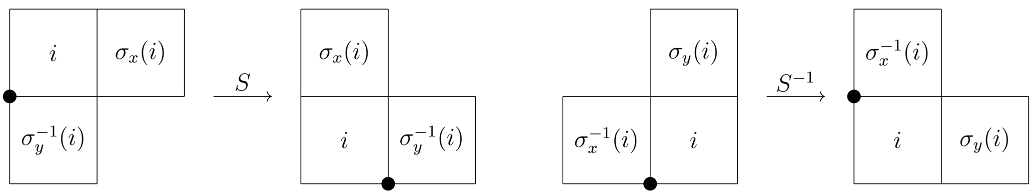

Figure II.4.: Application of S and S^{-1} to an origami.

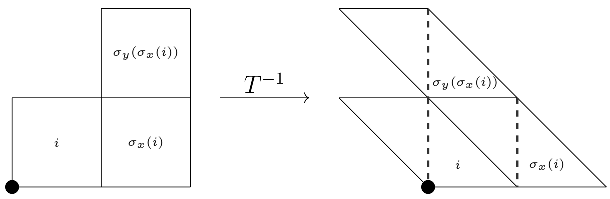

Figure II.5.: Application of T^{-1} to an origami.

Lemma II.3.1 can be used to compute the action of an arbitrary element of $\mathrm{SL}_2(\mathbb{Z})$. It also ensures that the application of a matrix in $\mathrm{SL}_2(\mathbb{Z})$ to an origami yields an origami again.

Chapter III.

The Graph of Saddle connections and Systoles of Origamis

1. Saddle Connections and Variants

The following chapter can be seen as a rough summary of sections 4 and 5 of *Systolic geometry of translation surfaces* [1]. As we have seen in Theorem I.5.3, for every translation surface X of genus $g \geq 2$ there exists a systole that is a concatenation of saddle connections. In the following, we introduce the *graph of saddle connections* ([1], Section 4). We will see how the systoles of the surface are related to the systoles of the graph of saddle connections. This will prove useful, as we can use methods from graph theory to compute $\mathrm{sys}(X)$ of the surface X.

Definition III.1.1: Let X be a translation surface. Let Γ be the graph whose vertices are the singularities of X and that has an edge between two vertices for every saddle connection of X connecting the corresponding singularities. Then Γ is called the *graph of saddle connections* of X. Γ can be turned into a weighted graph by assigning to each edge the length of the corresponding saddle connection.

An *edge path* $c = c_1 \dots c_n$ in Γ is the concatenation of the edges $c_1, \dots, c_n$ of Γ. The edge path c defines a path γ on X by considering the concatenation of the corresponding saddle connections γ_i and vice versa. We call γ the *realization* of c. Moreover, the *combinatorial length* of an edge path c is the number of edges it contains and we call c *trivial* if its combinatorial length is 0. If there are two edges c_i, c_{i+1} in c that are inverse to each other, we say that *c has backtracking*. An edge path that does not have backtracking is called *reduced*.

We call a non-trivial closed reduced edge path c of minimal length $l(c)$ in Γ a *systole* of Γ.

It would be convenient if there was a bijection between the systoles of a translation surface and the systoles of its graph of saddle connections. In general, this is not the case because systoles of Γ can belong to null-homotopic paths in X, as can be seen in [1] Example 5.6. Due to this, we need a good criterion when the systole of the graph is also a systole of the surface.

Definition III.1.2: Let $c = c_1, \ldots, c_n$ be a closed edge path and let $\gamma = \gamma_1, \ldots, \gamma_n$ be its realization. For two consecutive edges c_{i-1}, c_i let v_i be the vertex between the two edges. Moreover, let α_i, β_i be the angles around the corresponding singulary of v_i between the saddle connections γ_{i-1} and γ_i. We say *all angles are greater or equal to* π if $\alpha_i \geq \pi$ and $\beta_i \geq \pi$ for all $1 \leq i \leq n$.

Now we can make use of the following powerful theorem proven in [1] Theorem 4.5:

Theorem III.1.3: *Let c be a systole in the graph Γ and let γ be its realization. If the combinatorial length of c is not 3, then γ is a systole of the surface. If the combinatorial length of c is 3 and all angles of c are greater or equal to π, then γ is a systole of the surface.*

Theorem III.1.3 shows that in most cases the realization of a systole of the graph of saddle connections is a systole of the surface. We will later show that for all origamis up to and including degree 10, there is always a systole of combinatorial length not equal to three.

In the next section, we want to compute the systoles of origamis. We will need the following subgraphs of the graph of saddle connections.

Definition III.1.4: Let Γ be a graph of saddle connections of a surface X.

(i) Let $S \subset \mathbb{Z}^2$ be a finite set. Then Γ_S is the subgraph of Γ which contains all saddle connections whose directions lie in S.
(ii) Let $v \in \mathbb{Z}^2$ be a vector. Then $\Gamma_v := \Gamma_{\{v\}}$.
(iii) For a positive real number $\varepsilon > 0$ we let $\Gamma_{S_\varepsilon} = \Gamma_S$, where

$$S := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in A_1 \mid x^2 + y^2 \leq \varepsilon \right\}$$

and A_1 is the set of primitive vectors in the closed upper half plane, i.e.

$$\begin{aligned} A_1 :&= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \mid \gcd(x,y) = 1, y > 0 \text{ or } (y = 0 \text{ and } x > 0) \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \mid \gcd(x,y) = 1, y > 0 \right\} \cup \left\{ e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}. \end{aligned}$$

2. Computing Systoles of Origamis

While computing systoles of arbitrary translation surfaces of genus $g \geq 2$ may be difficult, we can use the permutations of origamis to easily compute their systoles. For this purpose, we will first present two algorithms introduced in [1] section 5. The first algorithm (Algorithm III.2.1) computes the saddle connections of an origami in a given direction $v_1 \in A_1$. The second algorithm (Algorithm III.2.3) invokes the first algorithm for all relevant $v \in A_1$ and produces a subgraph of the graph of saddle connections containing all systoles of the surface. Finally, we introduce an algorithm (Algorithm III.2.4) that, for a graph of saddle connections, returns $\mathrm{sys}(X)$ in the cases where the combinatorial length of a systole is not 3. In this chapter, we restrict to origamis of genus $g \geq 2$ and to singularities that are cone points.

To compute the saddle connections in a certain direction $v \in A_1$, the following algorithm applies an affine map $z \mapsto Az \quad (A \in \mathrm{SL}_2(\mathbb{Z}))$, to the origami O such that the saddle connections of O in direction v correspond to the saddle connections of $A \cdot O$ in direction e_1. We show in Remark III.2.2 that such a matrix A always exists. Consequently, the graphs Γ_v of O and Γ_{e_1} of $A \cdot O$ coincide. Note that it is sufficient to consider vectors in A_1 because we will later treat the union of all the graphs Γ_v as an undirected graph.

The algorithm uses the crucial fact that the cycles of the commutator $[\sigma_x, \sigma_y]$ are in correspondence with the singularities of O. To be more precise: Two squares i, j have the same singularity as their lower left corner if and only if i and j are in the same cycle of the commutator $[\sigma_x, \sigma_y]$. This is a direct consequence of the fact that for each square i with a singularity $[i]$ as its lower left vertex, the commutator $[\sigma_x, \sigma_y]$ defines a path around the singularity $[i]$ as a connected component of the preimage of the path $y^{-1}x^{-1}yx$ under the covering $p : O \to T$. Here x and y are horizontal/vertical paths on T that do not lie on an edge. A more rigorous explanation can be found in [12]. Note that the authors consider the commutator $[\sigma_y^{-1}, \sigma_x^{-1}]$ and the path $xyx^{-1}y^{-1}$. Additionally, we have seen in Lemma II.3.1 how the singularities of O relate to the singularities of $A \cdot O$. Now we have all the necessary tools to compute all saddle connections with directions in A_1.

Algorithm III.2.1 (Computation of the subgraph Γ_v): *Let $O = (\sigma_x, \sigma_y)$ be an origami and let $v \in A_1$.*

(i) *Choose an $A \in SL_2(\mathbb{Z})$ such that $Av = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Decompose A as a product of the generators (and their inverses) of $SL_2(\mathbb{Z})$ given in* II.1. *Compute the origami $A \cdot O = (\sigma'_x, \sigma'_y)$ and the bijection $\pi : \Sigma_O \to \Sigma_{A \cdot O}, [i] \mapsto [\pi([i])]$ between the singularities of O and $A \cdot O$ using Lemma II.3.1.*

(ii) *The vertices of Γ_v are the cycles of the commutator $[\sigma'_x, \sigma'_y] := \sigma'_x \circ \sigma'_y \circ \sigma'^{-1}_x \circ \sigma'^{-1}_y$. The cycles of the commutator are in a 1-1 correspondence with the singularities of $A \cdot O$. If a vertex corresponds to the singularity $[j]$ of $A \cdot O$, then label the vertex with $\pi^{-1}([j])$.*

(iii) *Make a list L of all squares whose lower left corners are singularities. This can be done as follows: $L := \{i \in \{1, \ldots, deg(O)\} \mid [\sigma'_x, \sigma'_y](i) \neq i\}$.*

(iv) *For i in L do:*
Let $k := min\{a \in \mathbb{N} \mid (\sigma'_x)^a(i) \in L\}$ and let $j = (\sigma'_x)^k(i)$.
Γ_v has an edge labeled with $k \cdot ||v||$ from the vertex $\pi^{-1}([i])$ to the vertex $\pi^{-1}([j])$.

Remark III.2.2: Note that the matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ in the first step of Algorithm III.2.1 always exists. The following proof shows how to explicitly construct this matrix.

Proof: Let $v = \begin{pmatrix} x \\ y \end{pmatrix} \in A_1$. Let $c, d \in \mathbb{Z}$ with $c = -y$ and $d = x$. Moreover let $a, b \in \mathbb{Z}$ be given via the extended Euclidian algorithm such that

$$1 = ax + by.$$

This is possible by Bézout's identity as x and y are coprime. Now let

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since the entries of A are integers and $\det(A) = ad - bc = ax + by = 1$ we have that $A \in \mathrm{SL}_2(\mathbb{Z})$. Then the computation

$$Av = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} 1 \\ -yx + xy \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

concludes the proof. $\square$

In the following, we describe an algorithm that repeatedly invokes Algorithm III.2.1 and merges the resulting graphs into one graph Γ_S containing the systoles of O. We only need to consider directions with lengths less or equal to the minimum cycle length of the horizontal and vertical permutations. This stems from the fact that the cycles of the permutations correspond to closed paths on the surface where the cycle lengths are exactly the lengths of the corresponding paths.

Algorithm III.2.3 (Computation of the graph $\boldsymbol{\Gamma_S}$): *Let $O = (\sigma_x, \sigma_y)$ be an origami.*

(i) *Let*

$$\varepsilon := min\{k \mid \sigma_x \text{ or } \sigma_y \text{ contain a } k\text{-cycle}\}$$

be the minimum cycle length of the permutations.

(ii) *Let $S := S_{\varepsilon^2}$ (as defined in Definition III.1.4). For each $v \in S$ compute Γ_v as described in Algorithm III.2.1.*

(iii) *Merge the obtained graphs Γ_v into one graph Γ_S as follows:*
- *The vertices of Γ_S are the singularities of O.*
- *For each $v \in S$, every pair of singularities $[i], [j]$ and every edge between $[i]$ and $[j]$ in Γ_v add an edge of the same weight between $[i]$ and $[j]$ to Γ_S.*

For every systole c of the graph whose realization γ is not null homotopic, γ is a systole of the surface. In cases where a combinatorial length of 3 does not occur, the criterion given in Theorem III.1.3 is sufficient to show that γ is indeed a systole.

To conclude this chapter we give an algorithm that, assuming that a combinatorial length of 3 does not occur, computes the length of the systole given the graph obtained via Algorithm III.2.3.

The idea is that a systole of the surface corresponds to a minimum cycle in Γ_S, so the algorithm computes for each edge e of the graph a minimum cycle containing e. Since it does this for every edge, we find a minimum cycle of the graph.

Algorithm III.2.4 (Computation of sys($\boldsymbol{O}$) given $\boldsymbol{\Gamma_S}$): *Let O be an origami and Γ_S the subgraph of the graph of saddle connections given by Algorithm III.2.3.*

(i) *Make a list E containing all edges of Γ_S.*

(ii) *Calculate a minimum cycle of Γ_S as follows:*
min_cycle $:= \infty$;

For each edge e of Γ_S:

- *Remove e from Γ_S.*
- *Let i and j be the vertices of the edge e. If $i = j$ let p be the empty path of length $l(p) = 0$. Else find a shortest path p from i to j (e.g. via Dijkstra).*
- *If $l(p) + l(e) \leq min\_cycle$ then $min\_cycle = l(p) + l(e)$.*
- *Add the edge e back to the graph Γ_S.*

(iii) *We have $sys(O) = min\_cycle$.*

Note that this algorithm can be extended to also give the combinatorial length of the systole. In case the combinatorial length is 3 it is technically possible to try to find another minimum cycle whose combinatorial length is not 3. If no other minimum cycle is found, then the angles of the edge path have to be examined.

Chapter IV.

Implementation

In this chapter we present a possible implementation of the algorithms from the previous chapter in GAP (Groups, Algorithms, Programming) [3]. While in theory each algorithm can be implemented separately, from a practical point of view it makes sense to interweave the implementations. For instance, instead of creating multiple graphs which are then merged, we can directly add the edges to the final graph, thus avoiding the merging process. Little details in the implementation like this can significantly boost the runtime. This can be felt especially when calculating the systole of either an origami with an enormous degree or when computing the systoles of many origamis.

Note that we only introduce the relevant parts of the implementation, for everything else we refer to the Appendix. We use the Origami Package [6] which provides many useful methods for computations related to origamis.

1. Implementation of SystoleLength

An origami O is represented by its permutation given by *HorizontalPerm(O)* and *VerticalPerm(O)*. We first implement the function *SystoleLength* which, given an origami, computes the length of its systole. We start by computing the minimal cycle length of the permutations. To avoid dealing with square roots later, we also save the squared minimal cycle length.

```
min_cycle_length_horizontal :=
    Minimum(List(MovedPoints(HorizontalPerm(O)), x ->
    CycleLength(HorizontalPerm(O), x)));
```

```
min_cycle_length_vertical :=
    Minimum(List(MovedPoints(VerticalPerm(O)), x ->
    CycleLength(VerticalPerm(O), x)));

min_cycle_length := Minimum(min_cycle_length_horizontal,
    min_cycle_length_vertical);
min_cycle_length_squared := min_cycle_length^2;
```

With this information, we can compute the list S which contains all relevant directions.

```
S := [[1, 0]];

# y > 0
x := -(min_cycle_length - 1);
y := 1;
while x^2 + y^2 <= min_cycle_length_squared do
    while x^2 + y^2 <= min_cycle_length_squared do
        if Gcd(x, y) = 1 then
            Add(S, [x, y]);
        fi;
        y := y + 1;
    od;
    y := 1;
    x := x + 1;
od;
```

As we mentioned at the beginning of the chapter, we immediately want to create the final graph. In the beginning, the graph has no edges but has as many vertices as there are singularities. The graph is represented by an adjacency list, meaning a list of lists, where each list represents a node and contains all outgoing edges. The edges are represented as records of the form

$$\text{rec (node := ..., weight := ...)}$$

where *node* is the other vertex of the edge. We also keep an additional list of all existing edges for Algorithm III.2.4. These edges are again records but contain both vertices u and v explicitly.

$$\text{rec(u := ..., v := ..., weight := ...)}$$

The exact implementation of the graph and the list of edges can be found in the Appendix in the section Graphs in GAP. The additional list of edges is empty in the beginning, too.

```
# create an empty graph with as much nodes as there are
   singularities
graph := CreateGraph(Length(Stratum(O)));

# we keep an extra list of the edges for algorithm 3
edges := [];
```

Stratum is a function provided by the Origami Package which just returns the stratum of an origami represented as a list of numbers that are exactly the multiplicities of the singularities.

Next, we need some way to identify the vertices with the cycles of the commutator. Since a different order of the cycles of the permutation represents the same permutation, the identification is ambiguous, i.e. we can not properly define what the nth cycle of a permutation is. To avoid this problem we convert the permutation into a list of lists, where each inner list represents a cycle. We do this as follows

```
# the argument passed to CyclesToList is the commutator of the
   permutations
cycle_list := CyclesToList(VerticalPerm(O)^-1 *
   HorizontalPerm(O)^-1 * VerticalPerm(O) * HorizontalPerm(O));
```

where *CyclesToList* is a helper function that can be found under the section Helper functions. Now the nth entry of *cycle_list* corresponds to the nth vertex of the graph, i.e. the nth entry of the adjacency list. Note that this is only correct for direction e_1. For any other direction, we need to compute a new cycle list from the original cycle list, by iteratively applying Lemma II.3.1, which we will see in detail later.

So far we only implemented steps of Algorithm III.2.3. Now we need to loop over all the directions in S and add the edges we compute to the graph.

```
# for every direction in S we apply Algorithm 1
for v in S do
   # we are now in this loop
od;
```

Inside this loop, we first calculate the matrix $A \in \mathrm{SL}_2(\mathbb{Z})$. We only do this if $v \neq [1, 0]$.

```
x := v[1];
y := v[2];

A_O := O;
cycle_list_mapped := StructuralCopy(cycle_list);

# if v is the unit vector we do not need to apply a matrix
if not v = [1, 0] then
    # find a matrix A in SL_2(Z) s.t. Av = (1, 0) via the
        extended Eucledian algorithm
    g := Gcdex(x, y);
    A := [[g.coeff1, g.coeff2], [-y, x]];

    # map the singularities from O to the singularities of A.O
        and compute the origami A.O
    mapping_result :=
        MapCyclesListAndOrigami(String(STDecomposition(A)),
        cycle_list_mapped, A_O);

    cycle_list_mapped := mapping_result.cycle_list_mapped;
    A_O := mapping_result.mapped_origami;
fi;
```

A_O denotes the origami obtained by applying the matrix A to O. The variable *cycle_list_mapped* contains the cycles of the commutator of A_O but already in the correct order, i.e. the first entry of *cycle_list_mapped* corresponds to the first singularity/vertex and so on. These two variables are obtained by the function *MapCycleListAndOrigami* which we will discuss now.

This method takes in three parameters. The first parameter is the decomposition of the matrix A into its generators (see II.1). This is done via the function *STDecompositon* provided by the Origami Package. The second parameter is the cycle list corresponding to the cycles of the commutator of the permutations of O. At the beginning *cycle_list_mapped* is just a copy of *cycle_list*, so we can use it as an input for the second parameter. The third and final parameter is the initial origami O. *MapCycleListAndOrigami* then returns a record containing the correctly mapped cycle list, i.e. the bijection between the singularities of O and the singularities of A_O.

The function itself is the following:

```
InstallGlobalFunction(MapCycleListAndOrigami,
   function(wordString, cycle_list, O)
   local i, j, letter, F, word;

   F := FreeGroup("S","T");
   word := ParseRelators(GeneratorsOfGroup(F), wordString)[1];

   # loop over the decompositon of A
   for letter in Reversed(LetterRepAssocWord(word)) do
      if letter = 1 then
         # we apply S to the origami and the singularities
         for i in [1..Length(cycle_list)] do
            Apply(cycle_list[i], j -> j ^
               (VerticalPerm(O)^-1));
         od;
         O := ActionOfS(O);
      elif letter = 2 then
         # only need to apply T to the origami
         O := ActionOfT(O);
      elif letter = -1 then
         # we apply S^-1 to the origami and the singularities
         for i in [1..Length(cycle_list)] do
            Apply(cycle_list[i], j -> j ^
               (HorizontalPerm(O)^-1));
         od;
         O := ActionOfSInv(O);
      elif letter = -2 then
         # only need to apply T to the origami
         O := ActionOfTInv(O);
      else
         Error("<word> must be a word in two generators");
      fi;
   od;

   return rec(cycle_list_mapped := cycle_list, mapped_origami :=
      O);
end);
```

We loop over the different generators appearing in the decomposition of A. Each generator then has its own case, in which we apply it to the origami and the singularities. We do this exactly as described in Lemma II.3.1. In the cases of T and its inverse we only need to apply it to the origami. Note that iteratively applying Lemma II.3.1 to the singularities yields the same commutator as if we had computed it from the origami A_O. The advantage of doing this iteratively is that we are directly computing the bijection as well as we are not potentially changing the order of the cycles.

Before we can finally start with the computation of the edges we first calculate the list L of all squares having a singularity in its lower left corner.

```
L := MovedPoints(VerticalPerm(A_O)^-1 * HorizontalPerm(A_O)^-1 *
    VerticalPerm(A_O) * HorizontalPerm(A_O));
```

The edges are now calculated as described in Algorithm III.2.1.

```
# in this step the edges are computed and added to the graph
for i in L do
    perm := HorizontalPerm(A_O);
    k := 1;
    j := i^perm;

    # find the smallest k such that sigma(i)^k is contained in L
    while not j in L do
        perm := HorizontalPerm(A_O) * perm;
        k := k + 1;
        j := i^perm;
    od;

    length := k * (x^2 + y^2)^0.5;
    # if i is contained in the nth cycle then LookUpIndex returns
        n, same for j
    index_i := LookupIndex(i, cycle_list_mapped);
    index_j := LookupIndex(j, cycle_list_mapped);

    AddEdge(graph, index_i, index_j, length);
    AddSingleEdge(edges, index_i, index_j, length);
od;
```

The only method that really needs an explanation here is *LookUpIndex*.

```
InstallGlobalFunction(LookupIndex, function(a, c)
    local i, j;
```

```
    for i in [1..Length(c)] do
        for j in [1..Length(c[i])] do
            if c[i][j] = a then
                return i;
            fi;
        od;
    od;
    Error("No index found!");
end);
```

This function takes in a singularity a, i.e. the number of the square whose lower left corner is exactly this singularity, and the mapped cycle list c. *LookUpIndex* then just returns the number of the cycle containing the singularity a. We do this for i and j and know to which vertices the edge we just computed belongs. Lastly, we add the edge to the graph and the list of edges via *AddEdge* (for the graph) and *AddSingleEdge* (for the list of edges). The implementation of these functions can be found in the Appendix.

With this step, we are done with the computation of the graph Γ_S. We return from the function with

```
    return MinimalCycle(graph, edges);
```

which calculates the minimal cycle, i.e. the length of the systole, which corresponds to Algorithm III.2.4.

2. Implementation of MinimalCycle

This implementation is rather straightforward. We loop over the list of edges, remove an edge from the graph, update *min_cycle* and add the edge back to the graph. In the end, we are left with a minimal cycle which corresponds to a systole if the combinatorial length is not three.

```
InstallGlobalFunction(MinimalCycle, function(G, E)
    local min_cycle, i, e, distance, combinatorial_length,
        temp_length, result;

    min_cycle := infinity;
    combinatorial_length := -1;
```

```
    for i in [1..Length(E)] do
        distance := 0;
        e := E[i];
        RemoveEdge(G, e.u, e.v, e.weight);

        # edge from a node to itself already is a path with minimum
            distance
        if e.u = e.v then
            # set distance to 0 because the weight is already added
                in the if statement below
            distance := 0;
            temp_length := 1;
        else
            result := ShortestPath(G, e.u, e.v);
            distance := result.length;
            # add one because we removed one edge
            temp_length := result.combinatorial_length + 1;
        fi;

        # update the minimal cycle if necessary
        if Float(min_cycle) > Float(distance + e.weight) then
            min_cycle := distance + e.weight;
            combinatorial_length := temp_length;
        fi;

        AddEdge(G, e.u, e.v, e.weight);
    od;

    # we save the combinatorial length to verify its a systole
    return rec(systole := min_cycle, combinatorial_length :=
        combinatorial_length);
end);
```

Note that we also return the combinatorial length of the edge path to verify if we are dealing with a systole or not. The function *ShortestPath* is just an implementation of Dijkstra's algorithm to find a shortest path between two nodes, which can be found here.

3. Computing Systolic Ratios for Lists of Origamis

So far we have only considered the length of the systole. In some sense, the systolic ratio is more interesting when comparing two origamis. We implement the function *SystolicRatio* by just calling *SystoleLength* and dividing by the degree. As always, we also return the combinatorial length of the systole.

```
InstallGlobalFunction(SystolicRatio, [IsOrigami], function(O)
   local systole_info;

   systole_info := SystoleLength(O);
   return rec(systolic_ratio := (systole_info.systole)^2 /
      DegreeOrigami(O), combinatorial_length :=
      systole_info.combinatorial_length);
end);
```

Consider an arbitrary list of origamis of genus $g \geq 2$. We can find the maximal systolic ratio of this list by using *SystolicRatio* on all origamis. For this we implement the function *MaximalSystolicRatioOfList*.

```
InstallGlobalFunction(MaximalSystolicRatioOfList, function(origamis)
   local max_sr, combinatorial_length, O, result, count,
      three_occured, max_origami;

   max_sr := -1.;
   combinatorial_length := -1;
   max_origami := Origami((),());
   count := 0;
   three_occured := false;

   for O in origamis do
      result := SystolicRatio(O);
      count := count + 1;
      if combinatorial_length = 3 then
         three_occured := true;
      fi;
      if(result.systolic_ratio > max_sr) then
         max_sr := result.systolic_ratio;
         combinatorial_length := result.combinatorial_length;
         max_origami := O;
      fi;
   od;
```

```
    return rec(systolic_ratio := max_sr, combinatorial_length :=
        combinatorial_length, origami := max_origami,
                count := count, three_occured := three_occured);
end);
```

This function additionally returns the first origami found for which the maximum is attained as well as *three_occured*, which is just an indicator if the combinatorial length of three occurred for any origami in the list.

4. Computing Systolic Ratios for fixed Degrees and Strata

Let $d \geq 3$. We are interested in the maximal systolic ratio of all origamis with degree d. For sufficiently small d this can easily be computed as follows: We loop over all possible permutations in S_d and check if the corresponding origami is connected. This is done via the function *AllOrigamisByDegree* provided by the Origami Package. Then we put all these origamis in a list, remove any with genus $g \leq 1$, and run *MaximalSystolicRatioOfList*.

```
InstallGlobalFunction(MaximalSystolicRatioByDegree, function(d)
    local origamis;

    origamis := Filtered(AllOrigamisByDegree(d), o -> Genus(o) >= 2);
    return MaximalSystolicRatioOfList(origamis);
end);
```

We can additionally fix a stratum $H(k_1, \ldots, k_n)$ and consider all origamis from *deg1* to *deg2*. Here *AllOrigamisInStratum* again is a function from the Origami Package.

```
InstallGlobalFunction(MaximalSystolicRatioInStratum, function(deg1,
    deg2, stratum)
    local max_sr, combinatorial_length, deg, origamis, result,
        max_origami, count, three_occured;

    max_sr := -1.;
    combinatorial_length := -1;
    max_origami := Origami((),());
```

```
    count := 0;
    three_occured := false;

    for deg in [deg1..deg2] do
        origamis := AllOrigamisInStratum(deg, stratum);
        result := MaximalSystolicRatioOfList(origamis);
        count := count + result.count;

        if result.three_occured then
            three_occured := true;
        fi;

        if result.systolic_ratio > max_sr then
            max_sr := result.systolic_ratio;
            combinatorial_length := result.combinatorial_length;
            max_origami := result.origami;
        fi;
    od;

    return rec(systolic_ratio := max_sr, combinatorial_length :=
        combinatorial_length, origami := max_origami, count := count,
        three_occured := three_occured);
end);
```

Chapter V.

Computational Results

In this chapter, we examine the systolic ratio for certain families of origamis using the implementations from the previous chapter and the appendix. We already mentioned in Chapter III, that for each origami up to degree 10, there always is a systole with combinatorial length not equal to three. This can be verified computationally with the implementation from the previous chapter. In the table below, we list the maximal systolic ratio for each degree up to degree 10, as well as a stratum in which the maximum is attained. The column *count* shows, how many origamis of degree *deg* with genus $g \geq 2$ exist.

deg	sys	SR	count	stratum
3	1	~ 0.33	3	[2]
4	$\sqrt{2}$	0.5	19	[1, 1]
5	$\sqrt{2}$	0.4	91	[1, 1]
6	$\sqrt{2}$	~ 0.33	612	[1, 1]
7	$\sqrt{2}$	~ 0.29	4155	[1, 1]
8	2	0.5	34455	[1, 1]
9	2	~ 0.44	314480	[1, 1]
10	$\sqrt{2}$	0.4	3202821	[1, 1]

Table V.1.: The maximal systolic ratio of all origamis up to degree 10.

While the stratum $H(1,1)$ is not the only stratum in which the maximum is attained for degree 4 to 10, the results suggest that the stratum $H(1,1)$ is a good candidate for maximal systolic ratios. This is not too surprising, as there are reasons to believe that the translation surface with maximal SR in $H(1,1)$ ([5], Conjecture 1.2) is not an origami but can be approximated by origamis in $H(1,1)$. Furthermore, it is assumed that this surface achieves the maximal systolic ratio under all translation surfaces.

In general, the *principal strata* $H(1,\dots,1)$ are of particular interest, as they have maximal dimension and lie dense in the space of translation surfaces of genus g. It is also expected that the maximal systolic ratio of translation surfaces of genus g is achieved in the principal strata.

1. Systoles for two Families of Normal Origamis

We have seen that we can choose an arbitrary finite group G with 2 generators r, s to construct an origami $O_{(G,r,s)}$ such that $\mathrm{Deck}(O_{(G,r,s)}) = G$. In this section we consider origamis coming from the dihedral group D_n and the group $G_{(n,k)}$ (Example II.2.4) for $n \geq 3$. Since we are particularly interested in the principal strata, we will restrict to $k = 1$ (see Theorem II.2.5).

For $3 \leq n \leq 9$ and generators r, s the origamis $O_{(G_{(n,1)},r,s)}$, coming from the group

$$G_{(n,1)} := \langle r, s \mid r^4 = s^{2^{n-2}} = 1, s^{-1}rs = r^{-1}\rangle,$$

have the following systoles:

n	stratum	deg	CL	sys	SR
3	$[4 \times 1]$	8	2	2	0.5
4	$[8 \times 1]$	16	2	2	0.25
5	$[16 \times 1]$	32	2	2	0.125
6	$[32 \times 1]$	64	2	2	0.0625
7	$[64 \times 1]$	128	2	2	0.03125
8	$[128 \times 1]$	256	2	2	0.015625
9	$[256 \times 1]$	512	2	2	0.0078125

Table V.2.: Systolic ratios of the origamis $O_{(G_{(n,1)},r,s)}$. CL denotes the combinatorial length of the systole.

It can be observed, that each origami has a systole of length 2 and of combinatorial length 2 (see e.g. Figure II.2) and that the systolic ratio tends to 0. This observation can be generalized for all $n \geq 3$.

Proposition V.1.1: *Let $G_{(n,1)}$ be the 2-group as given in Example II.2.4 with $k = 1$. Then for all $n \geq 3$ we have sys$(O_{(G_{(n,1)},r,s)}) = 2$.*

Proof: A presentation of $G_{(n,1)}$ is given by

$$\langle r, s \mid r^4 = s^{2^{n-2}} = 1, s^{-1}rs = r^{-1} \rangle.$$

Since $r^{-1} = s^{-1}rs$, we have $s = rsr$.

The commutator $[\sigma_x, \sigma_y]$, where σ_x, σ_y are the horizontal and vertical permutations, respectively, defines a walk in the directions down, left, up and right.

Assume we start in an arbitrary square i. Since O is in correspondence with the Cayley graph of $G_{(n,1)}$ with the generators r and s, we have that the square i corresponds to some group element $g \in G_{(n,1)}$. So a walk on O starting at i given by the commutator can also be interpreted as a walk on the Cayley graph given by $s^{-1}r^{-1}sr$ starting at g and ending in $gs^{-1}r^{-1}sr$.

One observes the following:

$$\begin{aligned} s^{-1}r^{-1}sr &\overset{s=rsr}{=} s^{-1}r^{-1}(rsr)r \\ &= s^{-1}(r^{-1}r)sr^2 \\ &= (s^{-1}s)r^2 \\ &= r^2 \end{aligned}$$

Hence, the walk defined by the commutator ends in the same square as the walk defined by walking right, twice.

Moreover, observe that O has 2^{n-1} many singularities (see Remark II.2.6) and that the group $G_{(n,1)}$ has order 2^n. Additionally, O is normal and thus, for a fixed singularity $[i]$, there are exactly

$$\frac{\deg(O)}{\#\Sigma_O} = \frac{2^n}{2^{n-1}} = 2$$

squares whose lower left corner is the singularity $[i]$. Since the cycles of the commutator $[\sigma_x, \sigma_y]$ are in correspondence with the singularities, the commutator must consist of transpositions and thus, for every square labeled with the group element g, the square labeled with gr^2 has the same singularity as its lower left corner.

Note that a square can not have the same singularity more than once. To see this, firstly assume that the square i with the singularity $[i]$ and corresponding group element a has the same singularity in the upper left corner, too. Assume that the upper edge of the square i is glued to the lower edge of the square i, i.e., $as = a$. Then $s = 1$, a contradiction. Therefore, the second square with the singularity $[i]$ must coincide with the square labeled with ar^2, i.e. $ar^2 = as$. But then $r^2 = s$, a contradiction. Hence, the singularity in the lower left corner of the square i can not also be in the upper left corner of the same square.

Now assume that the square i has the same singularity in the upper right corner. Then with a similar case distinction, we have that $asr = a \implies s = r^{-1}$ or $asr = ar^2 \implies sr = r^2 \implies s = r$, both contradictions.

Lastly, assume that i has the same singularity in its lower right corner. Since $1 \neq r$ and $r \neq r^2$, this is a contradiction, too.

Combining these facts we get that, starting in a square i with singularity $[i]$, we need to walk horizontally twice to reach the other unique square j with the same singularity $[i]$. Since the commutator can not be any further simplified, this is the minimum distance to walk from i to j. This gives a systole of the surface by Theorem III.1.3 because the combinatorial length is 2. □

Corollary V.1.2: *Define* $O_n := O_{(G_{(n,1)},r,s)}$. *Then* $\lim_{n\to\infty} SR(O_n) = 0$.

Proof: $G_{(n,1)}$ is a group of order 2^n and we have that $\mathrm{sys}(O_n) = 2$. Thus,

$$\lim_{n\to\infty} \mathrm{SR}(O_n) = \lim_{n\to\infty} \frac{\mathrm{sys}(O_n)^2}{\mathrm{area}(O_n)} = \lim_{n\to\infty} \frac{4}{2^n} = \lim_{n\to\infty} \frac{1}{2^{n-2}} = 0. \qquad \square$$

Now we want to examine the systoles of the origamis $O_{(D_n,r,s)}$ coming from the dihedral group

$$D_n := \langle r, s \mid r^n = s^2 = (rs)^2 = 1\rangle.$$

Remark V.1.3: Consider the origami given by the following permutations:

$$\sigma_x = (1, \dots, n)(n + 1, \dots, 2n),$$
$$\sigma_y = (n, n + 1)(n - 1, n + 2) \dots (2, 2n - 1)(1, 2n).$$

Then we have $\sigma_x^n = 1$, $\sigma_y^2 = 1$ and

$$(\sigma_x\sigma_y)^2 = (1, 2n - 1)(2, 2n - 2), \dots , (n - 1, n + 1)(n, 2n) = 1$$

because $(\sigma_x\sigma_y)^2$ consists of transpositions. Since this origami respects the relations of D_n and has degree $2n$, this origami is exactly $O_{(D_n,r,s)}$. Figure V.1 shows the origami $O_{(D_n,r,s)}$ for $n = 4$ and $n = 5$.

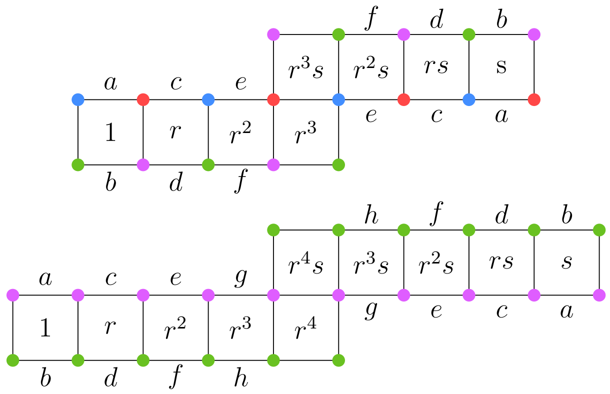

Figure V.1.: The origamis $O_{(D_4,r,s)}$ and $O_{(D_5,r,s)}$. Edges without a label are glued together with their opposite edge.

Below, we list a table containing the computed systoles of $O_{(D_n,r,s)}$ for $3 \leq n \leq 12$.

n	stratum	deg	CL	sys	SR
3	$[2,2]$	6	1	1	0.1667
4	$[1,1,1,1]$	8	2	2	0.5
5	$[4,4]$	10	1	1	0.1
6	$[2,2,2,2]$	12	2	2	0.3333
7	$[6,6]$	14	1	1	0.0714
8	$[3,3,3,3]$	16	2	2	0.25
9	$[8,8]$	18	1	1	0.0556
10	$[4,4,4,4]$	20	2	2	0.2
11	$[10,10]$	22	1	1	0.0455
12	$[5,5,5,5]$	24	2	2	0.1667

Table V.3.: Systolic ratios of the origamis $O_{(D_n,r,s)}$.

If we separate the table w.r.t. the parity of n, we can observe two distinct patterns for the systoles and the strata of $O_{(D_n,r,s)}$.

Proposition V.1.4: *Let $n \geq 3$ and let $O_{(D_n,r,s)}$ be the origami coming from the dihedral group D_n. Then we have*

$$O_{(D_n,r,s)} \in \begin{cases} H(\frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}-1), n \text{ even} \\ H(n-1, n-1), n \text{ odd} \end{cases}$$

and

$$sys(O_{(D_n,r,s)}) = \begin{cases} 2, n \ \mathit{even} \\ 1, n \ \mathit{odd} \end{cases}$$

Proof: Since $(rs)^2 = 1 \implies rs = s^{-1}r^{-1}$ and $s^2 = 1 \implies s = s^{-1}$ we have that

$$[s^{-1}, r^{-1}] = s^{-1}r^{-1}sr = rssr = rss^{-1}r = r^2$$

For the order of the commutator $[s^{-1}, r^{-1}]$ we have

$$\mathrm{ord}([s^{-1}, r^{-1}]) = \mathrm{ord}(r^2) = \begin{cases} n, n \text{ odd} \\ \frac{n}{2}, n \text{ even} \end{cases} .$$

because $\mathrm{ord}(r) = n$.

For normal origamis $O_{(G,r,s)}$, $\mathrm{ord}([s^{-1}, r^{-1}])$ coincides with the multiplicity of the singularities. Furthermore, we can directly compute the number of singularities.

$$\#\Sigma = \frac{\#G}{ord([s^{-1}, r^{-1}])}.$$

Thus, for n odd we have

$$\frac{2n}{n} = 2$$

singularities with multiplicity n. Hence $O_{(D_n,r,s)} \in H(n-1, n-1)$.

For n even we have

$$\frac{2n}{\frac{n}{2}} = \frac{4n}{n} = 4$$

singularities with multiplicity $\frac{n}{2}$ and therefore $O_{(D_n,r,s)} \in H(\frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}-1)$.

To determine the length of the systole of $O_{(D_n,r,s)}$ we again make a case distinction w.r.t. the parity of n.

Let n be odd. Observe that

$$([s^{-1}, r^{-1}])^{\frac{n+1}{2}} = (r^2)^{\frac{n+1}{2}} = r^{n+1} = r.$$

This implies that the squares labeled with the group elements 1 and r share the same singularity as their lower left corner. This gives a closed saddle connection of length 1. Since there can be no shorter saddle connection and the combinatorial length is 1, the length of the systole is 1.

Finally, let n be even. Since $[s^{-1}, r^{-1}] = r^2$, the squares labeled by 1 and r^2 have the same singularity as their lower left corner. This gives a closed, not null-homotopic geodesic of length 2. Assume that there is a shorter closed saddle connection, i.e. $([s^{-1}, r^{-1}])^k \in \{r, s, rs\}$ for some $k \in \mathbb{N}$.

Assume that $r^{2k} = r$. Then $r^{2k-1} = 1$ which can not be true because the the order of r is even and $2k - 1$ is odd.

Now assume that $r^{2k} = s$ or that $r^{2k} = rs$. Every element of D_n can be written as a unique product $r^k s^l$ with $k \in \{0, \dots, n-1\}$ and $l \in \{0, 1\}$. But we already have $s = s^1$, a contradiction.

Hence, the length of the systole of $O_{(D_n,r,s)}$ is 2. □

Corollary V.1.5: *Let $n \geq 3$. Then*

$$SR(O_{(D_n,r,s)}) = \begin{cases} \frac{2}{n}, n \ \textit{even} \\ \frac{1}{2n}, n \ \textit{odd.} \end{cases}$$

Corollary V.1.6: *Let $k \geq 2$. Then*

$$\lim_{k \to \infty} SR(O_{(D_{2k},r,s)}) = 0$$

and

$$\lim_{k \to \infty} SR(O_{(D_{2k-1},r,s)}) = 0.$$

2. Cyclic Covers of the $(n \times n)$-torus

In this section, we study a special class of origamis introduced in [11] that come from cyclic covers of the $(n \times n)$-torus.

Definition V.2.1: Let $n \in \mathbb{N}$. The *$(n \times n)$-torus T_n* is the set

$$(\mathbb{R}^2 \setminus \mathbb{Z}^2)/(n\,\mathbb{Z})^2.$$

Hence, the $(n \times n)$-torus can be seen as the set of orbits of the group action of $(n\,\mathbb{Z})^2$ on $\mathbb{R}^2 \setminus \mathbb{Z}^2$ defined by translation. Note that the fundamental group of T_n is isomorphic to the free group F_{n^2+1} (see [11], Satz I.2.11).

In [11] Rogovskyy studies normal coverings $p : X \to T_n$ that are cyclic with deck transformation group $\mathbb{Z}/d\,\mathbb{Z}$. These coverings are entirely determined by their monodromy $m : \pi_1(T_n) \to \mathbb{Z}/d\,\mathbb{Z}$ (see [11], Satz I.4.2).

Equivalently one can choose a basis B of the fundamental group $\pi_1(T_n)$ and only consider the image of every basis element in $\mathbb{Z}/d\mathbb{Z}$. This yields a vector in $(\mathbb{Z}/d\mathbb{Z})^N$, where $N = \operatorname{rank}(\pi_1(T_n)) = n^2 + 1$.

We will consider the following bases, as given by Rogovskyy:

Definition V.2.2: In the proof that $\pi_1(T_n) \cong F_{n^2+1}$ (see [11], Satz I.2.11) Rogovskyy uses the graph in Figure V.2 which is a deformation retract of T_n. Every edge of the graph not contained in the spanning tree passes through

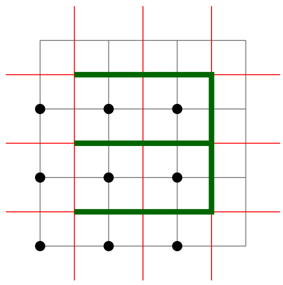

Figure V.2.: A graph (red lines) that is a deformation retract of T_n. The green lines form a spanning tree. This figure was provided by Alexander Rogovskyy.

exactly one edge of the lattice given by the $(n \times n)$-torus T_n (see Figure V.3). We call these edges of the lattice *slits* and denote them by $s_i, 1 \leq i \leq n^2 + 1$. Every slit induces a generator of the fundamental group $\pi_1(T_n)$ as follows:

Fix the base point $\times = (\frac{1}{2}, \frac{1}{2})$. The generator corresponding to s_i consists of a path on the spanning tree starting in $\times$, passing the slit s_i, and going back to the base point via the spanning tree. Figure V.3 shows the generator induced by s_8.

We set $N = \operatorname{rank}(\pi_1(T_n)) = n^2 + 1$ and call the basis $S := \{s_1, \ldots, s_N\}$ the *slit basis*.

Definition V.2.3: We again fix the base point $(\frac{1}{2}, \frac{1}{2})$. Moreover, we define the following paths (see Figure V.4):

- a is the horizontal path passing through the base point and going to the right.
- b is the vertical path passing through the base point and going upwards.
- For $(x, y) \in (\mathbb{Z}/n\mathbb{Z})^2$ the paths $l_{(x,y)}$ are defined as loops going counter-clockwise around the points (x, y).

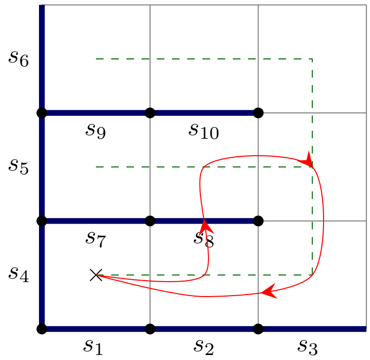

Figure V.3.: Visualization of the slit basis S. This figure was provided by Alexander Rogovskyy.

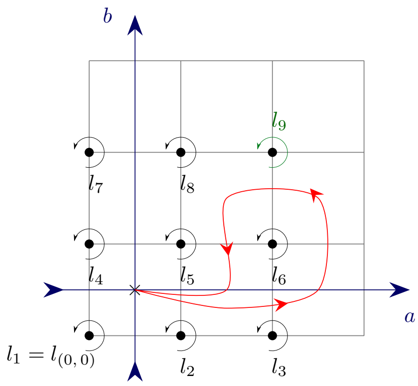

Figure V.4.: Visualization of the loop basis L. This figure was provided by Alexander Rogovskyy.

Fix a path starting at the base point, heading to the square with mid point $(x - \frac{1}{2}, y - \frac{1}{2})$. Then walk through the 4 squares around the point (x, y) and then back to the basepoint via the fixed path. Figure V.4 shows a possible choice for the loop l_6.

We enumerate the loops $l_{(x,y)}$ starting at 1 from the left to the right and from the bottom to the top (see Figure V.4).

These paths form a basis $L := \{a, b, l_1, \ldots, l_{n^2-1}\}$ called the *loop basis*. Note that the loop l_{n^2} does not belong to the basis because it can be expressed in terms of the elements of L (see [11], Bemerkung I.2.14).

Change of basis matrices can be found in [11] in the section *Bestimmung der Basiswechselmatrix* D_{BS}. Note that Rogovskyy denotes the slit basis with B and the loop basis with S.

Remark V.2.4: The surface X of a normal, cyclic covering $p : X \to T_n$ with deck transformation group $\mathbb{Z}/d\mathbb{Z}$ can be visualized as follows:

Let $v \in (\mathbb{Z}/d\mathbb{Z})^N$ be the vector representing the covering w.r.t. the slit basis. Take d copies of T_n where opposing edges are glued together. Moreover, glue these copies together such that passing through a slit s_i changes v_i-many copies of T_n. Then X is the resulting surface.

The cyclic covers of T_n introduced above naturally define origamis, as described in the following. Let us denote the cyclic torus cover by $p : X \to T_n$. This covering has degree d, where $\mathbb{Z}/d\mathbb{Z}$ is the deck group of p. Moreover, let $q : T_n \to T$ be the normal covering defined by sending each square of T_n onto one square. This covering has degree n^2. Then the composition of the coverings

$$o : X \xrightarrow{p} T_n \xrightarrow{q} T$$

defines an origami of degree dn^2. In general, the resulting origami is not normal. We denote by O_d^n the set of origamis induced by the coverings $p : X \to T_n$ up to isomorphism. Figure V.5 shows an example of an origami in O_2^5.

For d prime, Rogovskyy gives the following classification for the strata of o (see [11], Korollar IV.2.4).

Proposition V.2.5: *Let d be prime and $o \in O_d^n$. Moreover, let v be a vector representing o w.r.t. the loop basis. Then we have*

$$o \in H(\underbrace{d-1, \dots, d-1}_{k \text{ times}}),$$

where k denotes the number of non-zero elements in $(v_3, v_4, \dots, v_N, \sum_{i=3}^{N} v_i)$.

In this section, we restrict to the principal stratum $H(1,1)$, i.e. all origamis in O_2^n constructed as follows:

Let $n \geq 2$. Moreover, let $a, b, l_1, \dots, l_{n^2-1} \in \mathbb{Z}/2\mathbb{Z}$ such that

$$\#\{i \in \{1, \dots, n^2-1\} \mid l_i \neq 0\} \subset \{1, 2\}$$

Then the origamis induced by the monodromy vectors (w.r.t. the loop basis)

$$(a, b, l_1, \dots, l_{n^2-1})$$

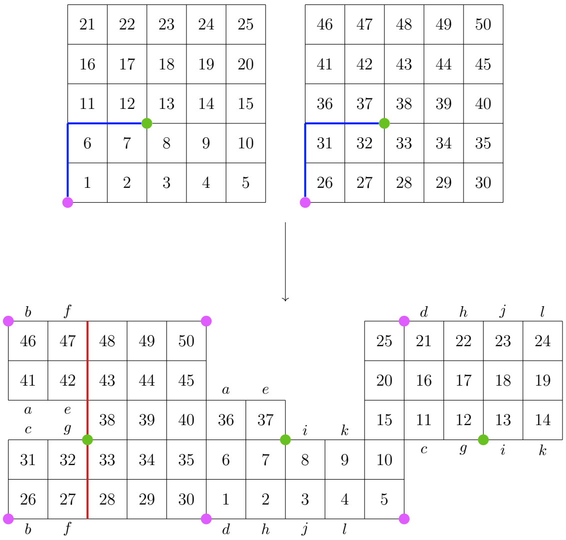

Figure V.5.: An origami in O_2^5 induced by the monodromy vector $v \in (\mathbb{Z}/d\mathbb{Z})^{26}$, where $v_6 = v_7 = v_{15} = v_{16} = 1, v_i = 0$ for $1 \leq i \leq 26, i \neq 6, 7, 15, 16$, w.r.t. the slit basis. The red line is a systole.

lie in the stratum $H(1, 1)$.

The table below shows the computed systolic ratios for the stratum $H(1, 1)$ up to $n = 12$, where *count* denotes how many cyclic covers for fixed n exist.

n	deg	sys	SR	count
2	8	2	0.5	24
3	18	$\sqrt{8}$	~ 0.44	144
4	32	4	0.5	480
5	50	5	0.5	1200
6	72	6	0.5	2520
7	98	7	0.5	4704
8	128	8	0.5	8064
9	162	9	0.5	12960
10	200	10	0.5	19800
11	242	11	0.5	29040
12	288	12	0.5	41184

Table V.4.: Systolic ratios of origamis in the stratum $H(1, 1)$ induced by cyclic covers of the $(n \times n)$-torus.

These computational results inspire the following theorem:

Theorem V.2.6: *Let $n \neq 3$. Then the maximal systolic ratio of all cyclic torus covers in $H(1,1)$ is $\frac{1}{2}$. The maximum is achieved at least once for every $n \neq 3$. If $n = 3$, then the maximal systolic ratio of a cyclic torus cover in $H(1,1)$ is $\frac{4}{9}$.*

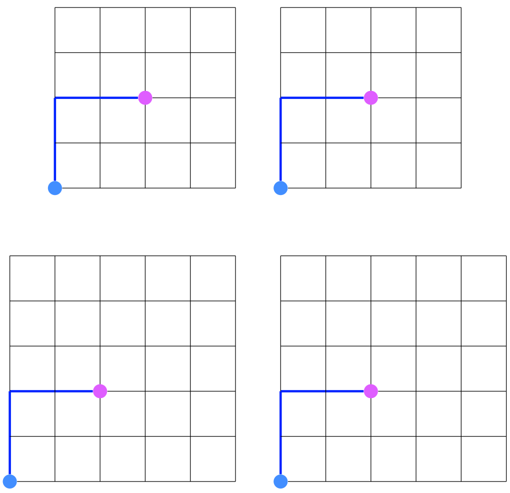

Figure V.6.: Cyclic covers in $H(1,1)$ that achieve the systolic ratio $\frac{1}{2}$. At the top: Origami of degree 32; At the bottom: Origami of degree 50. The copies of T_n are glued together along the blue slits.

Proof: Let $p : X \to T_n$ be a cyclic torus cover of degree 2 with 2 branch points. We first show that $\text{sys}(X) \leq n$. Let s_1 and s_2 be the singularities of X and $p(s_1), p(s_2)$ be their images on T_n. Let $\bar{c}$ be one of the two horizontal paths of length n that start in s_1. The image $p(\bar{c})$ of $\bar{c}$ under p is a closed horizontal path on T_n. Since $\#(p^{-1}(s_1)) = 1$, we have that $\bar{c}$ has to be a closed path, too. Moreover $p(\bar{c})$ is a generator of the fundamental group $\pi_1(T_n)$ and thus not null-homotopic. Hence, $\bar{c}$ is not null-homotopic and there is always a closed, not null-homotopic geodesic of length n, i.e. $\text{sys}(X) \leq n$.

In the following we present for every $n \in \mathbb{N}, n \neq 3$ a cyclic covering with a systole of length n. For n even, we choose a cyclic cover $p : X \to T_n$ with branch points $(0,0)$ and $(\frac{n}{2}, \frac{n}{2})$ (right image of Figure V.6). For n odd, we choose a cyclic cover $q : Y \to T_n$ with branch points $(0,0)$ and $(\frac{n-1}{2}, \frac{n-1}{2})$ (left image of Figure V.6). In both cases, the argument from above ensures the existence of a not null-homotopic closed geodesic of length n.

We need to show that there is no shorter closed path. We first consider the case n even. Let $s_1 = p^{-1}((0,0))$ and $s_2 = p^{-1}((\frac{n}{2}, \frac{n}{2}))$. Assume that c is a systole of X and $l(c) < n$. We can choose c such that it is a concatenation of saddle connections. Let $\bar{c} = p(c)$. Then we have $l(c) \geq l(\bar{c})$ and that $\bar{c}$ is a

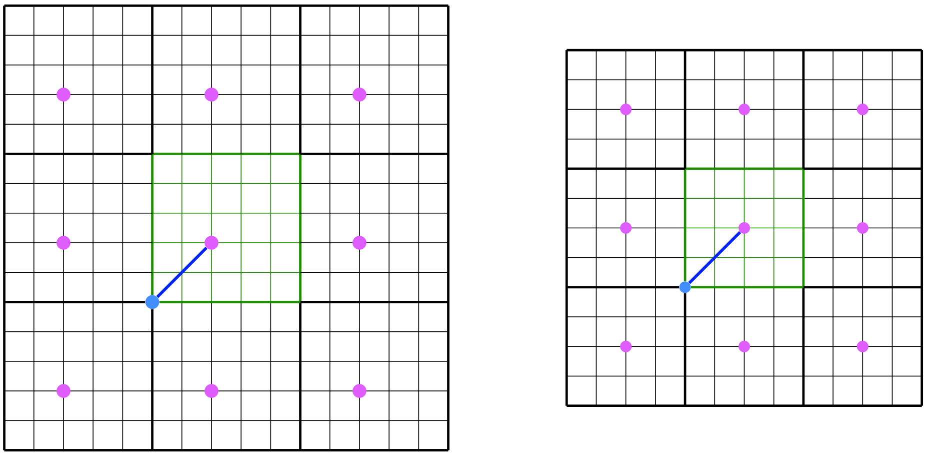

Figure V.7.: Sections of the universal covers of $\overline{T_n}$ where n is odd (left image) and n is even (right image).

concatenation of geodesic segments with endpoints in $\{p(s_1), p(s_2)\}$. Now if $\bar{c}$ contains a closed segment then $l(\bar{c}) \geq n$ because the shortest closed geodesic on T_n is the horizontal/vertical path.

Hence, assume that $\bar{c}$ consists of at least two segments whose distinct starting points and endpoints lie in $\{p(s_1), p(s_2)\}$. Then $l(\bar{c}) \geq 2 \cdot d(p(s_1), p(s_2))$. We need to show that $d(p(s_1), p(s_2)) \geq \frac{n}{2}$.

The right image of Figure V.7 shows a part of the universal cover of $\overline{T_n}$. A geodesic segment from $p(s_1)$ to $p(s_2)$ lifts to a Euclidean segment h which connects a purple and a blue point in the Euclidean plane. The shortest possible path connecting the singularities has a length of

$$\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right)^2} = \frac{\sqrt{2}}{2}n.$$

We have that $\frac{\sqrt{2}}{2}n > \frac{n}{2}$ for $n > 0$ because this is equivalent to $\sqrt{2}n > n$ which is obviously true for $n > 0$. Therefore, $d(p(s_1), p(s_2)) > \frac{n}{2}$.

Now, we proceed similarly for the surface Y in the case that n is odd. Let $s_1 = q^{-1}((0,0))$ and $s_2 = q^{-1}((\frac{n-1}{2}, \frac{n-1}{2}))$. Assume that c is a systole on Y with length less than n which is a concatenation of saddle connections. We set $\bar{c} = q(c)$. With the same reasoning, we have that either $l(\bar{c}) \geq n$ or that $\bar{c}$ consists of at least two segments whose distinct starting points and endpoints lie in $\{q(s_1), q(s_2)\}$. The left image of Figure V.7 again shows the universal cover of $\overline{T_n}$. A geodesic segment connecting $q(s_1)$ and $q(s_2)$ also lifts to a

Euclidean segment h connecting a blue point and a purple point. The shortest segment connecting a blue and a purple point has a length of

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2} = \frac{\sqrt{2}}{2}(n-1).$$

We need to show that for $n > 3$

$$\frac{\sqrt{2}}{2}(n-1) > \frac{n}{2}.$$

We prove this via induction. For $n = 4$ we have $\frac{\sqrt{2}}{2}(4-1) = \frac{3}{2}\sqrt{2} > 2 = \frac{4}{2}$. For $n \to n+1$ we have

$$\begin{aligned}\frac{\sqrt{2}}{2}((n+1)-1) &= \frac{\sqrt{2}}{2}n \\ &= \frac{\sqrt{2}}{2}(n-1) + \frac{\sqrt{2}}{2} \\ &> \frac{n}{2} + \frac{\sqrt{2}}{2} \\ &= \frac{n+\sqrt{2}}{2} \\ &> \frac{n+1}{2}.\end{aligned}$$

Hence, $d(q(s_1), q(s_2)) > \frac{n}{2}$ and therefore, for $n \neq 3$, there always exists a cyclic torus cover with a systole of length n. Since the covering $X \to T_n \to T$ has degree $2n^2$ we get a systolic ratio of $\frac{1}{2}$.

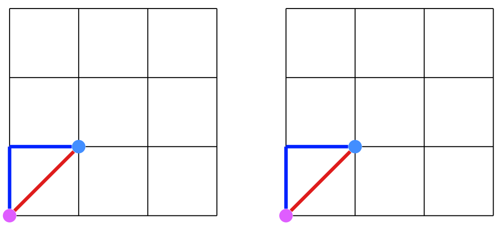

Figure V.8.: An example of a cyclic cover in $H(1,1)$ for $n = 3$. The red line is a systole.

Now let $n = 3$. Consider a cyclic cover $p : X \to T_n$ of degree 2 with 2 branch points. W.l.o.g. fix the image of a singularity at $(0,0)$. We then have the following possibilities for the second singularity:

- The second image of the singularity is on an edge of the 3×3 squares. Then we have a closed geodesic on X of length 2.
- The second image of the singularity is on one of the four inner points. Then we have a closed geodesic on X of length $2\sqrt{2}$.

In both cases, the combinatorial length is not equal to 3. Theorem III.1.3 ensures that these closed geodesics are not null-homotopic. This implies that the maximal length of the systole is $2\sqrt{2}$.

Finally, let $q : Y \to T_n$ be a cyclic cover as shown in Figure V.8. We see that the shortest closed geodesic has a length of $2\sqrt{2}$. We call it γ. In this case, we can even waive Theorem III.1.3 and show that γ is not null-homotopic with the following argument. Consider the surface $Y \setminus \gamma$. Both squares are still connected and for each point x in a square we can find a path starting in x and passing a slit. Hence, $Y \setminus \gamma$ is still connected and as a consequence γ is not null-homotopic. This gives a systolic ratio of $\frac{4}{9}$. $\square$

We end this section with a short prospect about systoles of cyclic torus covers in other principal strata.

Let $p : X \to T_n$ be a cyclic cover of degree 2 with $2k$ branch points. Let $s_1, \dots, s_{2k}$ be the singularities of X and $p(s_1), \dots, p(s_{2k})$ the branch points on T_n. Then the systoles on X can either be a lift of a horizontal/vertical, closed curve on T_n that starts in a singularity or a lift of the shortest path connecting two branch points and of the path back in the other copy such that the resulting combined path is not null-homotopic on X. In either case, Theorem III.1.3 applies.

To simplify notation, we assume that the length of the square T_n is 1. In the first case we have $\operatorname{sys}(X) = 1$. In the other case let

$$\operatorname{mindist}(S) := \min\{d(x_1, x_2) \mid x_1 \neq x_2, x_1, x_2 \in S\}.$$

Then

$$\operatorname{sys}(X) = 2 \cdot \operatorname{mindist}(p(s_1), \dots, p(s_{2k})).$$

Hence for all cyclic covers of the $(n \times n)$-torus X in $H(\underbrace{1, \dots, 1}_{2k})$ the maximal systole is

$$\operatorname{sys}\operatorname{Max}_k = \min\{1, R\}$$

where

$$R := \max\{2 \cdot \operatorname{mindist}(S) \mid S \subset \overline{T_n} \setminus T_n, \#S = 2k\}.$$

Hence, the maximum systolic ratio of cyclic torus covers in $H(\underbrace{1,\dots,1}_{2k})$ is

$$\frac{(\text{sys Max}_k)^2}{2}$$

Note that if S consists of irrational points, then the corresponding surface is no origami. We can improve the systolic ratio by considering the hexagonal torus, i.e. instead of a square we consider the parallelogram formed by two equilateral triangles.

Example V.2.7: Let $2k = n^2$. Distribute the points $x_1, \dots, x_{2k}$ on the n^2 vertices of the squares.

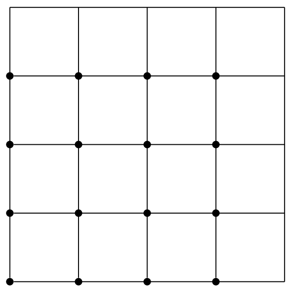

Figure V.9.: An example of the distribution for $k = 8$ and $n = 4$.

Then $\text{mindist}(x_1, \dots, x_{2k}) = \frac{1}{n} = \frac{1}{\sqrt{2k}}$. Hence,

$$\begin{aligned} \text{sys}(X) &= 2 \cdot \frac{1}{\sqrt{2k}} \\ \implies \text{SR}(X) &= \frac{\frac{2}{k}}{2} = \frac{1}{k}. \end{aligned}$$

This gives a linear lower bound for maximal systoles in the principal strata.

Appendix

Graphs in GAP

CreateGraph creates a graph with i nodes and no edges.

```
InstallGlobalFunction(CreateGraph, function(i)
   local G;
   G := [];
   for i in [1..i] do
      Add(G, []);
   od;
   return G;
end);
```

AddEdge adds an undirected edge from a to b with weight w to the graph G.

```
InstallGlobalFunction(AddEdge, function(G, a, b, w)
   Add(G[a], rec(node := b, weight := w));
   Add(G[b], rec(node := a, weight := w));
end);
```

RemoveEdge removes the edge from a to b with weight w from the graph G.

```
InstallGlobalFunction(RemoveEdge, function(G, a, b, w)
   local j, k;

   Remove(G[a], Position(G[a], rec(node := b, weight := w)));
   Remove(G[b], Position(G[b], rec(node := a, weight := w)));
end);
```

ShortestPath returns a shortest path from *src* to *dst* in the graph G.

```
InstallGlobalFunction(ShortestPath, function(G, src, dest)
```

```
    local pq, dist, p, u, i, weight, v, prev, S, node;
    pq := [];
    dist := [];
    prev := [];

    for p in [1..Length(G)] do
        Add(dist, infinity);
        Add(prev, -1);
    od;

    PQueue_push(pq, rec(first := 0., second := src));
    dist[src] := 0;

    while(not IsEmpty(pq)) do
        u := PQueue_pop(pq).second;

        for i in G[u] do
            v := i.node;
            weight := i.weight;

            if (Float(dist[v]) > Float(dist[u] + weight)) then
                dist[v] := dist[u] + weight;
                PQueue_push(pq, rec(first := dist[v], second := v));
                prev[v] := u;
            fi;
        od;
    od;

    S := [];
    node := dest;
    if not (prev[node] = -1) or node = src then
        while not (prev[node] = -1) do
            Add(S, node, 1);
            node := prev[node];
        od;
    fi;

    return rec(length := dist[dest], combinatorial_length :=
        Length(S));
end);
```

AddSingleEdge adds an edge from a to b with weight w to a list of edges E.

```
InstallGlobalFunction(AddSingleEdge, function(E, a, b, w)
    Add(E, rec(u := a, v := b, weight := w));
end);
```

Priority Queue in GAP

We implement a priority queue w.r.t. the lexographical order in GAP for the shortest path algorithm. *PQueue_push* adds an element to the queue, *PQueue_top* returns the first element in the queue and *PQueue_pop* returns the first element in the queue and removes it.

```
InstallGlobalFunction(PQueue_push, function(q, elem)
    local low, high, mid;

    low := 1;
    high := Length(q);

    while low < high do
        mid := Int((low + high) * 0.5);
        if q[mid] < elem then
            low := mid + 1;
        else
            high := mid;
        fi;
    od;

    Add(q, elem, low);
end);
```

```
InstallGlobalFunction(PQueue_top, function(q)
    return q[1];
end);
```

```
InstallGlobalFunction(PQueue_pop, function(q)
    local temp;

    temp := q[1];
    Remove(q, 1);
```

```
    return temp;
end);
```

Helper functions

CyclesToList takes in a permutation *sigma* and returns a list of lists, where each list corresponds to one cycle of *sigma*.

```
InstallGlobalFunction(CyclesToList, function(sigma)
    local L, k, i, cycle_list, curr, temp;

    L := List(MovedPoints(sigma));
    cycle_list := [];
    curr := 1;

    while(Length(L) > 0) do
        k := 1;
        i := L[1];

        Add(cycle_list, []);
        Add(cycle_list[curr], i);
        while(not (i ^ (sigma ^ k) = i)) do
            temp := i ^ (sigma ^ k);
            Remove(L, Position(L, temp));
            Add(cycle_list[curr], temp);
            k := k + 1;
        od;

        Remove(L, Position(L, i));
        curr := curr + 1;
    od;

    return cycle_list;
end);
```

ArrayWithOnes returns a list with all possibilities arranging $2k$ or $2k-1$ ones in an array of length n.

```
InstallGlobalFunction(ArrayWithOnes, function(k, n)
    local result, i, j, temp, combination;
```

```
    result := [[0], [1]];
    if n = 1 then
        return result;
    fi;

    for i in [2..n] do
        temp := [];
        for j in [1..Length(result)] do
            combination := StructuralCopy(result[j]);
            if n - i >= 2*k - 1 - Sum(combination) then
                Add(combination, 0);
                Add(temp, StructuralCopy(combination));
                Remove(combination, Length(combination));
            fi;
            Add(combination, 1);
            if not (Sum(combination) > 2 * k) then
                Add(temp, StructuralCopy(combination));
            fi;
        od;
        result := StructuralCopy(temp);
    od;
    return result;
end);
```

Normal Origamis

GenerateOrigamiByFpGroup takes in a finitely presented, 2-generated group G, generators r, s and returns the normal origami coming from the Cayley graph $C(G, \{r, s\})$.

```
InstallGlobalFunction(GenerateOrigamiByFpGroup, [IsFpGroup],
    function(G, r, s)
    local horizontalPerm, verticalPerm, i, j, elemTimes_r,
        elemTimes_s, elements, elem;

    elements := Elements(G);

    horizontalPerm := [1..Order(G)];
    verticalPerm := [1..Order(G)];
```

```
    for i in [1..Order(G)] do
        elemTimes_r := elements[i] * r;
        elemTimes_s := elements[i] * s;
        j := 1;
        for elem in elements do
            if elemTimes_r = elem then
                horizontalPerm[i] := Position(elements, elem);
            fi;
            if elemTimes_s = elem then
                verticalPerm[i] := Position(elements, elem);
            fi;
        od;
    od;

    return Origami(PermList(horizontalPerm), PermList(verticalPerm));
end);
```

G_n_k returns the group $G_{(n,k)}$ introduced in Example II.2.4.

```
    InstallGlobalFunction(G_n_k, function(n, k)
    local G, r, s;

    G := FreeGroup("r", "s");
    r := G.1;
    s := G.2;
    G := G / [r^(2^(k+1)), s^(2^(n-k-1)), s^-1*r*s/r^-1];

    return G;
end);
```

D_n returns the dihedral group D_n.

```
    InstallGlobalFunction(D_n, function(n)
    local G, r, s;

    G := FreeGroup("r", "s");
    r := G.1;
    s := G.2;
    G := G / [r^n, s^2, r*s*r*s];

    return G;
```

```
end);
```

Cyclic Covers

MaximalSystolicRatioOfCyclicTorusCover returns the maximal systolic ratio of all cyclic covers in the stratum $H(\underbrace{1,\ldots,1}_{2k})$ for a specific n.

```
InstallGlobalFunction(MaximalSystolicRatioOfCyclicTorusCover,
   function(k, n)
   local max_sr, combinatorial_length, O, result, count,
      three_occured, max_origami,
      a, b, l_list, vec, max_vec, i;

   max_sr := -1.;
   combinatorial_length := -1;
   max_origami := Origami((),());
   count := 0;
   three_occured := false;
   max_vec := [];

   l_list := ArrayWithOnes(k, n^2 - 1);
   for a in [0..1] do
      for b in [0..1] do
         for i in [1..Length(l_list)] do
            vec := [a, b];
            Append(vec, l_list[i]);
            O := CyclicTorusCoverOrigami(n, 2, vec *
               Inverse(BaseChangeLToS(n)));
            result := SystolicRatio(O);
            count := count + 1;
            if combinatorial_length = 3 then
               three_occured := true;
            fi;
            if(result.systolic_ratio > max_sr) then
               max_sr := result.systolic_ratio;
               combinatorial_length :=
                  result.combinatorial_length;
               max_origami := O;
               max_vec := StructuralCopy(vec);
```

```
                fi;
            od;
        od;
    od;

    return rec(systolic_ratio := max_sr, combinatorial_length :=
        combinatorial_length, origami := max_origami,
                count := count, three_occured := three_occured,
                    monodromy := max_vec);
end);
```

Bibliography

[1] Tobias Columbus et al. *Systolic geometry of translation surfaces*. 2018. DOI: 10.48550/ARXIV.1809.10327. URL: https://arxiv.org/abs/1809.10327.

[2] George K. Francis and Jeffrey R. Weeks. "Conway's ZIP Proof". In: *The American Mathematical Monthly* 106.5 (1999), pp. 393–399. DOI: 10.1080/00029890.1999.12005061. eprint: https://doi.org/10.1080/00029890.1999.12005061. URL: https://doi.org/10.1080/00029890.1999.12005061.

[3] *GAP – Groups, Algorithms, and Programming, Version 4.11.1*. The GAP Group. 2021. URL: https://www.gap-system.org.

[4] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002. ISBN: 0521795400.

[5] Chris Judge and Hugo Parlier. "The maximum number of systoles for genus two Riemann surfaces with abelian differentials". English. In: *Comment. Math. Helv.* 94.2 (2019), pp. 399–437. ISSN: 0010-2571. DOI: 10.4171/CMH/463.

[6] L. Junk, P. Kattler, and A. Thevis. *Origami, Computing Veech groups of origamis, Version 1.0.0*. https://PascalKattler.github.io/Origami/. GAP package. May 2020.

[7] Robion C. Kirby. "Stable Homeomorphisms and the Annulus Conjecture". In: *Annals of Mathematics* 89.3 (1969), pp. 575–582. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970652 (visited on 08/02/2022).

[8] Stefan Kühnlein. *Einführung in die Geometrie und Topologie*. 2010. URL: https://www.math.kit.edu/iag3/lehre/geotopo2010w/media/eingeotop.pdf.

[9] Daniel Massart. "A short introduction to translation surfaces, Veech surfaces, and Teichmüller dynamics". In: *Surveys in Geometry I*. Ed. by A. Papadopoulos. Springer Nature, 2021. URL: https://hal.archives-ouvertes.fr/hal-03300179.

[10] Anja Randecker. *Unendliche Translationsflaechen*. 2013. URL: https://www.math.kit.edu/iag3/~randecker/media/unendliche_translationsflaechen.pdf.

[11] Alexander Rogovskyy. *Origamis, die zyklisch ueber dem $n \times n$-Torus faktorisieren*. 2022.

[12] Jan-Christoph Schlage-Puchta and Gabriela Weitze-Schmithuesen. *Totally non congruence Veech groups*. 2018. DOI: 10.48550/ARXIV.1802.05024. URL: https://arxiv.org/abs/1802.05024.

[13] Jan-Christoph Schlage-Puchta and Gabriela Weitze-Schmithüsen. "Finite translation surfaces with maximal number of translations". English. In: *Isr. J. Math.* 217 (2017), pp. 1–15. ISSN: 0021-2172. DOI: 10.1007/s11856-017-1436-8. URL: https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.764.2463.

[14] Gabriela Schmithüsen. "Origamis with non congruence Veech groups". In: (2007). DOI: 10.48550/ARXIV.0704.0416. URL: https://arxiv.org/abs/0704.0416.

[15] Gabriela Schmithüsen. "Veech Groups of Origamis". 2005.

[16] Michael Spivak. *Calculus On Manifolds: A Modern Approach To Classical Theorems Of Advanced Calculus*. Benjamin Cummings, 1965.

[17] Andrea Hannah Thevis. *On the geometry of p-origamis and beyond*. 2022. DOI: http://dx.doi.org/10.22028/D291-35962.

[18] Gabriela Weitze-Schmithüsen. "The Deficiency of being a Congruence Group for Veech Groups of Origamis". In: *International Mathematics Research Notices* 2015.6 (Dec. 2013), pp. 1613–1637. DOI: 10.1093/imrn/rnt268. URL: https://doi.org/10.1093%2Fimrn%2Frnt268.

[19] Alex Wright. "Translation surfaces and their orbit closures: an introduction for a broad audience". English. In: *EMS Surv. Math. Sci.* 2.1 (2015), pp. 63–108. ISSN: 2308-2151. DOI: 10.4171/EMSS/9.

[20] David Zmiaikou. "Origamis and permutation groups". Theses. Université Paris Sud - Paris XI, Sept. 2011. URL: https://tel.archives-ouvertes.fr/tel-00648120.