Geometric Group Theory

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in Summer 2022

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Contents

General Information

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Chapter 0

Introduction

In geometry group theory, we discover a relationship between groups and geometric objects, which may be linked to one another.

The main idea for this course is finding nice metric spaces, on which a given group acts. Metric properties of the space then teach us something about the group. But this is not a one way road: algebraic properties of the group will also give information about a given metric space.

Example: Groups that you may have encountered throughout your studies are the symmetric group S_n, finite cyclic groups, i.e. $(\mathbb{Z}/n\mathbb{Z}, +)$, infinite cyclic groups, i.e. $(\mathbb{Z}, +)$, or matrix groups like $\mathrm{Sl}(n, \mathbb{Z})$, $\mathrm{Gl}(n, \mathbb{Z})$ or $O(n, \mathbb{R})$.

Something which you might not have seen until now are fundamental groups. A fundamental group is a group associated to a topological space, consisting of equivalence classes of paths in the topological space modulo homotopy (i.e. continuous deformations of one path into another path).

Further examples of groups would be braided groups, mapping class groups and automorphism groups, e.g. $\mathrm{Aut}(F_n)$.

For this course, we will be mostly interested in infinite groups. More precisely in infinite groups that are finitely generated. Among other questions that we are going to tackle, we will ask ourselves, what a good way for describing infinite, finitely generated groups would be. For spoiled readers it may already be known that presentations are of no use.

In a first approach, we will try to think of ways to draw such a group. More precisely, we assign to our group G and a chosen set S of generators a graph (see Figure 0.1 for two example graphs). This graph is called the *Cayley graph* $\Gamma = \Gamma(G, S)$. Unfortunately, the graph Γ depends on the set S of generators. "Looking from further away", meaning looking up to quasi-isometry, identifies graphs that should be considered the same. This leads to "coarse geometry" or

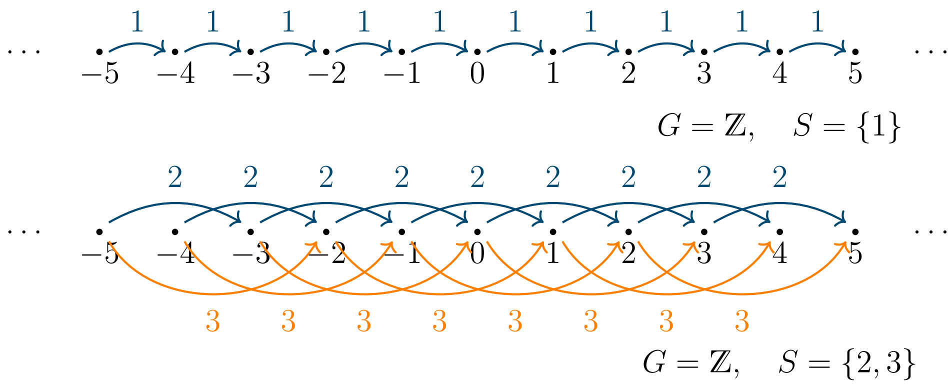


Figure 0.1: Some examples of Cayley graphs for the group $\mathbb{Z}$.

"large scale geometry". A reasonable goal would then be finding invariants of quasi-isometry.

Our more general goal is relating group theoretical properties of a given group to geometric properties of a space, on which it acts.

Chapter I

Cayley Graphs

In this chapter, we want to associate to an infinite, finitely generated group G with set of generators S a graph $\Gamma = \Gamma(G, S)$. On this graph $\Gamma(G, S)$, the group acts in a "good way" and the graph already knows a lot about the group.

1 The Category of Graphs

Definition I.1.1 (Graph): Let V and E be disjoint sets, let $\delta \colon E \to V \times V$ be a map and let $\iota \colon E \to E$ be a map. If for any e in E it holds $o(e) = t(\iota(e))$, $t(e) = o(\iota(e))$, $\iota(\iota(e))$ and $\iota(e) \neq e$, then the quadruple $\Gamma = (V, E, \delta, \iota)$ is called an *unoriented graph*.

The elements of V are called *vertices* and the elements of E are called *edges*. The *boundary map* δ assigns to an edge e a tuple of vertices $(o(e), t(e))$, the so-called *origin of* e and *terminus of* e. The *inverse map* ι assigns to an edge e the *inverse edge* $\bar{e} := \iota(e)$.

For any edge e in E, the set $\{e, \bar{e}\} =: [e] =: [\bar{e}]$ is called *geometric edge*.

For a choice of a subset E_+ of E, denote by E_- the image of E_+ under the inverse map. If for a subset E_+ of E it holds that $E = E_+ \cup E_-$ and $E_+ \cap E_- = \varnothing$, then E_+ is called an *orientation of* E.

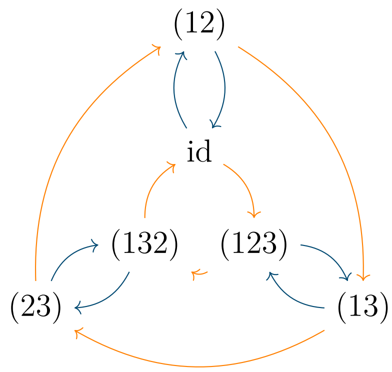
In the following, we will write $\delta = o \times t$, where $o \colon E \to V$, $e \mapsto o(e)$ and $t \colon E \to V$, $e \mapsto t(e)$.

Remark I.1.2: Let $\Gamma = (V, E, \delta, \iota)$ be a graph and let E_+ be an orientation of Γ. The triple $(\Gamma_+, E_+, \delta_+)$ with $\delta_+ := \delta|_{E_+ \times V}$ fully determines our original graph Γ. We call Γ_+ an *oriented graph*.

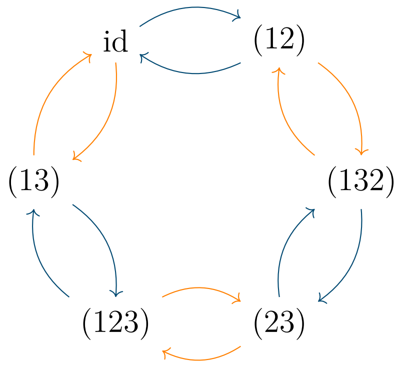
Let M be a set. We may consider maps $h_V \colon V \to M$ and $h_E \colon E \to M$. These are called *vertex-labellings* respectively *edge-labellings*.

Definition I.1.3 (Generalised Cayley Graph): Let $(G, \cdot)$ be a group and let S be a subset of G. Let $V = G$, let $E_+ = G \times S$ and let $\delta_+ \colon G \times S \to G \times G$, $(g, s) \mapsto (g, g \cdot s)$. Then $\Gamma^+(G, S)$ respectively $\Gamma(G, S) = (V, E, \delta, \iota)$ is called *generalised oriented Cayley graph* respectively *generalised Cayley graph*.

Example I.1.4 (The Symmetric Group on Three Letters): Recall that the symmetric group on three letters is $S_3 = \{\mathrm{id}, (12), (13), (23), (123), (132)\}$. Choosing the set of generators $S = \{(12), (123)\}$, we obtain the Cayley graph



Choosing the set of generators $S = \{(12), (13)\}$ yields the Cayley graph



From "very far away" both graphs look very similar: Both look like a point. Choosing the set $S = \{(12)\}$, which does *not* generate S_3, yields yet another different graph—this time an honest generalised Cayley graph. Later we will say that graph were not connected.

Remark I.1.5: The generalised Cayley graph $\Gamma(G, S)$ has the following natural edge labelling with values in $S \cup S^{-1}$:

$$h_E \colon E \longrightarrow S \cup S^{-1}, \qquad e \longmapsto \begin{cases} s, & \text{if } e \in E_+ = G \times S \text{ and } e = (g, s), \\ s^{-1}, & \text{if } e \in E_- \text{ and } \bar{e} = (g, s) \in G \times S. \end{cases}$$

Here, as usual, we denote $S^{-1} := \{s^{-1} \mid s \in S\}$.

Definition I.1.6 (Combinatorial Structure): Let $\Gamma = (V, E, \delta, \iota)$ be a graph.

(i) Let v_1 and v_2 be vertices. If there is some edge e such that $o(e) = v_1$ and $t(e) = v_2$, then v_1 and v_2 are called *neighbours*.

(ii) Let $[e_1]$ and $[e_2]$ be geometric edges. If the intersection of $\{o(e_1), t(e_1)\}$ and $\{o(e_2), t(e_2)\}$ is non-empty, the geometric edges are called *neighbours*.

(iii) Let e respectively $[e]$ be an edge respectively geometric edge. If it holds that $o(e) = t(e)$, then e respectively $[e]$ is called a *loop*.

(iv) Let v_1 and v_2 be distinct vertices. If there are distinct edges e and e' such that $o(e) = v_1 = o(e')$ and $t(e) = v_2 = t(e')$, then Γ has *multiple edges between v_1 and v_2*.

(v) If Γ has no loops and no multiple edges, then Γ is called a *combinatorial graph*.

Remark I.1.7: Often times, combinatorial graphs are often defined as pairs (V, E) with and arbitrary set V and a subset E of $\mathfrak{P}(V)$ such that every element of E has order 2.

Definition I.1.8 (Basic Definitions for Graphs): Let $\Gamma = (V, E, \delta, \iota)$ be a graph and let x be a vertex of Γ.

(i) The set $E_x = \{e \in E \mid o(e) = x\}$ is called the *star of x*.

(ii) The cardinality of the star of x is called the *valency* or *order of x*, denoted $\mathrm{val}(x)$.

(iii) Let k be a natural number. If for any vertex v of Γ it holds $\mathrm{val}(v) = k$, then the graph is called k-regular.

(iv) Let $e_1, \ldots, e_n$ be edges of Γ. If for $1 \leq i \leq n-1$ it holds $t(e_i) = o(e_{i+1})$, then $c = (e_1, \ldots, e_n)$ is called an *edge-path*. We denote $o(c) := e_1$ and $t(c) = e_n$.

(v) An edge path $c = (c_1, \ldots, c_n)$ with $o(e_1) = t(e_n)$ is called a *cycle*.

(vi) If for any vertices x_1 and x_2 of Γ there is an edge path c with $o(c) = x_1$ and $t(c) = x_2$, then Γ is called *path-connected*.

In the following, we aim at defining a morphism of graphs to make the class of graphs into a category. For a sensible concept of a morphism, we'd need two maps f_V and f_E that rendered the diagrams

$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ V \times V & \xrightarrow[f_V \times f_V]{} & V' \times V' \end{array} \qquad\qquad \begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota'} \\ E & \xrightarrow[f_E]{} & E' \end{array}$$

commutative. Indeed, this turns out to be the "correct" definition.

Definition I.1.9: Let $\Gamma = (V, E, \delta, \iota)$ and $\Gamma' = (V', E', \delta', \iota')$ be two graphs and let $f_V : V \to V'$ and $f_E : E \to E'$ be two maps. If it holds that $\delta' \circ f_E = (f_V \times f_V) \circ \delta$, i.e. $o' \circ f_E = f_V \circ o$ and $t' \circ f_E = f_V \circ o$, and if $\iota' \circ f_E = f_E \circ \iota$, i.e. if for any edge e of Γ it holds $\overline{(f_E(e))} = f_E(\bar{e})$, then the pair $f = (f_V, f_E)$ is called a *morphism of graphs*. In this case, we write $f : \Gamma \to \Gamma'$ to indicate the categorical nature of this definition. We denote $\mathrm{Mor}(\Gamma, \Gamma') := \{f : \Gamma \to \Gamma' \text{ is a morphism}\}$.

Remark I.1.10: We obtain a category called Graphs as follows: The objects are all graphs, for two graphs Γ_1 and Γ_2, the set of morphisms $\mathrm{Mor}(\Gamma_1, \Gamma_2)$ is as defined above, for three graphs Γ_1, Γ_2 and Γ_3, we define a composition

$$\begin{aligned} \circ\colon\ \mathrm{Mor}(\Gamma_2, \Gamma_3) \times \mathrm{Mor}(\Gamma_1, \Gamma_2) &\longrightarrow \mathrm{Mor}(\Gamma_1, \Gamma_3), \\ (g = (g_V, g_E), f = (f_V, f_E)) &\longmapsto g \circ f := (g_V \circ f_V, g_E \circ f_E) \end{aligned}$$

and for each graph Γ, we define its identity morphism id_Γ in $\mathrm{Mor}(\Gamma, \Gamma)$ to be $\mathrm{id}_\Gamma := (\mathrm{id}_V, \mathrm{id}_E)$.

To show that the graphs indeed form a category in this way, we have to check that the composition of graph morphisms indeed yields a new morphism of graphs. Then, one has to check associativity for the composition of morphisms, which follows immediately from the associativity for composition of maps on vertices and edges. Finally, one has to check that the identity morphism acts trivially.

Reminder I.1.11: Let Γ_1, Γ_2 and Γ be graphs.

(i) Let $f : \Gamma_1 \to \Gamma_2$ be a morphism of graphs. If there is a morphism $g : \Gamma_2 \to \Gamma_1$ such that $g \circ f = \mathrm{id}_{\Gamma_1}$ and $f \circ g = \mathrm{id}_{\Gamma_2}$, then f is called an *isomorphism*. Note that in this case, g is unique and usually denoted f^{-1}.

(ii) The set of automorphisms $\mathrm{Aut}(\Gamma) := \{f : \Gamma \to \Gamma \text{ is an isomorphism}\}$ turns into a group with composition of morphisms as law of composition.

Proposition I.1.12 (Isomorphism via Bijectivity): *Let Γ_1 and Γ_2 be graphs and let $f = (f_V, f_E)\colon \Gamma_1 \to \Gamma_2$ be a morphism of graphs. The maps f_V and f_E are bijective if and only if f is an isomorphism.*

Definition I.1.13 (Subgraphs): Let $\Gamma = (V, E, \delta, \iota)$ be a graph and let $\Gamma' = (V', E', \delta', \iota')$ be a quadruple of two sets V' and E' and two maps $\delta'\colon V' \to V'$, $\iota'\colon E' \to E'$. If V' is a subset of V, if E' is a subset on E, if δ' is the restriction $\delta|_{V'}$ and if the restriction $\iota|_{E'}$ equals ι', then Γ' is called a *subgraph of* Γ.

Remark I.1.14: Let $\Gamma = (V, E, \delta, \iota)$ and $\Gamma' = (V', E', \delta', \iota')$ be combinatorial graphs and let $f_V\colon V \to V'$ be a map.

(i) The map f_V uniquely determines a morphism of graphs $f = (f_V, f_E$ with $f_E((a, b)) = (f_V(a), f_V(b)$ if and only if for any neighbouring vertices x and y in Γ, their images $f_V(x)$ and $f_V(y)$ are neighbouring vertices in Γ'.

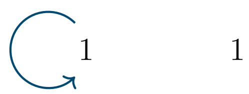
(ii) The map f_V determines an isomorphism of graphs if and only if f_V is bijective and if for any neighbouring vertices $f_V(x)$ and $f_V(y)$, their preimages x and y are neighbouring vertices in Γ.

2 Cayley Graphs

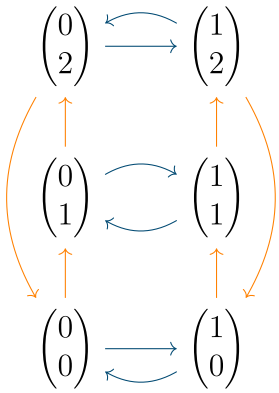
In this section, we want to relate properties of a tuple (G, S) of a group G and a set of generators S to properties of its Cayley graph $\Gamma = (G, S)$. Furthermore, we want to see how G acts on $\Gamma(G, S)$ and we try to find criteria to decide if a given graph is a Cayley graph.

Remark I.2.1: Let G be a group. Two elements g and h of G are neighbours in $\Gamma(G, S)$ if and only if $g^{-1}h$ belongs to $S \cup S^{-1}$.

Example I.2.2 (Generalised Cayley Graphs): (i) Consider the trivial group $G = \{1\}$ and the sets of generators S and $\varnothing$. Those lead to the generalised Cayley graphs



(ii) Consider the group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and the subset $S = \{(1, 0), (0, 1)\}$. Then we obtain the generalised Cayley graph



Proposition I.2.3 (First Properties of Cayley Graphs): *Let G be a group, let S be a subset of G and let $\Gamma = \Gamma(G, S)$ be the corresponding generalised Cayley graph.*

(i) *There are loops in Γ if and only if the identity belongs to S.*

(ii) *There are multiple edges in Γ if and only if there is some element s in S, whose inverse also belongs to S.*

(iii) *The graph Γ is combinatorial if and only if $S \cap S^{-1} = \varnothing$.*

(iv) *The graph Γ is connected if and only if S generates G.*[1]

Proof: The statements in (i)-(iii) are clear.

"$\Longrightarrow$": Assume Γ is connected. For any element g in G there is a path $c = (e_1, \ldots, e_n)$ in Γ such that $o(e_1) = 1$ and $t(e_n) = g$. Let $h_E \colon E \to S \cup S^{-1}$ be the natrual edge-labelling of Γ defined in Remark I.1.5. Define $s_i := h_E(e_i)$. Then $t(e_i) = o(e_i)s_i$ and thus $g = 1s_1 \cdots s_n$ for s_i in $S \cup S^{-1}$, i.e. $G = \langle S \rangle$.

"$\Longleftarrow$": Assume that S generates G. We have to show that for any distinct elements g and h in G, there is an edge-path from h to g. Since G is generated by S, there are some elements $s_1, \ldots, s_n$ in S such that $h^{-1}g = s_1 \cdots s_n$. Recursively, we now define

$$e_i := \begin{cases} (h \cdot s_1 \cdots s_{i-1}, s_i), & \text{if } s_i \in S, \\ \iota(h \cdot s_1 \cdots s_i, s_i^{-1}), & \text{if } s_i \in S^{-1}. \end{cases}$$ □

Remark I.2.4: Let S be a generating system for the group G. Then $\Gamma = \Gamma(G, S)$ is called *Cayley graph.*

[1]Recall that a subset S of a group G is said to generate the group, if it holds that $G = \bigcap (H \subseteq G \mid H \text{ is a subgroup of } G \text{ with } S \subseteq H) = \{s_1 \cdots s_k \mid s_i \in S \text{ or } s_i \in S^{-1}\}$

Definition I.2.5: Let G be a group, let C be a category, let X be an object in C and let $\mathrm{Aut}(X)$ be the group of automorphisms of X.

(i) A group homomorphism $\rho\colon G \to \mathrm{Aut}(X)$ is called an *action of* G *on* X.

(ii) Let $\rho_1\colon G \to \mathrm{Aut}(X_1)$ and $\rho_2\colon G \to \mathrm{Aut}(X_2)$ be two actions. If there is an isomorphism $f\colon X_1 \to X_2$ with $\kappa_f \circ \rho_1 = \rho_2$, then ρ_1 and ρ_2 are called *equivalent*.

Here, $\kappa_f\colon \mathrm{Aut}(X_1) \to \mathrm{Aut}(X_2)$ denotes conjugation with f, i.e. $h \mapsto f \circ h \circ f^{-1}$.

Example I.2.6: For the category $\mathsf{C} = \mathsf{Graphs}$, an action ρ of a group G on a graph $\Gamma = (V, E, \delta, \iota)$ can equivalently be described as a pair of group homomorphisms $\rho_V\colon G \to \mathrm{Perm}(V)$ and $\rho_E\colon G \to \mathrm{Perm}(E)$ such that for any g in G and e in E it holds $o(\rho_E(g)(e)) = \rho_V(g)(o(e))$, $t(\rho_E(g)(e)) = \rho_V(g)(t(e))$ and $\overline{\rho_E(g)(e)} = \rho_E(g)(\bar{e})$.

Remark I.2.7 (Action on the Cayley Graph): A group G acts on a Cayley graph $\Gamma = \Gamma(G, S)$ of the group via left-multiplication. Namely, we have an action described by (ρ_V, ρ_E) with

$$\rho_V\colon G \longrightarrow \mathrm{Perm}(V), \qquad g \longmapsto (h \mapsto gh),$$
$$g_E\colon G \longrightarrow \mathrm{Perm}(E), \qquad g \longmapsto ((h, s) \mapsto (gh, s), \overline{(h, s)} \mapsto \overline{(gh, s)}).$$

Reminder I.2.8: Let G be a group and let M be a set. For an action $\rho\colon G \to \mathrm{Perm}(M)$ of G on M and some element x of M, we denote

(i) $g \cdot x := \rho(g)(x)$,

(ii) $\mathrm{Stab}_G(x) := \{g \in G \mid g \cdot x = x\}$, called *stabiliser of* x,

(iii) $Gx := \mathrm{orb}_G(x) := \{g \cdot x \mid g \in G\}$, called *orbit of* x,

(iv) If it holds $G \cdot x = M$, then the action is called *transitive*,

(v) If for any x in X it holds that $\mathrm{Stab}_G(x) = \{1\}$, then ρ is called *free* or *fixed-point free*,

(vi) If ρ is injective, the action is called *faithful*.

Proposition I.2.9 (Properties of Actions by Left-Multiplication): *Let G be a group and let S be a set of generators for G. The action $\rho = (\rho_V, \rho_S)$ of G on $\Gamma(G, S)$ by left-multiplication defined in Remark 2.6 we have the following:*

(i) *The action ρ is free, i.e. ρ_V and ρ_E are free.*

(ii) *The action* ρ *is* vertex-transitive*, i.e.* ρ_V *is transitive.*

(iii) *The action* ρ *acts* without inversions*, i.e. for any* g *in* $G - \{1\}$ *and for any edge* e *it holds* $\rho_E(g)(e) \neq \bar{e}$*. In particular,* ρ_E *acts free on geometric edges.*

Proof: (i) Suppose g and h are group elements such that g belongs to $\mathrm{Stab}_G(h)$. Since $g \cdot h = h$, g must be the identity, i.e. ρ_V is free.

Suppose g is an element of G and e is an edge of such that g belongs to $\mathrm{Stab}_G(e)$. Then $g \cdot o(e) = o(e)$, which means that g also belongs to $\mathrm{Stab}_G(o(e))$ and hence, g must be the identity. Thus, ρ_E is free.

(ii) For group elements h_1 and h_2, choose $h = h_2 h_1^{-1}$. Then $g \cdot h_1 = h_2$, thus h_1 and h_2 are in the same orbit. Because h_1 and h_2 were arbitrary, the action ρ_V is transitive.

(iii) By definition of ρ_E we have that an edge e belongs to E_+ if and only if $\rho_E(g)(e)$ belongs to E_+. But this implies that $\rho_E(g)(e) \neq \bar{e}$. □

Theorem 1: *Let* $\Gamma = (V, E, \delta, \iota)$ *be a combinatorial graph.*

(i) *An action* $\rho\colon G \to \mathrm{Aut}(\Gamma)$ *is equivalent to the action via left-multiplication if and only if* ρ *is free, vertex-transitive and without inversions.*

(ii) *The graph* Γ *is the Cayley graph for some group if and only if* $\mathrm{Aut}(\Gamma)$ *contains a subgroup which acts freely, vertex-transitively and without inversions.*

Proof: (i) "$\Longrightarrow$": This was shown in Proposition I.2.9.

"$\Longleftarrow$": Suppose $\rho = (\rho_V, \rho_E)$ has the stated properties. As the first step, we aim to find a suitable set of generators S. Let x be any vertex and let $\widehat{S} := \{g \in G \mid gx \text{ is a neighbour of } x\}$.

This set $\widehat{S}$ is closed under inversion, as for some s in $\widehat{S}$, there is an edge e in E such that $o(e) = x$ and $t(e) = sx$. For the edge $s^{-1}x$ we obtain $o(s^{-1}e) = s^{-1}x$ and $t(s^{-1}e) = x$, such that x and $s^{-1}x$ are neighbours.

Furthermore, no element in $\widehat{S}$ is not self-inverse. For an e as above, we get $s^{-1}e \neq \bar{e}$, since ρ acts without inversions. If s were equal to s^{-1}, there were two geometric edges between x and sx, which cant be, since Γ is combinatorial.

Because $\widehat{S}$ is closed under inverses, but does not contain self-inverse element, there is a subset S of $\widehat{S}$ such that $\widehat{S} = S \cup S^{-1}$.

As the second step, we aim to find a morphism $f\colon \Gamma' := \Gamma(G, S) \to \Gamma$. As usual, we denote $\Gamma' = (V', E', \delta', \iota')$. Note that, by choice of S, the graph Γ' is combinatorial.

We define $f_V : V' = G \to V$, $h \mapsto h \cdot x$. To check that f_V indeed defines a graph homomorphism, one has to check that neighbouring vertices are mapped to neighbouring vertices. Suppose h_1 and h_2 are neighbours in Γ'. Then $s = h_1^{-1}h_2$ belongs to $S \cup S^{-1}$ and thus x and sx are neighbours in Γ. Applying h_1 yields that h_1x and $h_1sx = h_2x$ are neighbours in Γ. The other direction, i.e. going from neighbours in Γ to Γ', works just the same way.

As the third step, we want to show that f is an isomorphism. For this, it remains to show that f_V is bijective.

For the injectivity, suppose that $h_1x = h_2x$. Then $h_2^{-1}h_1x = x$, i.e. $h_2^{-1}h_1$ belongs to $\mathrm{Stab}_G(x)$. Since ρ_V is free, $h_2^{-1}h_1 = 1$ and thus $h_1 = h_2$.

For the surjectivity, suppose y is a vertex. Because ρ_V is transitive, there is some h in G with $hx = y$, which is another way of writing $f_V(h) = y$.

As the fourth step, we have to show that f induces an equivalence between the actions of G on Γ_1 and Γ_2. For this, we have to show that for any g in G it holds $\rho(g) \circ f = f \circ \rho'(g)$. On vertices it holds

$$\rho(g)(f_v(h)) = g \cdot f_V(h) = g \cdot h \cdot x = f_V(g \cdot h) = f_V(\rho'(g)(h)).$$

Since Γ and Γ' are combinatorial, we obtain the analogue statement for f_E.

(ii) "$\Longrightarrow$": Suppose Γ is a Cayley graph. Then the action $\rho: G \to \mathrm{Aut}(\Gamma)$ by left-multiplication is free and thus, in particular, ρ is injective. The image of ρ is a subgroup of the automorphism group with the desired properties by Proposition I.2.9. "$\Longleftarrow$": This follows from (i). $\square$

3 Topological Realisation of Graphs

So far, graphs are combinatorial objects. Now, we want to consider geometric spaces, on which a group acts. For this, we "glue" edges between vertices.

Reminder I.3.1 (Topological Space): Let X be a set.

(i) The system $\mathfrak{T} := \mathfrak{P}(X)$ is called *discrete topology on X*.

(ii) If $\mathfrak{T}$ is a topology on X and Y is a subset of X, then $\mathfrak{T}' := \{U \cap Y \mid U \in \mathfrak{T}\}$ yields a topology on Y, called *trace topology* or *subset topology* or *relative topology* or *induced topology*. We have the following characteristic property: If $(Z, \widehat{\mathfrak{T}})$ is any other topological space, and if $i: Y \hookrightarrow X$ denotes the inclusion map, a map $f: Z \to Y$ is continuous if and only if $i \circ f$ is continuous.

(iii) Suppose that $(X, \mathfrak{T})$ is a topological space and let $q\colon X \to Y$ be surjective. Then $\overline{\mathfrak{T}} = \{U \subseteq Y \mid q^{-1}(U) \in \mathfrak{T}\}$ defines a topology on Y, called *quotient topology.* If $(Z, \widehat{\mathfrak{T}})$ is another topological space, and if $f\colon Y \to Z$ is a map, then f is continuous if and only if $f \circ q$ is continuous.

(iv) Let $(X_i, T_i)_{i\in I}$ be a family of topological spaces. On the disjoint union $\dot\bigcup_{i\in I} X_i$, the set

$$\mathfrak{T} := \left\{ U \in \mathfrak{P}(\bigcup_{i\in I} X_i) : \text{For all } i \in I : U \cap X_i \in \mathfrak{T}_i \right\}$$

is a topology. The space $(\dot\bigcup_{i\in I} X_i, \mathfrak{T})$ is called *topological sum of the* $(X_i, \mathfrak{T}_i)_{i\in I}$.

(v) We consider intervals $[a, b]$ with the trace topology of the Euclidean topology on $\mathbb{R}$.

Definition I.3.2 (Topological Realisation): Let $\Gamma = (V, E, \delta, \iota)$ be a graph.

(i) For an edge e, we define $X_e := [0, 1] \times \{e\}$ as a copy of $[0, 1]$. We define $X := \dot\bigcup_{e\in E} X_e \,\dot\cup\, V/{\sim}$, where $\sim$ is the equivalence relation generated by the following requirements: For any edge e and any t from $[0, 1]$, $X_e \ni (t, e) \sim (1 - t, \bar{e}) \in X_{\bar{e}}$, for any edge e, $X_e \ni (0, e) \sim o(e)$ and for any edge $X_e \ni (1, e) \sim t(e)$.

(ii) The set $\Gamma^{\text{top}} := X$ turns into a topological space as follows: Take the discrete topology on V and the topology as segment on each X_e, then take the topology as topological sum on $\dot\bigcup_{e\in E} X_e \,\dot\cup\, V$, then take the quotient topology for the surjective map $q\colon \dot\bigcup_{e\in E} X_e \,\dot\cup\, V \to \dot\bigcup_{e\in E} X_e \,\dot\cup\, V/{\sim}$.

(iii) Consider the maps $i_V\colon V \to \Gamma^{\text{top}}$, $v \mapsto [v]_\sim$; $\chi_e\colon [0, 1] \to \Gamma^{\text{top}}$, $t \mapsto [(t, e)]_\sim$. They are continuous, i_V is surjective and $\chi_e|_{[0,1)}$ is injective. We write v for $[v]_\sim \in \Gamma^{\text{top}}$ and e for $\chi_e([0, 1]) \subseteq \Gamma^{\text{top}}$ and (t, e) for $[(t, e)] \in \Gamma^{\text{top}}$ and call them vertices respectively edges of Γ^{top}.

The topological space Γ^{top} is called *topological realisation of* Γ.

Remark I.3.3: (i) Every morphism $f = (f_V, f_E)$ between two graphs Γ_1 and Γ_2 defines a continuous map $f^{\text{top}}\colon \Gamma_1^{\text{top}} \to \Gamma_2^{\text{top}}$ that maps a vertex $f_V(v)$ and (t, e) to $(t, f_E(e))$.

(ii) We have $(f \circ g)^{\text{top}} = f^{\text{top}} \circ g^{\text{top}}$ and $(\mathrm{id}_\Gamma)^{\text{top}} = \mathrm{id}_\Gamma^{\text{top}}$.

Proof: We only show the first assertion. To show that f^{top} is well-defined, one has to check that the map respects the gluing from Definition I.3.2. For example, we have

$$f^{\text{top}}(1 - t, i) = (1 - t, f_E(\bar{e})) = (1 - t, \overline{f_E(e)}) = (t, f_E(e)) = f^{\text{top}}(t, e).$$

Similarly for the other statements. For the continuity of f^{top}, consider the following commutative diagram,

$$\begin{array}{ccc} \dot{\bigcup}_{e\in E_1} X_e \,\dot{\cup}\, V_1 & \longrightarrow & \dot{\bigcup}_{e\in E_2} X_e \,\dot{\cup}\, V_2 \\ \Big\downarrow q^1 & & \Big\downarrow q_2 \\ \Gamma_1^{\text{top}} & \xrightarrow[f^{\text{top}}]{} & \Gamma_2^{\text{top}} \end{array}$$

where on the top row, v is mapped to v and (t, e) is mapped to $(t, f_E(e))$, which yields a continuous map. Now f^{top} is continuous due to the characteristic property of the quotient maps. □

Corollary I.3.4: *We have a functor* Graphs → TopSpaces *defined on objects by* $\Gamma \mapsto \Gamma^{\text{top}}$ *and on morphisms by* $(f\colon \Gamma_1 \to \Gamma_2) \mapsto \bar{f} = f^{\text{top}}\colon (\Gamma_1^{\text{top}} \to \Gamma_2^{\text{top}}$. *This functor is covariant. We say that* $\bar{f} = f^{\text{top}}$ *is the topological realisation of* f.

This follows immediately from Remark I.3.3.

4 Graphs as Metric Spaces

In this section, we want to define a metric on the topological realisation $\Gamma^{\top}$ of a graph Γ. As an idea for this, we want to assign length 1 to each edge.

Reminder I.4.1: Let X be a set. A map $d\colon X \times X \to \mathbb{R}$, which for any elements x, y and z of X satisfies that $d(x, y) \geq 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ is called a *pseudo-metric*. If in addition it holds for any x and y from X that $d(x, y) = 0$ if and only if $x = y$, then d is called a *metric*.

If d is a metric on the set X, then d induces a topology on X, whose basis are the open balls with respect to d. More precisely, a subset U of X is open with respect to this induced topology if and only if for every x in U there is an $\varepsilon > 0$ such that $B(x, \varepsilon) := \{y \in X \mid d(x, y) < \varepsilon\} \subseteq U$.

Remark I.4.2: Suppose that we are given a connected topological space X (i.e. X cannot be decomposed into two non-empty disjoint open subsets), an open cover $(U_i)_{i\in I}$ of X (i.e. for any i in I, the set U_i is an open subset of X and $X = \bigcup_{i\in I} U_i$) and for each i in I a metric $d_i\colon U_i \times U_i \to \mathbb{R}$ such that for any indices i and j and any x and y from $U_i \cap U_j$ it holds $d_i(x, y) = d_j(x, y)$. Then

$$d(x, y) := \inf \left\{ \sum_{k=0}^{n-1} d_{i_k}(x_k, x_{k+1}) : n \in \mathbb{N}, x_0 = x, x_n = y, x_k, x_{k+1} \in U_{i_k} \right\}$$

defines a pseudo-metric on X.

Proof: Let x and y be points of X. Define the set

$$S(x,y) := \{(x_0,\dots,x_n) \mid n \in \mathbb{N}, x = x_0, y = x_n, \\ \forall\, k \in \{0,\dots,n-1\}\, \exists\, i_k : x_k, x_{k+1} \in U_{i_k}\}$$

and for $\omega = (x_0,\dots,x_n)$ in S, denote $\ell(\omega) := \sum_{k=0}^{n-1} d_{i_k}(x_k,x_{k+1})$. Evidently, $d(x,y) \geq 0$ and $d(x,y) = d(y,x)$. Suppose now x, y and z are points of X and let $\omega_1 = (x = x_0, x_1, \dots, x_n = y) \in S(x,y)$, $\omega_2 = (y = y_0, y_1, \dots, y_m = z) \in S(y,z)$. Then $\omega_3 = (x = x_0, \dots, x_n = y_0, y_1, \dots, y_m = z)$ belongs to $S(x,z)$ and $\ell(\omega_3) = \ell(\omega_1) + \ell(\omega_2)$. Hence, $d(x,z) \leq d(x,y) + d(y,z)$, which establishes the triangular inequality.

It remains to show that d is well-defined, i.e. $S(x,y) \neq \varnothing$. Consider the sets $V_x := \{y \in X \mid S(x,y) \neq \varnothing\}$ and $W_x := \{y \in X \mid S(x,y) = \varnothing\}$. For any i in I we have that $U_i \subseteq V_x$ or $U_i \subseteq W_x$, thus

$$V_x = \bigcup_{i\in I} V_x \cap U_i = \bigcup_{i \in I'} U_i$$

for $I' = \{i \in I \mid V_x \cap U_i \neq \varnothing\}$ and similarly $W_x = \bigcup_{i\in I - I'} U_i$. Thus V_x and W_x are open, disjoint and they satisfy $X = V_x \,\dot\cup\, W_x$. Because X is connected, we must have that $X = V_x$ or $X = W_x$. As x belongs to V_x, V_x is non-empty, which enforces $X = V_x$. □

Remark I.4.3 (Graph Metric): Let $\Gamma = (V,E,\delta,\iota)$ be a connected graph and let Γ^{top} be its topological realisation. For fixed $r < 1/2$, choose the following open subset of X: For each e in E, let $U_e = \chi_e((0,1))$, for each v in V let $U_{v,r} := \bigcup(\chi_e([0,r)) \mid e \in E, o(e) = v)$. Define on them the following metrics. On U_e define the metric d_e via $d_e((t_1,e),(t_2,e)) := |t_1 - t_2|$, and on $U_{v,r}$ define $d_{v,r}$ via

$$d_{v,r}((t_1,e_1),(t_2,e_2)) := \begin{cases} |t_1 - t_2|, & \text{if } e_1 = e_2, \\ t_1 + t_2, & \text{if } e_1 \neq e_2, \end{cases}$$

where $o(e_1) = o(e_2)$. Observe that for $e_1 = \bar{e}_2$ we obtain $d_{e_1} = d_{e_2}$ on $U_{e_1} = U_{e_2}$ and that if $U_e \cap U_{v,r}$ is non-empty, we have $o(e) = v$ or $t(e) = v$ and the metric coincide and finally that $U_{v_1,r} \cap U_{v_2,r}$ is empty. Hence, we can glue the metric by Remark I.4.2. This yields a pseudo-metric on $X = \Gamma^{\text{top}}$.

Proposition I.4.4: *Let $\Gamma = (V,E,\delta,\iota)$ be a connected graph. Then, the pseudo-metric from Remark I.4.3 is in fact a metric, called the* Graph metric *for Γ.*

Proof: We have to show that for any x, y in $X = \Gamma^{\text{top}}$ with $d(x, y) = 0$ it holds that $x = y$. Let $\omega = (x = x_0, x_1, \dots, x_n = y)$ be an element of $S(x, y)$. If $x_0, x_1, \dots, x_n$ lie in the same U_e, then

$$\ell(\omega) = d_e(x_0, x_1) + \cdots + d_e(x_{n-1}, x_n) \geq d_e(x_0, x_n) > 0.$$

In the same way, we obtain that if all the $x_0, \dots, x_n$ are contained in the same $U_{v,r}$, then $\ell(\omega) \geq d_{U_{v,r}}(x, y)$.

If not all $x_0, \dots, x_n$ are contained in the same U_e or $U_{v,r}$, then there is some index i in $\{0, \dots, n-2\}$ such that for some edge e and some vertex v we have that x_i, x_{i+1} in U_e x_{i+1}, x_{i+2} in $U_{v,r}$ and $x_i \notin U_{v,r}$ or that x_i, x_{i+1} in $U_{v,r}$, x_{i+1}, x_{i+2} in U_e and $x_{i+2} \notin U_{v,r}$. Without loss of generality, we may assume the first. We denote $x_i = (t_1, e)$, $x_{i+1} = (t_2, e)$ and $x_{i+2} = (t_3, \tilde{e})$ with $o(e) = o(\tilde{e}) =: v$. Then $t_1 > r$, since x_i doesn't belong to $U_{v,r}$ and thus

$$\ell(\omega) \geq d_e(x_i, x_{i+1}) + d_{V,r}(x_{i+1}, x_{i+2}) = |t_1 - t_2| + t_2 + t_3 \geq t_1 - t_2 + t_2 + t_3 \geq t_1 > r.$$

In all three cases, the lengths of the sequences are bounded below be positive constants, hence $d(x, y) > 0$. □

Remark I.4.5 (First Properties of the Graph Metric): Let Γ be a connected graph, let $X = \Gamma^{\text{top}}$ be its topological realisation and let d be the graph metric on X. For any x and y from X it holds:

(i) If $x, y \in \chi_e([0,1])$, i.e. if $x = (t_1, e)$ and $y = (t_2, e)$, then $d(x, y) = |t_2 - t_1|$.

(ii) If x and y are vertices, then

$$d(x, y) = \min\{n \in \mathbb{N} \mid \text{There is an edge path } \omega = (x_0 = x, \dots, x_n = y)\}.$$

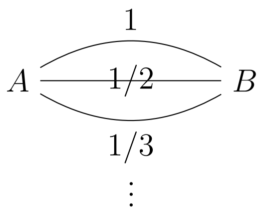
(iii) If $x = (t_1, e_1)$ and $y = (t_2, e_2)$ with $e_1 \neq e_2$ and $e_1 \neq \bar{e}_2$, then

$$\begin{aligned} d(x, y) = \min\{&t_1 + t_2 + d(o(e_1), o(e_2)), t_1 + 1 - t_2 + d(o(e_1), t(e_2)), \\ &1 - t_1 + t_2 + d(t(e_1), o(e_2)), 1 - t_1 + 1 - t_2 + d(t(e_1), t(e_2))\}. \end{aligned}$$

This can be shown with arguments similar to those used to show Proposition I.4.4.

Remark I.4.6 (Graph Metric for Graphs with Edge-Weights): Suppose Γ is a connected graph, E_+ the choice of an orientation and $\omega \colon E_+ \to \mathbb{R}_{>0}$ an edge-labelling. If there is a positive constant C such that for all edges e in E_+ it holds $\omega(e) \geq C$, then we obtain, in a similar fashion to Proposition I.4.4, a metric on $X = \Gamma^{\text{top}}$ such that the length of the geometric edge $\{e, \bar{e}\}$ is $\omega(e)$ for any e in E_+.

Example I.4.7: The constant C in Remark I.4.6 is needed. Consider the graph



with $V = \{A, B\}$, $E_+ = \mathbb{N}$, $o(e) = A$ and $t(e) = B$ for any e in E_+ and $\omega\colon E_+ \to \mathbb{R}_{>0}$, $n \mapsto 1/n$. Then $d(A, B) = 0$, even though $A \neq B$. Hence, in this case we end up with a pseudo-metric.

Example I.4.8:

Remark I.4.9 (Two Different Topologies): If Γ has a vertex x with valency $\mathrm{val}(x) = \infty$, then the topology on $X = \Gamma^{\mathrm{top}}$ defined by the graph metric is different to the original topology.

Proposition I.4.10 (Graph Morphisms are Contractions): *Let Γ_1 and Γ_2 be graphs and let X_1 and X_2 be their topological realisations equipped with the corresponding graph metrics d_1 respectively d_2. Furthermore, let $f = (f_V, f_E)\colon \Gamma_1 \to \Gamma_2$ be a graph morphism and let $\bar{f}\colon X_1 \to X_2$ be its topological realisation. Then $\bar{f}$ is a contraction, i.e. for any points x and y of X_1 it holds that $d(f(x), f(y)) \leq d(x, y)$.*

Proof: Recall that $\bar{f}(t, e) = (t, f_E(e))$. Observe the following:

(i) If e belongs to E_1 and v belongs to V_1, then $\bar{f}(U_e) = U_{f_E(e)}$ and $\bar{f}(U_{v,r}) \subseteq U_{f_V(v),r}$.

(ii) For $x = (t_1, e)$ and $y = (t_2, e)$ in the same open edge U_e it holds $d_2(f(x), f(y)) = d_2((t_1, f_E(e)), (t_2, f_E(e))) = |t_1 - t_2| = d_1(x, y)$. Similarly, if x and y belong to the same open star $U_{v,r}$, then $d_2(f(x), f(y)) \leq d_1(x, y)$.

For arbitrary points x and y in X_1, let $\omega = (x = x_0, x_1, \ldots, x_n = y)$ be a chain in $S(x, y)$. Then also $\bar{f}(\omega) = (f(x) = f(x_0), f(x_1), \ldots, f(x_n) = f(y))$ is a chain going from $f(x)$ to $f(y)$. For the distance $d_2(f(x), f(y))$ we find

$$d_2(f(x), f(y)) \leq \ell(\bar{f}(\omega)) = \sum_{i=1}^{n} d_2(f(x_{i-1}), f(x_i)) \leq \sum_{i=1}^{n} d_1(x_{i-1}, x_i) = \ell(\omega).$$

By taking the infimum, we obtain that $d_2(f(x), f(y)) \leq d_1(x, y)$. $\square$

Notation I.4.11: Let Γ be a graph, let $\Gamma^\top$ be its topological realisation. If we consider the topological realisation equipped with the topology induced by the graph metric, we will denote this space Γ^{geom}. In this case, we write f^{geom} for the topological realisation $\bar{f}$ of a graph morphism and call it its *geometric realisation.*

Corollary I.4.12 (of Proposition I.4.10): *Let Γ_1, Γ_2 be graphs and let $f\colon \Gamma_1 \to \Gamma_2$ be a graph morphism. Then, its geometric realisation $\bar{f} = f^{\mathrm{geom}}\colon \Gamma_1^{\mathrm{geom}} \to \Gamma_2^{\mathrm{geom}}$ is continuous.*

Proof: This follows from Proposition I.4.10, since $\bar{f}$ is a contraction. □

Corollary I.4.13: *Let Γ_1, Γ_2 be graphs and let $f\colon \Gamma_1 \to \Gamma_2$ be an isomorphism of graphs. Then, $f^{\mathrm{geom}}\colon (\Gamma_1^{\mathrm{geom}}, d_1) \to (\Gamma_2^{\mathrm{geom}}, d_2)$ is an isometry.*

Here, of course, d_1 and d_2 denote the respective graph metrics.

5 Isometry Group and Quotient Graphs

In this section, we want to see that the isometry group of the geometric realisation of a graph equals the isomorphism group of said group. Further, we want to study objects that result of quotients by subgroups of the isomorphism group.

In this section, $\Gamma = (V, E, \delta, \iota)$ will denote a graph with geometric realisation $X = \Gamma^{\mathrm{geom}}$ and graph metric d. Similarly for graphs Γ_1 and Γ_2. Furthermore, we want to assume for this section that all graphs are connected.

Proposition I.5.1 (Isomorphism and Isometries):

(i) *Suppose that Γ_1 has a vertex v of valency $\mathrm{val}(v) \neq 2$. Then for each isometry $\bar{h}\colon X = \Gamma_1^{\mathrm{geom}} \to Y = \Gamma_2^{\mathrm{geom}}$ there is a graph isomorphism $h\colon \Gamma_1 \to \Gamma_2$ such that $\bar{h} = h^{\mathrm{geom}}$.*

(ii) *Γ_1 and Γ_2 are isomorphic if and only if $\Gamma_1^{\mathrm{geom}}, d_1)$ and $(\Gamma_2^{\mathrm{geom}}, d_2)$ are isometric.*

Note that if $\bar{h}$ doesn't preserve vertices, then we don't stand a chance.

Reminder I.5.2 (Connected Components): Let $(X, \mathfrak{T})$ be a topological space and let x be a point in X. Then the union of all connected sets in X containing x is called the *connected component of X*. It is equivalently described as the unique largest connected subset of X containing x. Here, "largest" is to be understood with resepct to inclusion.

Example I.5.3 (Connected Components of Punctures Neighbourhoods): (i) Let x be a vertex of the graph Γ with $\mathrm{val}(x) = n$, let $\varepsilon \in (0, 1/4)$ and let $U = B(x, \varepsilon) - \{x\}$. Then U has n connected components.

(ii) Let $x = (t, e)$ for some t in $(0, 1)$ and some edge e of Γ, let $\varepsilon \in (0, \min\{1, 1-t\})$ and let $U = B(x, \varepsilon) - \{x\}$. Then U has two connected components.

Lemma I.5.4 (Isometries Preserve Valencies): *Suppose that* $\bar{h}\colon X = \Gamma_1^{\mathrm{geom}} \to Y = \Gamma_2^{\mathrm{geom}}$ *is an isometry and* $x = v$ *is a vertex in* V_1 *with valency* $\mathrm{val}(x) \neq 2$. *Then* $y = \bar{h}(x)$ *is a vertex* w *in* V_2 *with the same valency.*

Proof: The number of connected components of the punctured balls $B(x, \varepsilon) - \{x\}$ and of $B(\bar{h}(x), \varepsilon) - \{\bar{h}(x)\}$ has to be equal. □

Proof: (i) We are given an isometry $\bar{h}\colon X \to Y$ and we know there is a point $x_0 = v$ in V_1 with valency $\mathrm{val}(x) \neq 2$. By Lemma I.5.4, $y_0 = \bar{h}(x_0)$ is again a vertex w in V_2. Observe that x in X is a vertex if and only if $d(x_0, x)$ is a natural number; same for a vertex y in Y. Hence $\bar{h}$ preserves vertices, i.e. $\bar{h}(V_1) = V_2$. But this means in particular that $\bar{h}$ preserves open edges. More precisely, for any e in E_1 and $U_e = \{(t, e) \mid t \in (0, 1)\}$ we have $\bar{h}(U_e) = U_{\tilde{e}}$ for some $\tilde{e}$ in E_2.

Defining $h_V := \bar{h}|_{V_1}\colon V_1 \to V_2$ and $h_E\colon E_1 \to E_2$, $e \mapsto \tilde{e}$, where $\tilde{e}$ is chosen such that $\tilde{h}(t, e) = (\tilde{t}, \tilde{e})$ yields a graph morphism, as $\bar{h}$ being an isometry ensures that $\tilde{t} = t$.

This means in total that $\bar{h}$ is the geometric realisation of the graph isomorphism $h = (h_V, h_E)$.

(ii) "$\Longrightarrow$": This follows from Proposition I.4.10.

"$\Longleftarrow$": Let $\bar{h}\colon X \to Y$ be an isometry. If Γ_1 or Γ_2 has a vertex of valency different from 2, then $\bar{h}$ is the geometric realisation of some graph isomorphism, and thus $\Gamma_1 \cong \Gamma_2$.

If not, then $\Gamma_1 \cong \mathsf{Circ}_n$ for $n \in \mathbb{N}_0 \cup \{\infty\}$. Observe that

$$\mathrm{diam}(\mathsf{Circ}_n^{\mathrm{geom}}) = \sup\{d(x, y) \mid x, y \in \mathsf{Circ}_n^{\mathrm{geom}}\} = \begin{cases} n/2, & \text{if } n \in \mathbb{N}, \\ \infty, & \text{if } n = \infty. \end{cases}$$

which has to be preserved by the isometry $\bar{h}$. Hence Γ_1 and Γ_2 are isomorphic. Do be careful however. In the latter case, $\bar{h}$ does not have to be an isomorphism. □

Example I.5.5 (Graphs whose valencies are all two): The only connected possible offenders, i.e. graphs whose vertices all have vertices 2, are the following:

(i) Circ_n, a circle with n vertices, where n is a natural number.

(ii) $\mathsf{Circ}_\infty = \mathrm{Cay}(\mathbb{Z}, \{1\})$.

Definition I.5.6 (Isometry Group): The set

$$\mathrm{Isom}(\Gamma) = \{\bar{h}\colon (\Gamma^{\mathrm{geom}}, d) \to (\Gamma^{\mathrm{geom}}, d) \text{ isometry}\}$$

is the *isometry group of* Γ. Here, d denotes the graph metric.

Corollary I.5.7 (of Proposition I.5.1, Automorphisms via Isometries): *If* Γ *is not isomorphic to* Circ_n *for* $n \in \mathbb{N} \cup \{\infty\}$*, then* $\mathrm{Aut}(\Gamma) \cong \mathrm{Isom}(\Gamma)$.

Reminder I.5.8 (Quotients of Sets by Group Actions): (i) Let X be a set and let "$\sim$" be an equivalence relation. Then $X/\!\sim\, = \{[x] \mid x \in X\}$ is the set of equivalence classes with respect to "$\sim$" and $q\colon X \to X/\!\sim$, $x \mapsto [x]$ is the canonical projection. Let Y be any set and let $f\colon X \to Y$ be a map. If for any x_1 and x_2 in X with $x_1 \sim x_2$ it holds that $f(x_1) = f(x_2)$, we say that f *equivariant with respect to "$\sim$"*.

Observe that $X/\!\sim, q$ has the following universal property: For any map $f\colon X \to Y$ being equivariant with respect to "$\sim$", there is one and only one map $\bar{f}\colon X/\!\sim\, \to Y$ such that $f = \bar{f} \circ q$. We say that f factors through $X/\!\sim$.

A fancy way of saying this is that $X/\!\sim$ is a universal object. With respect to what functor?

(ii) Let now X be a set and let $\rho\colon G \to \mathrm{Perm}(X)$ be a group action. Then "If $Gx = Gy$, then $x \sim y$" declares an equivalence relation on X and we denote $X/\!\sim\, := G\backslash X = \{Gx \mid x \in X\}$. Again, we have the canonical projection $q\colon X \to G\backslash X$, $x \mapsto Gx$. A map $f\colon X \to Y$ which for all x in X and g in G satisfies that $f(gx) = f(x)$ is called *G-invariant.*

(iii) If X is a topological space and $\rho\colon G \to \mathrm{Perm}(X)$ is a group action, then $G\backslash X$ comes with the quotient topology and we have everything as in (i) just with continuous maps.

Definition I.5.9 (Quotients of Graphs by Group Actions): Let $\Gamma = (V, E, \delta, \iota)$ be a graph, let G be a group and let $\rho\colon G \to \mathrm{Aut}(\Gamma)$ be a group action without inversions. Then, the data $\bar{V} := G\backslash V = \{Gv \mid v \in V\}$, $\bar{E} := G\backslash E = \{Ge \mid e \in E\}$, $\bar{o}(Ge) = Go(e)$, $\bar{t}(Ge) = Gt(e)$, $\bar{\iota}(Ge) = G\iota(e)$ makes up the *quotient graph*, denoted $G\backslash\Gamma := \rho\backslash\Gamma := \bar{\Gamma} = (\bar{V}, \bar{E}, \bar{\delta} = \bar{o} \times \bar{t}, \bar{\iota})$.

Here, we use the following notations: $\rho = (\rho_V, \rho_E)$ with $gv := \rho_V(g)(v)$, $ge := \rho_E(g)(v)$.

Remark I.5.10 (Well-definednes): Observe the following:

(i) For any g in G, the map $\rho(g) = (\rho_V(g), \rho_E(g))$ is a graph morphism, thus $o(ge) = go(e)$, $t(ge) = gt(e)$ and $\iota(ge) = g\iota(e)$. This shows precisely that $\bar{o}$, $\bar{t}$, $\bar{\iota}$ from Definition I.5.9 are well-defined.

(ii) The quotient graph $\bar{\Gamma}$ is indeed a graph. This is seen as follows: For any edge e of Γ we have $\bar{o}(Ge) = Go(e) = Gt(\iota(e)) = \bar{t}(G\iota(e)) = \bar{t}(\bar{\iota}(Ge))$ and $\bar{\iota}(\bar{\iota}(Ge)) = Ge$ by similar arguments.

(iii) Since G acts without inversions, it holds $\iota Ge \neq Ge$ for any edge e.

Example I.5.11 (Some Quotient Graphs): (i) Let $\Gamma = \mathrm{Cay}(\mathbb{Z}, \{1\}) = \mathsf{Circ}_\infty$ and consider the action ρ_1 given by $\rho_{1,v} \colon \mathbb{Z} \to \mathrm{Perm}(\mathbb{Z})$, $1 \mapsto (z \mapsto z+1)$. Its quotient graph by ρ_1 is a graph with one vertex and one edge. The quotient by the action ρ_3 declared via $\rho_{3,v} \colon \mathbb{Z} \to \mathrm{Perm}(\mathbb{Z})$, $1 \mapsto (z \mapsto z+3)$ is a graph with three vertices and three edges.

(ii) Let $\Gamma = \mathrm{Cay}(G, S)$ be the Cayley graph of some group with finite set of generators S and let $\rho \colon G \to \mathrm{Aut}(\Gamma)$ be the action by left multiplication. Then the quotient graph $G\backslash\Gamma$ again has only one vertex, since G acts transitively on G by the cancellation law. For each generator, we obtain an edge. Hence, $G\backslash\Gamma$ is a rose with $\#S$ many leaves.

(iii) *Sketch missing.* Consider the action $\rho \colon \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(\Gamma)$, given by $\rho_E(1) = (e_1 e_2 e_3)$. Then $G\backslash\Gamma$ is *Sketch missing.*

Remark I.5.12 (Quotient Group): Let G be a group, let $\Gamma = (V, E, \delta, \iota)$ be a graph and let $\rho \colon G \to \mathrm{Aut}(\Gamma)$ be a group action. Then we have the following:

(i) The maps $q_V \colon V \to G\backslash V$, $v \mapsto Gv$ and $q_E \colon E \to G\backslash E$, $e \mapsto Ge$ make up a graph homomorphism $q = (q_V, q_E) \colon \Gamma \to G\backslash\Gamma$.

(ii) The graph morphism q is G-invariant, i.e. for any group element g, $q \circ \rho(g) = q$.

(iii) The graph morphism q has the universal property. More precisely: For each G-invariant graph morphism $f \colon \Gamma \to \Gamma'$, there is a unique graph morphism $\bar{g} \colon G\backslash V \to \Gamma'$ with $\bar{f} \circ q = f$.

(iv) The geometric realisation q^{geom} is continuous, open and G-invariant.

(v) For the geometric realisations it holds $(G\backslash\Gamma)^{\mathrm{geom}} \cong G\backslash\Gamma^{\mathrm{geom}}$. To be more precise we have the commutative diagram

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle q^{\mathrm{geom}}} \swarrow & & \searrow {\scriptstyle q} \\ (G\backslash\Gamma)^{\mathrm{geom}} & \longleftrightarrow & G\backslash\Gamma^{\mathrm{geom}} \end{array}$$

Proof: The first two statements are clear. As for the third statement, $\bar{f}_V : Gv \mapsto f(v)$ and $\bar{f}_E : Ge \mapsto fe$ give rise to a well-defined graph morphism and it is the only possible map.

As for (iv), we already know that q^{geom} is continuous by (Corollary I.4.13) and G-invariant by definition. It remains to show that q^{geom} is open. Let thus U be open in Γ^{geom} and let $y = f(x)$ be a point in $f(U)$. Now we distinguish cases for x.

If $x = (t, e)$ for some edge e and some t in $(0, 1)$, y is the point $y = (t, Ge)$. In this case, choose $\varepsilon < \delta_x := \min\{t, 1-t\}$ such that $B(x, \varepsilon) \subseteq U$. Then $B(f(x), \varepsilon)$ is contained in $f(U)$.

If x is a vertex, then $y = Gv$. Choose $\varepsilon < \delta_x := 1/4$ such that $B(x, \varepsilon)$ is contained in U. Then, for any t' in $(0, \varepsilon)$ and e in E with $o(e) = v$ we have that $x' = (t', e)$ belongs to U, and thus $f(x') = (t, Ge)$ belongs to $f(U)$. Hence, for any t' in $(0, \varepsilon)$ and Ge with $\bar{o}(Ge) = Gv$ it holds $(t', Ge) \in f(U)$. This means that $B(f(x), \varepsilon)$ is contained in $f(U)$.

As for (v), we have to show that the pair $((G\backslash\Gamma)^{\mathrm{geom}}, q^{\mathrm{geom}})$ satisfies the universal property, i.e. for any other topological space Y and a G-invariant map $f\colon \Gamma^{\mathrm{geom}} \to Y$, there is a unique map $\bar{f}\colon (G\backslash\Gamma)^{\mathrm{geom}} \to Y$ such that $f = \bar{f} \circ q^{\mathrm{geom}}$.

Let thus $f\colon \Gamma^{\mathrm{geom}} \to Y$ be a continuous and G-invariant map. Then

$$\bar{f}\colon (G\backslash\Gamma)^{\mathrm{geom}} \longrightarrow Y, \qquad (t, Ge) \longmapsto f((t, e))$$

is our candidate. It remains to show that $\bar{f}$ is continuous. For an open subset U of Y it holds $\bar{f}^{-1}(U) = q^{\mathrm{geom}}(f^{-1}(U))$ by the surjectivity of q^{geom} and by the openness of q^{geom}, the preimage of U under $\bar{f}$ is open, which shows that $\bar{f}$ is continuous. □

6 Trees

In this section, we want to show that a graph is a tree if and only if that graph is contractible. For this section, let $\Gamma = (V, E, \delta, \iota)$ be a graph, let $\omega = (e_1, \ldots, e_n)$ be an edge-path and for each vertex v, denote by ω_v the constant edge-path with origin and terminus v.

Definition I.6.1 (Basic Definitions):

(i) If for the edge-path ω it holds $o(\omega) = t(\omega)$, then ω is called *closed*.

(ii) If for all indices $i \in \{1, \ldots, n\}$ it holds $e_i \neq \bar{e}_{i+1}$, then ω has *no backtracking*.

(iii) If the edge-path ω has no backtracking and if for any distinct indices i and j it holds that $o(e_i) \neq o(e_j)$ and $t(e_i) \neq t(e_j)$, then ω is called *simple*.

(iv) The edge-path $\bar{\omega} = (\bar{e}_n, \ldots, \bar{e}_1)$ is called *inverse edge-path*.

(v) For the additional edge-path $\omega' = (f_1, \ldots, f_m)$ with $o(f_1) = t(e_1)$, the path $\omega\omega' = (e_1, \ldots, e_n, f_1, \ldots, f_m)$ is called *product* or *concatenation*.

(vi) For the edge-path ω, the number $\ell(\omega) = n$ is called *combinatorial length* or briefly *length of* ω.

(vii) If the edge-path ω is closed and has no backtracking, then ω is called a *cycle*.

Note that some authors require cycles to be simple as well.

Definition I.6.2 (Tree): Let $\Gamma = (V, E, \delta, \iota)$ be a graph. If V is non-empty, if Γ is connected and if Γ has no simply cycle ω of length greater then zero, then Γ is called a *tree*.

Proposition I.6.3 (Basic Properties):

(i) *Let $\omega = (e_1, \ldots, e_n)$ be a cycle of length $\ell(\omega) \geq 1$. Then ω contains a simple cycle.*

(ii) *A graph Γ is a tree if and only if V is non-empty and if for any two vertices u and v there is a unique edge-path without backtracking from u to v.*

Proof: Statement (i) is clear. For (ii), there are two assertions. "$\Longrightarrow$": By assumption, V is non-empty, and it is easy to see that V is connected. Remains to show that V doesn't contain simple cycles of positive length.

Suppose Γ contained a simple cycle $\omega = (e_1, \ldots, e_n)$ with origin e_1 and end e_1. Then both ω and the constant edge-path ω_v were both without backtracking, contradicting the assumption.

"$\Longleftarrow$": The existence of a path $\omega_{u,v}$ from u to v follows from the connectedness of Γ, as we can remove backtracking inductively. Assume now there were two paths $\omega_{u,v} = (e_1, \ldots, e_n)$ and $\omega'_{u,v} = (f_1, \ldots, f_m)$ without backtracking from u to v. Denote $\omega := \omega_{u,v}\bar{\omega}'_{u,v}$. If the length of ω were zero, then both $\omega_{u,v}$ and $\bar{\omega}_{u,v}$ were empty.

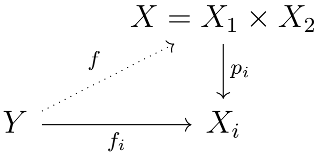
If not, then we inductively rempte backtracking to obtain a simple cycle. □

Reminder I.6.4: Let $(X_1, \mathfrak{T}_1)$ and $(X_2, \mathfrak{T}_2)$ be topological spaces and denote by p_i the projection $p_i\colon X_1 \times X_2 \to X_i$.

(i) The sets of the form $\{p_i^{-1}(U) \mid U \in \mathfrak{T}_i\}$ make up a subbasis for a topology on $(X_1 \times X_2)$. It is the coarsest topology rendering continuous the projections p_1 and p_2, called *product topology.*

A sequence $(x_n, y_n)_{n \in \mathbb{N}}$ converges to (x, y) in $X \times Y$ with respect to the product topology if and only if x_n converges to x in X and y_n converges to y in Y.

We have the following universal property: If Y is another topological space and if there are continuous maps $f_i \colon Y \to X_i$, then there is one and only one continuous map $f \colon Y \to X$ such that $p_i \circ f = f_i$. This is captured by the following diagram:



(ii) Denote by I the closed unit interval and let $f_1, f_2 \colon X \to Y$ be continuous maps. If there is a continuous map $H \colon X \times I \to Y$ such that for any x it holds $H(x, 0) = f_1(x)$ and $H(x, 1) = f_2(x)$, then f_1 and f_2 are called *homotopic.*

Let $f \colon X \to Y$ be a continuous map. If there is a continuous map $g \colon Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y, then f is called a *homotopy equivalence.*

If f is homotopic to a contant map, i.e. if there are y in Y and a continuous map $H \colon X \times I \to Y$ such that for any x in H it holds $H(x, 0) = f(x)$ and $H(x, 1) = y$, then f is called *null-homotopic.*

(iii) Let $(X, \mathfrak{T})$ be a topological space. If id_X is null-homotopic, i.e. if there are a point x_0 in X and a continuous map $H \colon X \times I \to X$ such that for any x in X it holds $H(x, 0) = x$ and $H(x, 1) = x_0$, then the space is called *contractible.*

Let T denote a tree, let X be its geometric realisation $X = T^{\mathrm{geom}}$ and let x_0 be a point in X. For any point x in X there is a unique geodesic c_x from x_0 to x. Denote $d_x := d(x_0, x)$ and define

$$H \colon X \times I \longrightarrow X, \qquad (x, t) \longmapsto c_x(d_x t).$$

Then, for any x in X it holds $H(x, 0) = x$ and $H(x, 1) = x_0$. It remains to show that H is indeed continuous. Instead of verifying continuity in this special situation, we will move to a more general statement.

Definition I.6.5 (Geodesics and Friends): Let (X, d) be a metric space.

(i) A continuous map $\alpha \colon [a, b] \to X$ is called a *path.*[2]

[2]For this notion it is sufficient for X to be merely a topological space.

(ii) Let $c\colon [a,b] \to X$ be a map. If c is isometric, i.e. if for any t_1, t_2 in $[a,b]$ it holds $d(c(t_1), c(t_2)) = |t_1 - t_2|$, then c is called a *geodesic*.[3] The image $c[a,b]$ is called a *geodesic segment*. Observe that in this case $b - a = d(c(t_1, c(t_2))$.

(iii) Let $c\colon [a,b] \to X$ be a geodesic. If there is a constant $\lambda > 0$ such that for any t_1, t_2 in $[a,b]$ it holds $d(c(t_1), c(t_2)) = \lambda|t_1 - t_2|$, then c is called a *constant speed geodesic*. Its image is again a geodesic segment.

(iv) Let $c\colon [a,b] \to X$ be a map. If for any point t in $[a,b]$ there is an open neighbourhood U of t such that $c|_U$ is a geodesic, then c is called a *local geodesic*.

Do be warned. Sometimes local geodesics are called geodesics, e.g. in the context of translation surfaces or differential geometry in general.

Definition I.6.6 (Geodesic Spaces): Let (X, d) be a metric space.

(i) If for any two points x and y in X there is a geodesic segment between them, then the space is called a *geodesic space*.

(ii) If for any two points x and y in X there is a unique geodesic segment between then, then the space is called *uniquely geodesic space*.

(iii) Let x_1, x_2 and x_3 be points in X and let $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$ be geodesic segments between the respective points. Their union $\Delta := [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ is called a *geodesic triangle with vertices x_1, x_2 and x_3*.

Example I.6.7: (i) The plane $(\mathbb{R}^2, d_E)$, equipped with the Euclidean metric d_E, is a uniquely geodesic space. In this space, geodesics are precisely lines, geodesic triangles are ordinary triangles as known from elementary geometry.

(ii) Trees are uniquely geodesic spaces. This was shown on Exercise Sheet 2.

Definition I.6.8 (Comparison Triangle): Let (X, d) be a metric space and let $\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ be a geodesic triangle. Choose in $(\mathbb{R}^2, d_E)$ a geodesic triangle $\bar{\Delta} = [\bar{x}_1, \bar{x}_2] \cup [\bar{x}_2, \bar{x}_3] \cup [\bar{x}_3, \bar{x}_1]$ with vertices $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ such that $d(x_i, x_j) = d(\bar{x}_i, \bar{x}_j)$. Then $\bar{\Delta}$ is called *comparison triangle for* Δ.

For each p in $[x_1, x_2]$ denote by $\bar{p} \in [\bar{x}_1, \bar{x}_2]$ the unique point with $d(\bar{x}_1, p) = d(x_1, p)$ which is equivalent to $d(\bar{x}_2, \bar{p}) = d(x_2, p)$. Similarly for p in $[x_2, x_3]$ and p in $[x_3, x_1]$.

[3] Note that geodesics are in particular paths, because the isometric property implies sequential continuity and thus continuity.

Definition I.6.9 (CAT0 Space): Let (X, d) be a geodesic space. If for all geodesic triangles $\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ and for all p and q in Δ it holds that $d(p, q) \leq d(\bar{p}, \bar{q})$, where $\bar{p}$ and $\bar{q}$ are corresponding points in the comparison triangle Δ, the space (X, d) is called CAT(0).

Example I.6.10: The plane $(\mathbb{R}^2, d_E)$ is a CAT(0) space.

In the following, we will show that Trees are CAT(0) spaces, and then we will show that CAT(0) spaces are contractible.

Proposition I.6.11: *Let T be a tree and let $X := T^{\text{geom}}$ with graph metric d. Then, we have the following: For any points x_1, x_2 and x_3 in X and the geodesic segments $[x_1, x_2]$, $[x_2, x_3]$ with $[x_1, x_2] \cap [x_2, x_3] = \{x_2\}$, it holds $[x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$.*

Proof: It follows from Proposition I.6.3 that the geodesic between two points is the unique path without backtracking. □

Proposition I.6.12 (Trees are CAT0): *Let (X, d) be a geodesic space such that (X, d) is uniquely geodesic and such that it holds "For any x_1, x_2 and x_3 in X with $[x_1, x_2] \cap [x_2, x_3] = \{x_2\}$, $[x_1, x_2] \cup [x_2, x_3] = [x_1, x_3]$". Then we have:*

(i) *Any geodesic triangle $\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ has a* triple point, *i.e. there is some point m in Δ with $[x_1, x_2] = [x_1, m] \cup [m, x_2]$, $[x_2, x_3] = [x_2, m] \cup [m, x_3]$ and $[x_3, x_1] = [x_3, m] \cup [m, x_1]$.*

(ii) *(X, d) is CAT0.*

Proof: (i) We show that $[x_1, x_2] \cap [x_2, x_3] \cap [x_1, x_3] = \{m\}$ for some point m in X. Denote $d_1 := d(x_2, x_3)$, $d_2 = d(x_1, x_3)$ and $d_3 = d(x_1, x_2)$. Furthermore, let $c_3 \colon [0, d_3] \to X$ and $c_2 \colon [0, d_2] \to X$ be geodesics with $c_3(0) = x_1 = c_2(0)$ and $c_3(d_3) = x_2$ and $c_2(d_2) = x_3$.

Let $t_0 := \max\{t \in [0, d_3] \mid c_3(t) \in [x_1, x_3]\} = \max\{t \in [0, 1] \mid c_3(t) = c_2(t)\}$. We claim that $m := c_3(t) = c_2(t)$ does the trick.

Firstly, observe that for $t \leq t_0$ we have $c_1(t) = c_2(t)$, as $c_2|_{[t,t_0]} = [x, m] = c_1|_{[t,t_0]}$. If on the other hand $t > t_0$, then $c_3(t)$ doesn't belong to $[x_1, x_3]$ and thus $[x_1, x_2] \cap [x_1, x_3] = c_3([0, t_0]) = c_2([0, t_0])$.

Secondly, observe that $[x_2, m] = c_3([t_0, d_3]) \cap c_2([t, d_2]) = [m, x_3] = \{m\}$. Hence, $[x_2, m] \cup [m, x_3] = [x_2, x_3]$ and $[x_1, x_2] \cap [x_1, x_3] \cap [x_2, x_3] = \{m\}$ as $[x_1, x_2] \cap [x_1, x_3] = c_3([0, t_0]) = c_2([0, t_0]$ and $[x_2, x_3] = c_3([t_0, d_3) \cup c_2([t_0, d_2])$.

(ii) Let $\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ be a geodesic triangle and let m be the triple point of this triangle. Let $\bar{\Delta} = [\bar{x}_1, \bar{x}_2] \cup [\bar{x}_2, \bar{x}_3] \cup [\bar{x}_3, \bar{x}_1]$ be the comparison triangle in $(\mathbb{R}^2, d_E)$. Finally, let p, q be two points on different sides of Δ and let $\bar{p}$, $\bar{q}$ be the corresponding points on $\bar{\Delta}$. We will now distinguish cases.

Case 1: Assume p and q lie on the same leg of Δ, without loss of generality we may assume that they lie on $[x_1, m]$. Furthermore, without loss of generality we may assume that $d(x_0, p) \leq d(x_1, q)$. Choose $\bar{q}'$ on $[\bar{x}_1, \bar{x}_2]$ such that $d(\bar{x}_1, \bar{q}') = d(x_1, q) = d(\bar{x}_1, \bar{q})$. Then

$$d(p, q) - d_E(\bar{p}, \bar{q}') \leq d_E(\bar{p}, \bar{q}),$$

since $\bar{q}$ and $\bar{q}'$ describe an equilateral triangle in our comparison triangle, and then elementary arguments do the trick.

Case 2: Assume p and q lie on different legs. Without loss of generality, we may assume that p lies on $[m, x_2]$ and that q lies on $[m, x_3]$. As we are in a unique geodesic space, we know that

$$d(x_2, m) + d(x_3, m) = d(x_2, x_3) = d_E(\bar{x}_2, \bar{x}_3) \leq d_E(\bar{x}_2, \bar{p}) + d_E(\bar{p}, \bar{q}) + d_E(\bar{p}, \bar{x}_3).$$

Because $d_E(\bar{x}_2, p) = d(x_2, m) - d(m, p)$ and $d_E(p, x_3) = d(x_3, m) - d(m, q)$, we obtain by plugging in and cancelling that

$$d(p, q) = d(m, p) + d(m, q) \leq d_E(\bar{p}, \bar{q}).$$

Therefore, (X, d) is a CAT(0)-space. □

Definition I.6.13 ($\mathbb{R}$-Tree): Let (X, d) be a geodesic space. If (X, d) is uniquely geodesic and if for any x_1, x_2, x_3 in X with $[x_1, x_2] \cap [x_2, x_3] = \{x_2\}$ it holds that $[x_1, x_2] \cup [x_2, x_3] = [x_1, x_3]$, then (X, d) is called an $\mathbb{R}$-*tree*.

In particular, the previous proposition shows that $\mathbb{R}$-trees are CAT(0).

Proposition I.6.14 (CAT(0)-Spaces are Convex): *Every CAT(0)-space (X, d) is convex, i.e. for any pair of constant speed geodesics $c, c' \colon [0, 1] \to X$ the point-wise distance function $t \mapsto c(t) - c'(t)$ is a convex function, which means that for any $t \in [0, 1]$ it holds $d(c(t), c'(t)) \leq (1 - t)d(c(0), c'(0)) + td(c(1), c'(1))$.*

Proof: We establish our claim in two steps. First, assume that c and c' share the same starting point, that is $c(0) = c'(0)$. We consider the triangle with edges $c(0)$, $c(1)$ and $c'(1)$ and its comparison triangle with edges $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$, where the length of the side from $\bar{x}_1$ to $\bar{x}_2$ is $d_3 = d(c(0), c(1))$ and where the length of the side from x_1 to x_3 is $d_2 = d(c'(0), c'(1))$. Then it holds

$$d(\bar{c}'(t), \bar{c}(t)) = td(\bar{x}_2, \bar{x}_3) = td(x_2, x_3) = t(c(1), c'(1)).$$

As our space is CAT(0), it follows $d(c(t), c'(t)) \leq d(\bar{c}(t), \bar{c}'(t)) \leq td(c(1), c'(1))$.

Secondly, we allow $c(0)$ to be distinct from $c'(0)$. Denoting by c'' the the geodesic from $c(0)$ to $c'(1)$, we obtain from the first consideration that $d(c(t), c''(t)) \leq ts(c(1), c''(1))$ and for applied to the inverse path $\bar{c}''$ to c'', we get $d(c''(1-t), c'(1-t)) \leq td(c(0), c'(0))$. By triangular inequality it holds

$$\begin{aligned} d(c(t), c'(t)) &\leq d(c(t), c''(t)) + d(c''(t), c'(t)) \\ &\leq td(c(1), c''(1)) \leq td(c(1), c''(1)) + (1-t)d(c(0), c'(0)) \end{aligned}$$

which we wanted to show. □

Remark I.6.15: On an exercise sheet, you will show that if (X, d) is CAT(0), then (X, d) is uniquely geodesic.

Definition I.6.16 (Geodesics Vary Continuously With Their Endpoints): Let (X, d) be a uniquely geodesic space. If for any constant speed geodesic $c\colon [0,1] \to X$ from x to y and any sequence $(c_n\colon [0,1] \to X)_{n\in\mathbb{N}}$ of constant speed geodesics with $\lim_{n\to\infty} c_n(0) = c(0) = x$ and $\lim_{n\to\infty} c_n(1) = c(1) = y$ it holds $\lim_{n\to\infty} \|c_n - c\|_\infty = 0$, then we say that in (X, d), *geodesics vary continuously with their endpoints.*

Proposition I.6.17: *In any CAT(0)-space (X, d), geodesics vary continuously with their endpoints.*

Proof: Let $c\colon [0,1] \to X$ and $(c_n\colon [0,1] \to X)_{n\in\mathbb{N}}$ be constant speed geodesics, let $x = c(0)$ and $y = c(1)$ and assume that $c_n(0) \to c(0)$ as well as $c_n(1) \to c(1)$. By convexity, for any $t \in [0,1]$ it holds

$$d(c(t), cn(t)) \leq (1-t)d(c(0), c_n(0)) + td(c(1), c_n(1)) \leq d(x, c_n(0)) + d(y, c_n(1))$$

which, by assumption, implies uniform convergence. □

Theorem 2 (CAT(0)-Spaces are Contractible): *Any CAT(0)-space (X, d) is contractible.*

Proof: Fix a point x_0 in X and let $c_x\colon [0,1] \to X$ be the unique constant speed geodesic from x to x_0. Define

$$H\colon X \times [0,1] \longrightarrow X, \qquad (x,t) \longmapsto c_x(t).$$

By Proposition I.6.17 this map H is continuous. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to x and suppose $(t_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ converging to t. For $c_n := c_{x_n}$ we obtain by Proposition I.6.17 that c_n converges to c uniformly, i.e. in particular $c_n(t_n) \to c(t)$. □

Corollary I.6.18 (Trees are Contractible): *The geometric realisation of a tree is contractible.*

In the following, we want to show that also the converse is true, i.e. contractible graphs are trees.

Proposition I.6.19: *Let Γ be a graph, let T be a subtree of Γ and let Γ/T be the graph obtained by collapsing T[4] with collapse map $p\colon \Gamma \to \Gamma/T$. Then p is an homotopy equivalence.*

Proof: This will be an exercise on Exercise Sheet 5. □

Proposition I.6.20 (Contractible Graphs are Trees): *Let X be the geometric realisation of a graph Γ. If X is contractible, then Γ is a tree.*

Lemma I.6.21 (Topological Basics): *Let $f_1, f_2\colon X \to Y$ be continuous maps between topological spaces. If f_1 and f_2 are homotopic, we write $f_1 \sim f_2$.*

(i) *Being homotopic is an equivalence relation. If $f_1, f_2\colon X \to Y$ are continuous and homotopic and $g_1, g_2\colon Y \to Z$ are continuous and homotopic, then $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic.*

(ii) *Let $f\colon X \to Y$ be a homotopy equivalence. Then X is contractible if and only if Y is contractible.*

(iii) *If X is a contractible space and if $\gamma\colon [0,1] \to X$ is a closed path, i.e. $\gamma(0) = \gamma(1)$, then γ is null-homotopic.*

Proof: Statements (i) and (ii) will be on exercise sheets. As for assertion (iii), suppose X is contractible. Then there is some point x_0 in X such that $\mathrm{id}_X \sim \mathbf{x}_0$, where $\mathbf{x}_0$ denotes the path that is constantly x_0. Hence, by (i) for the path γ it holds $\mathbf{x}_0 \circ \gamma \sim \mathrm{id}_X \circ \gamma = \gamma$, i.e. γ is null-homotopic. □

[4] See Exercise 1 on Exercise Sheet 3.

To show the above proposition, we will export the main argument into a lemma.

Lemma I.6.22 (Cycles in Graphs): *Let $c = (c_1, \dots, c_n)$ be a simple cycle in the graph Γ of length $n \geq 1$ and let $\gamma_c\colon [0, n] \to \Gamma^{\text{geom}} =: X$, $k + s \mapsto (s, e_{e_{k+1}})$, where $k \in \{0, \dots, n-1\}$ and $s \in [0, 1]$ a path realising the cycle c. Then γ_c is not null-homotopic.*

Proof: In the arguments, we will leave a gap to be filled later. As first step, we show that we may assume that $n = 1$. Suppose $n \geq 2$ and consider the tree spanned by the edges $e_1, \dots, e_{n-1}$. By Proposition I.6.20, the projection map $p\colon \Gamma^{\text{geom}} \to (\Gamma/T)^{\text{geom}}$ is a homotopy equivalence. In particular the path γ_c is null-homotopic if and only if $p \circ \gamma_c$ is null-homotopic, but $p \circ \gamma_c$ is the realisation (up to reparametrisation) of a loop.

As second step, we show the statement for $n = 1$, i.e. for a cycle $c = (e_1)$ with $o(e_1) = t(e_1) = v_0$. Suppose there were a homotopy $H\colon [0, 1] \times [0, 1] \to X$ with $H(1, s) = \gamma_c(s)$ and $H(0, s) = x$ for some point x in X. We may assume that $x = v_0$. Define $\tilde{H}\colon [0, 1] \times [0, 1] \to e_1^{\text{geom}} = \{(t, e_1) \mid t \in [0, 1]\}$ with

$$(t, s) \longmapsto \begin{cases} H(t, s), & \text{if } H(t, s) \in e_1^{\text{geom}}, \\ v_0, & \text{otherwise.} \end{cases}$$

This is continuous, since of an open subset U of e_1^{geom} it holds

$$\tilde{H}^{-1}(U) = \begin{cases} H^{-1}(U), & \text{if } v_0 \notin U, \\ H^{-1}(U) \cup H^{-1}(\Gamma^{\text{geom}} - e_1^{\text{geom}}), & \text{if } v_0 \in U. \end{cases}$$

Therefore, $\tilde{H}$ is a homotopy in e^{geom}, which yields that e^{geom} is contractible. But e^{geom} is homeomorphic to $\mathbb{S}^1$, which is not contractible as we will see later. □

Proof (of Proposition I.6.21): Assume $X = \Gamma^{\text{geom}}$ were contractible. Then there were x_0 in X and a homotopy map $H\colon [0, 1] \times X \to X$ between id_X and $\mathbf{x}_0$. Thus, firstly X were non-empty and X were connected due to the maps $H_x\colon [0, 1] \to X$, $t \mapsto H(t, x)$. Furthermore, Γ had no simply cycle, due to Lemma I.6.23 (cycles in graphs are not null-homotopic) and Lemma I.6.21 (loops on contractible spaces are null-homotopic). □

Theorem 3: *A graph is a tree if and only if its geometric realisation is contractible.*

7 Free Groups

Consider the set $X = \{x, y\}$ and let $W(X)$ be the set of all words with letters in X. For two words w_1 and w_2 in W, the concatenation gives a new word in W. For example, $w_1 = xyx$ and $w_2 = yxxy$, their concatenation is $w_1 \star w_2 := xyxyxxy$. If we now add to our alphabet the corresponding inverse letters $X' = \{x^{-1}, y^{-1}\}$, then $W(X \cup X')$ turns into a group with concatenation of words.

For this section, the letter X will always denote some set.

Definition I.7.1 (The Monoid of Words):

(i) The set $W(X) := \{(a_1, \dots, a_n) \mid n \in \mathbb{N}, a_1, \dots, a_n \in X\} \cup \{\varepsilon\}$ denotes the set of *words with letters in* X and ε denotes the empty word.

For a word $w = (a_1, \dots, a_n)$, the number n is called *length of* w, denoted $\operatorname{len}(w)$. For ε, we define $\operatorname{len}(\varepsilon) = 0$.

We write

$$\begin{aligned} \star\colon W(X) \times W(X) &\longrightarrow W(X), \\ ((a_1, \dots, a_n), (b_1, \dots, b_m)) &\longmapsto (a_1, \dots, a_n, b_1, \dots, b_m) \end{aligned}$$

for the *concatenation of words* and define $\star((a_1, \dots, a_n), \varepsilon) := (a_1, \dots, a_n)$ as well as $\star(\varepsilon, (b_1, \dots, b_m)) := (b_1, \dots, b_m)$. Observe that "$\star$" is associative and ε is a neutral element, i.e. $(W(X), \star)$ carries the structure of a monoid.

(ii) Identify X with the set $X \times \{1\}$ and define $X' := X \times \{-1\}$. For an element x of X we write $x = (a, 1)$ and we call $x^{-1} := (a, -1)$ in X' the inverse. Similarly, for an element $x = (a, -1)$ of X', we denote $x^{-1} := (a, 1)$.

If a word $w \in W(X \cup X')$ does not contain a subword of the form xx^{-1} for $x \in X \cup X'$, we call the word w *reduced*.

We write $w \overset{(1)}{\to} w'$ if w' is obtained from w by a single cancellation of a subword xx^{-1}. Similarly, we write $w \to w'$, if there is a finite sequence $w_1, \dots, w_k$ such that $w \overset{(1)}{\to} w_1 \overset{(1)}{\to} \dots \overset{(1)}{\to} w_k = w'$.

Observe that for every word w in $W(X \cup X')$, there is a word w' in $W(X \cup X')$ such that w' is reduced and $w \to w'$.

Example I.7.2: Consider the word $babb^{-1}a^{-1}c^{-1}ca$. By cancellation, we could obtain $baa^{-1}c^{-1}a$ and then $bcc^{-1}a$ and then ba. But we could also have proceeded in a different way, e.g. we could come to $babb^{-1}aa$, then $babb^{-1}$ and then ba.

It is thus obvious that "reducing sequences" are not unique, but their outcomes better be!

Proposition I.7.3 (Uniqueness of the Reduced Form): *Let w in $W(X \cup X')$ be a word. Then w has a unique reduced form.*

Proof: We show the statement via induction on the length $n = \mathrm{len}(w)$.

If $n = 0$, then $w = \varepsilon$, where there is nothing to be done.

Suppose now the claim held for words of the length n and assume w had length $n + 1$. If w were reduced, then the claim held. If w were not reduced, there were a subword xx^{-1} for some x from our alphabet. By cancelling this subword xx^{-1} from w, we could obtain the word w_1. We show that any reduced form $\hat{w}$ of w is also a reduced form of w_1. This then yields the claim by induction.

Let $\hat{\omega}$ be a reduced form of w. If at some point in the cancellation sequence we cancel this pair xx^{-1}, then we can change the order of cancellation and start with this cancellation, i.e. $\hat{w}$ is also a reduced form of w_1.

If the subword xx^{-1} we started with is never cancelled, then at least one of the individual letters has to be cancelled "from the left" respectively "from the right", because $\hat{w}$ is reduced. In both cases, we obtain the same word if we cancel the initial pair. Hence we obtain the claim by the first case. □

Definition I.7.4 (Equivalence):

(i) Let w be a word in $W(X \cup X')$. Then w^{red} denotes the reduced form of w.

(ii) If two words w_1, w_2 in $W(X \cup X')$ have the same reduced form $w_1^{\mathrm{red}} = w_2^{\mathrm{red}}$, we call both words equivalent. This declares an equivalence relation "$\sim$" and by $[w]$ we denote the equivalence class of w with respect to "$\sim$" .

Proposition I.7.5: *Let w_1, w_1', w_2, w_2' be words in $W(X \cup X')$ such that $w_1 \sim w_1'$ and $w_2 \sim w_2'$. Then $w_1 \star w_2$ is equivalent to $w_1' \star w_2'$.*

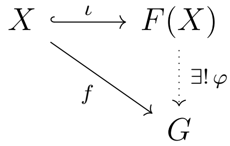
Proof: Denote $\hat{w} = (w_1 \star w_2)^{\mathrm{red}}$ and $\hat{w}' := (w_1' \star w_2')^{\mathrm{red}}$. To obtain $(w_1 \star w_2)^{\mathrm{red}}$, proceed as follows: First, cancel as much as possible in w_1. Then, cancel as much as possible in w_2. Then cancel in the result what can be cancelled. Hence

$$\hat{w} = (w_1^{\mathrm{red}} \star w_2^{\mathrm{red}})^{\mathrm{red}} = ({w_1'}^{\mathrm{red}} \star {w_2'}^{\mathrm{red}})^{\mathrm{red}} = \hat{w}'. \qquad \square$$

Theorem 4 (Free Group): *Let X be a set and let X' be the corresponding disjoint copy.*

(i) *The set $F(X) := W(X \cup X')/{\sim}$ with the operation "$\cdot$" defined by $[w_1] \cdot [w_2] = [w_1 \star w_2]$ is a group called* free group*.*

(ii) *The map $\iota\colon X \mapsto F(X)$, $x \mapsto [x]$ is an embedding and we have the following universal property: For any group G and a map of set $f\colon X \to G$, there is one and only one group homomorphism $\varphi\colon F(X) \to G$ such that $\varphi \circ \iota = f$, i.e. for any group G and any map of sets f, we have the following commutative diagram:*



(iii) *If (H, ι') is a group H together with a map $\iota'\colon X \to H$ with the same property as $(F(X), \iota)$, there is a unique group isomorphism $\theta\colon F(X) \to H$ such that $\theta \circ \iota = \iota'$.*

Proof: (i) We have already shown that "$\cdot$" is well-defined. Furthermore, it is associative as "$\star$" is and $1 = [\varepsilon]$ is a neutral element for this law of composition on $F(X)$. For a word $w = (x_1, \dots, x_n)$ for $x_i \in X \cup X'$, the word $w' = (x_1^{-1}, \dots, x_n^{-1})$ is its inverse.

(ii) We define a map $\varphi'\colon W(X \cup X') \to G$ as follows: If $w = (x_1, \dots, x_n)$ is a word with $\mathrm{len}(w) \geq 1$, send w to $f(x_1) \cdots f(x_n)$ and if $\mathrm{len}(w) = 0$, w must be the empty word and we should send w to 1_G. Here, for $x = a^{-1} \in X'$ we denote $f(x) = (f(a))^{-1}$. In particular, we have $f(x)f(x^{-1}) = 1_G$.

Then clearly $\varphi'(w_1 \star w_2) = \varphi'(w_1) \cdot \varphi'(w_2)$ and for equivalent words w_1 and w_2 it holds $\varphi'(w_1) = \varphi'(w_2)$, because φ' plays nicely with inverses. Now defining $\varphi\colon F(X) \to G$, $[w] \mapsto [\varphi'(w)]$ does the trick. Because of the way X is embedded in $F(X)$, there is no other choice for φ'.

(iii) This is shown as usually. □

Notation I.7.6: In the following, we will confound w with its equivalence class $[w]$ and by abuse of notation, we write $w_1 = w_2$ if indeed $w_1 \sim w_2$. In particular, $w_1 x x^{-1} w_2 = w_1 w_2^{-1}$ for any x in $X \cup X'$. One can identify $[w]$ with w^{red}. If X is the finite set $\{x_1, \dots, x_n\}$, one usually writes $F(x_1, \dots, x_n)$ for $F(\{x_1, \dots, x_n\})$ and calls this group the free group on n generators.

Example I.7.7: For the empty set X, the free group $F(X)$ is the trivial group.

If X is a singleton $\{x\}$, then $F(X) = \{x^k \mid k \in \mathbb{Z}\}$ is isomorphic to $(\mathbb{Z}, +)$.

If X is the set $\{x, y\}$, then $F(X) = \{1, x, y, x^{-1}, y^{-1}, xx, xy, xy^{-1}, yx, yy, \dots\}$. The Cayley graph of $F(X)$ with $S = \{x, y\}$ is the four-valent tree.

Proposition I.7.8: *Let G be a group and let S be a subset of G. The free group $F(S)$ is isomorphic to G if and only if the Cayley graph of G is a tree.*

Proposition I.7.9 (Groups as Quotients of Free Groups): *Each group G is a quotient group of a free group. That is $G \cong F(X)/N$ for some set X and some normal subgroup N of $F(X)$.*

Proof: Let S be a generating system of G. If everything else fails, we can always pick S to be G itself. Let $f\colon S \hookrightarrow G$ be the embedding. By the universal property of $F(S)$ there is one and only one group homomorphism $\varphi\colon F(S) \to G$ such that for the embedding $\iota\colon S \to F(S)$ it holds $\varphi \circ \iota = f$. Since S is a generating set of G, the homomorphism φ is surjective and by the homomorphism theorem, $G \cong F(S)/\ker(\varphi)$. ☐

Definition I.7.10: Let G be a group and let R be some subset of G. Then

$$\langle\langle R\rangle\rangle := \bigcap (N \mid N \triangleleft G \text{ with } R \subseteq N) = \left\{ \prod_{i=1}^{n} g_i r_i g_i^{-1} : g_i \in G, r_i \in R \cup R^{-1} \right\}$$

is called the *normal subgroup normally generated by R*.

Let now X be a set, let R be a subgroup of $F(X)$ and let $G = F(X)/\langle\langle R\rangle\rangle$. Then we call $\langle X|R\rangle$ a *presentation of G*. If both $R = \{r_1, \dots, r_k\}$ and $X = \{x_1, \dots, x_n\}$ is finite, then we also write $\langle x_1, \dots, x_n | r_1, \dots, r_k\rangle$ instead of $\langle X|R\rangle$.

We further write $\langle X | r_1 = r_1', \dots, r_k = r_k'\rangle$ for the presentation $\langle X|R\rangle$ where $R = \{r_1', {r_1'}^{-1}, \dots, r_k', {r_k'}^{-1}\}$.

We also write $G = \langle X|R\rangle$ to mean $G = F(X)/\langle\langle R\rangle\rangle$.

Example I.7.11: Consider the group $G = \langle x, y \mid xy = yx\rangle$, i.e. $R = \{xyx^{-1}y\}$. Then one can show that $G = \mathbb{Z}^2$.

Definition I.7.12 (Commutator Subgroup, Abelianisation): Let G be a group. Then the set $[G,G] := \langle\{[g_1, g_2] = g_1 g_2 g_1^{-1} g_2 \mid g_1, g_2 \in G\}\rangle$ is called the *commutator subgroup of G* and is indeed a subgroup. It has the following properties:

(i) The commutator subgroup is a normal subgroup of G, and $G/[G,G]$ is abelian.

(ii) The quotient $G/[G,G]$ is the "biggest abelian image of G", more precisely: $G/[G,G]$ together with the quotient map $q\colon G \to G/[G,G]$ has the following universal property. For any abelian group A and any homomorphism $\varphi\colon G \to A$ the is one and only one homomorphism $\bar{\varphi}\colon Q \to A$ such that $\bar{\varphi} \circ q = \varphi$.

A pair (Q, q) consisting of an abelian group Q and a morphism $q\colon G \to Q$ such that the property (ii) holds is called *abelianisation of* G. Any two abelianisations (Q_1, q_1) and (Q_2, q_2) are uniquely isomorphic to each other, i.e. there is a unique isomorphism $\varphi\colon Q_1 \to Q_2$ such that $\varphi \circ q_1 = q_2$. In this case, we thus denote by G^{ab} "the" abelianisation of G.

Proof: (i) Let g be an element of G and let a commutator $[g_1, g_2]$ be given. Then a quick calculation shows that $[gg_1g^{-1}, gg_2g^{-1}] = g[g_1, g_2]g^{-1}$, thus $[G, G]$ is a normal subgroup, hence we can form $G/[G, G]$. For elements $\bar{a}$ and $\bar{b}$ in $G/[G, G]$ it holds $\bar{a}\bar{b} = \bar{b}\bar{a} \Leftrightarrow aba^{-1}b^{-1} \in [G, G]$, i.e. $G/[G, G]$ is abelian.

(ii) By the Fundamental Theorem on Homomorphisms, the map φ descends to the quotient if and only if $[G, G]$ is contained in $\ker\varphi$. As for any g_1, g_2 in G it holds

$$\varphi([g_1, g_2]) = \varphi(g_1)\varphi(g_2)\varphi(g_1)^{-1}\varphi(g_2)^{-1} = 1_A,$$

we obtain the desired universal property.

(iii) This is the same argument as always. □

Corollary I.7.13 (Properties of Commutator): *Let G be a group.*

(i) *The commutator $[G, G]$ is the smallest normal subgroup of G such that the quotient is abelian. More precisely: For any normal subgroup N of G, whose quotient G/N is abelian, contains $[G, G]$.*

(ii) *If S is a generating system of G, then $[S, S] := \{[s_1, s_2] \mid s_1, s_2 \in S\}$ generates $[G, G]$ as a normal subgroup, i.e. $[G, G] = \langle\langle [S, S] \rangle\rangle$.*

Proof: Statement (i) follows directly from Definition I.7.12(ii). As for (ii): We have that $N := \langle\langle [S, S] \rangle\rangle$ is a subgroup of $[G, G]$. Furthermore, in $A := G/N$ for two elements $\bar{a}$ and $\bar{b}$ of A, we may write $\bar{a} = \bar{s}_1 \cdots \bar{s}_k$, $\bar{b} = \bar{s}_{k+1}, \ldots, \bar{s}_{k+\ell}$ with suitable $s_1, \ldots, s_{k+\ell}$ in $S \cup S^{-1}$. By definition of N the elements $\bar{s}_i$, $\bar{s}_j$ commute, i.e. $\bar{a}\bar{b} = \bar{b}\bar{a}$. Hence $[G, G] \subseteq N = \langle\langle [S, S] \rangle\rangle$ by Definition I.7.12(ii).□

Proposition I.7.14 (Presentation of $\mathbb{Z}^n$):

(i) *For the free group $G = F(x_1, \ldots, x_n)$, it holds $G^{\mathrm{ab}} \cong \mathbb{Z}^n$.*

(ii) *$F(x_1, \ldots, x_n)/\langle\langle x_i x_j x_i^{-1} x_j^{-1} \mid i, j \in \{1, \ldots, n\} \rangle\rangle \cong \mathbb{Z}^n$.*

In particular, $G = \langle x_1, \ldots, x_n \mid x_i x_j = x_j x_i, i, j \in \{1, \ldots, n\} \rangle$ is a presentation of $\mathbb{Z}^n$.

Definition I.7.15 (Free Product): Let $G_1 = \langle X_1 | R_1 \rangle$ and $G_2 = \langle X_2 | R_2 \rangle$ be two group presentations. Then $G_1 \star G_2 := \langle X_1 \cup X_2 | R_1 \cup R_2 \rangle$ is called *free product of G_1 and G_2.*

Proposition I.7.16 (Universal Property of Free Product): *Let G_1 and G_2 be two group presentations. Then $G_1 \star G_2$ has the following properties:*

(i) *$G_1 \star G_2$ depends on the chosen representations only up to unique isomorphism.*

(ii) *The free product $G_1 \star G_2$ comes with natural embeddings $\iota_1 \colon G_1 \hookrightarrow G_1 \star G_2$ and $\iota_2 \colon G_2 \hookrightarrow G_2$ such that we have the following universal property: For any group H together with morphisms $\psi_1 \colon G_1 \to H$ and $\psi_2 \colon G_2 \to H$, there is one and only one morphism $\psi \colon G_1 \star G_2 \to H$ such that $\psi_i = \psi \circ \iota_i$.*

Proof: As for assertion (ii): We define a map

$$\hat{\iota}_1 \colon F(X_1) \longrightarrow F(X_1 \cup X_2) / \langle\langle R_1 \cup R_2 \rangle\rangle = G_1 \star G_2$$

via $x \mapsto [x]$. In particular, for some r in R_1 we have that $\hat{\iota}_1(r) = [r] = 1_{G_1 \star G_2}$. This means that $\langle\langle R_1 \rangle\rangle$ is contained in $\ker(\hat{\iota}_1)$, i.e. $\hat{\iota}_1$ descends to a map $\iota_1 \colon G_1 \to G_1 \star G_2$. We do the same for G_2. Given morphisms ψ_1 and ψ_2 as described above, we define a map

$$\hat{\psi} \colon F(X_1 \cup X_2) \longrightarrow H, \qquad x \longmapsto \begin{cases} \psi_1([x]), & \text{if } x \in X_1, \\ \psi_2([x]), & \text{if } x \in X_2. \end{cases}$$

By the same argument as for the $\hat{\iota}_i$, i.e. $R_1 \cup R_2 \subseteq \ker \hat{\psi}$, $\hat{\psi}$ descends to a map $\psi \colon G_1 \star G_2 \to H$. A short calculation shows that this map ψ has the desired properties.

Now the uniqueness-party in the first assertion follows from the universal property in (ii). More precisely, if (H, ι_1', ι_2') also has the universal property in (ii), then there exists a unique isomorphism $h \colon G_1 \star G_2 \to G'$ such that $h \circ \iota_1 = \iota_1'$ and $h \circ \iota_2 = \iota_2'$. □

Definition I.7.17 (Amalgamated Product): Suppose we are given group presentations $G_1 = \langle X_1 | R_1 \rangle$ and $G_2 = \langle X_2 | R_2 \rangle$ and another group presentation $U = \langle X_3 | R_3 \rangle$ together with group homomorphisms $\alpha_1 \colon U \to G_1$ and $\alpha_2 \colon U \to G_2$. Let $R_3' := \{\widehat{\alpha}_1(u) = \widehat{\alpha}_2(u) \mid u \in U\}$, where $\widehat{\alpha}_i(u)$ is a preimage of $\alpha_1(u)$ in $F(X_i) \subseteq F(X_1 \cup X_2)$. Then

$$G_1 \star_U G_2 := \langle X_1 \cup X_2 | R_1 \cup R_2 \cup R_3' \}\rangle$$

is called the *amalgamated product of G_1 and G_2 over U with respect to α_1 and α_2.*

Proposition I.7.18 (Universal Property of Amalgamated Product): *In the situation of Definition I.7.12 it holds:*

(i) $G_1 \star_U G_2$ *does not depend on the chosen representation.*

(ii) *There are natural morphisms* $\varphi_1 \colon G_1 \to G_1 \star_U G_2$ *and* $\varphi_2 \colon G_2 \to G_1 \star_U G_2$ *such that* $\varphi_1 \circ \alpha_1 = \varphi_2 \circ \alpha_2$ *with the following universal property: For any other group* H *with morphisms* $\psi_1 \colon G_1 \to H$ *and* $\psi_2 \colon G_2 \to H$ *with* $\psi_1 \circ \alpha_1 = \psi_2 \circ \alpha_2$*, there is one and only one morphism* $\psi \colon G_1 \star_U G_2 \to H$ *such that* $\psi_1 = \psi \circ \varphi_1$ *and* $\psi_2 = \psi \circ \varphi_2$*.*

Proof: As for the second assertion, consider the free product $G_1 \star G_2$ together with the embeddings $\iota_i \colon G_i \hookrightarrow G_1 \star G_2$ and let $p \colon G_1 \star G_2 \to G_1 \star_U G_2$ be the quotient map. Define $\varphi_1 := p \circ \iota_1$ and $\varphi_2 := p \circ \iota_2$. Then we obtain that $\varphi_1 \circ \alpha_1 = \varphi_2 \circ \alpha_2$, since for all u in U it holds that

$$\varphi_1(\alpha_1(u)) = p(\iota_1(\alpha_1(u))) = p(\iota_2(\alpha_2(u))) = \varphi_2(\alpha_2(u))$$

by the additional relations used for passing from $G_1 \star G_2$ to $G_1 \star_U G_2$.

If we are now given the group H with the stated homomorphisms, Proposition I.7.16 yields the existence of homomorphisms $\hat{\psi} \colon G_1 \star G_2 \to H$ such that $\hat{\psi} \circ \iota_1 = \hat{\psi} \circ \iota_2$. It remains to show that $\hat{\psi}$ descends to our wanted map ψ. It holds

$$\hat{\psi}(\iota_1(\alpha_1(u))) = \psi_1(\alpha_1(u)) = \psi_2(\alpha_2(u)) = \hat{\psi}(\iota_2(\alpha_2(u))),$$

which establishes the claim. The first claim follows from (ii) as usual. □

Chapter II

A Topological Crash Course

1 Fundamental Groups

In this section, X, X_1, X_2 and Y always denote topological spaces with points $\star \in X$, $\star_1 \in X_1$ and $\star_2 \in X_2$.

Definition II.1.1:

(i) Let X be a topological space and let $\star$ be a point of X. Then, the tuple $(X, \star)$ is called a *punctured topological space*. Let $(X_1, \star_1)$ and $(X_2, \star_2)$ be punctured topological spaces and let $f\colon X_1 \to X_2$ be a map. If f is continuous with $f(\star_1) = \star_2$, then f is called a *morphism of punctured spaces*.

(ii) Let E be a subset of some topological space X and let $f, g\colon X \to Y$ be continuous maps between topological spaces such that $f|_E = g|_E$. A continuous map $H\colon X \times I \to Y$ that for any x in X satisfies that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ and which fulfils $H(x, t) = f(x) = g(x)$ for any x in E and any t in $[0, 1]$ is called a *homotopy between f and g relative to E*. In this case we write $f \sim_E g$.

(iii) The set
$$\pi_1(X, \star) := \{\gamma\colon [0, 1] \to X \mid \gamma(0) = \gamma(1)\}/{\sim_{[0,1]}}$$
is called *fundamental group of X at $\star$*.

Example II.1.2 (Composition, Reparametrisation): Let $\varphi\colon [0, 1] \to [0, 1]$ be a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then $\varphi \sim_E \mathrm{id}_{[0,1]}$.

If $f, f'\colon X_1 \to X_2$ and $g, g'\colon X_2 \to X_3$ are continuous maps such that $f \sim_E f'$ and $g \sim_{f(E)} g'$, then $g \circ f \sim_E g' \circ f'$.

Let $\gamma\colon [0, 1] \to X$ be a path and let $\varphi\colon [0, 1] \to [0, 1]$ be a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$, then $\gamma \circ \varphi \sim_{\{0,1\}} \gamma$.

Proof: We need to give homotopies for the considered maps. For the first statement,

$$H\colon [0,1]\times[0,1] \longrightarrow [0,1], \qquad (s,t) \longmapsto (1-t)s + t\varphi(s)$$

does the trick. For the second one, suppose $H_1\colon X_1 \times [0,1] \to X_2$ is a homotopy from f to f' relative to E and suppose $H_2\colon H \times [0,1] \to X_3$ is a homotopy from g to g' relative to $f(E)$, then $H\colon X_1 \times [0,1] \to X_3$, $(x,t) \mapsto H_2(H_1(x,t),t)$ is a homotopy suitable to our claim. The third assertion directly follows from the first and second assertion. □

Proposition II.1.3 (Fundamental Group): *For two paths $\gamma_1, \gamma_2\colon [0,1] \to X$ with $\gamma_1(1) = \gamma_2(0)$, the path*

$$\gamma_1 \cdot \gamma_2\colon [0,1] \longrightarrow X, \qquad t \longmapsto \begin{cases} \gamma_1(2t), & \text{if } 0 \le t < 1/2, \\ \gamma_2(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

is called composition of γ_1 and γ_2. *If $\gamma_1', \gamma_2'\colon [0,1] \to X$ are different paths with $\gamma_1(0) = \gamma_1'(0)$, $\gamma_1(1) = \gamma_1'(1)$, $\gamma_2(0) = \gamma_2'(0)$ and $\gamma_2(1) = \gamma_2'(1)$ and if $\gamma_1 \sim_{\{0,1\}} \gamma_1'$ and $\gamma_2 \sim_{\{0,1\}} \gamma_2'$, then $\gamma_1 \cdot \gamma_2$ is homotopic to $\gamma_1' \cdot \gamma_2'$, i.e. "$\cdot$" is well-defined on homotopy classes.*

The set $\pi_1(X, \star)$ together with the law of composition defined by composition of paths turns into a group, called fundamental group of X at $\star$.

Proof: Suppose $H_1, H_2\colon [0,1]\times[0,1] \to X$ are homotopies between γ_1 and γ_1' respectively γ_2 and γ_2'. Then

$$H\colon [0,1]\times[0,1] \longrightarrow X, \qquad (s,t) \longmapsto \begin{cases} H_1(2t,s), & \text{if } 0 \le t \le 1/2, \\ H_2(2t-1,s), & \text{if } 1/2 \le t \le 1 \end{cases}$$

is the desired homotopy between the path compositions.

Now to the law of composition. First, we check associativity of path composition. Let thus $\gamma_1, \gamma_2, \gamma_3\colon [0,1] \to X$ be paths with $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$. Then

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)\colon [0,1] \longrightarrow X, \qquad t \longmapsto \begin{cases} \gamma_1(2t), & \text{if } 0 \le t < 1/2, \\ \gamma_2(4t-2), & \text{if } 1/2 < t \le 3/4, \\ \gamma_3(4t-3), & \text{if } 3/4 \le t \le 1 \end{cases}$$

and

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3\colon [0,1] \longrightarrow H, \qquad t \longmapsto \begin{cases} \gamma_1(4t), & \text{if } 0 \le t \le 1/4, \\ \gamma_2(4t-1), & \text{if } 1/4 \le t \le 1/2, \\ \gamma_3(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Using the map

$$\varphi\colon I \longrightarrow I, \qquad s \longmapsto \begin{cases} 1/2s, & \text{if } 0 \leq s \leq 1/2, \\ s - 1/4, & \text{if } 1/2 \leq s \leq 3/4, \\ 2s - 1 & \text{if } 3/4 \leq s \leq 1 \end{cases}$$

it follows from Exercise 1.2 that $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3) \circ \varphi$ is homotopic to $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$ with respect to $E = \{0, 1\}$.

Secondly, we verify that $[\bigstar]$ is the identity element. Using arguments similar to those above, one shows that for any path $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = \star$ we have $\gamma \cdot \bigstar \sim_{\{0,1\}} \gamma \sim_{\{0,1\}} \bigstar \cdot \gamma$.

Thirdly, we show that for any path $\gamma\colon [0,1] \to X$ we have $\gamma \cdot \gamma_- = [\bigstar] = [\gamma_- \cdot \gamma]$, where γ_- denotes the inverse path declared by $s \mapsto \gamma(1-s)$. A suitable homotopy is given by

$$H\colon [0,1] \times [0,1] \longrightarrow X, \qquad (s,t) \longmapsto \begin{cases} \gamma(2ts), & \text{if } 0 \leq s \leq 1/2, \\ \gamma(2t - 2ts), & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

In total, we have thus shown that $(\pi_1(X, \star), \cdot)$ is indeed a group. □

Proposition II.1.4 (Functoriality):

(i) *Every morphism $f\colon (X_1, \star_1) \to (X_2, \star_2)$ between punctured spaces induces a group homomorphism $\pi_\star(f) := f_\star\colon \pi_1(X_1, \star_1) \to \pi_1(X_2, \star_2)$ defined by $[\gamma] \mapsto [f \circ \gamma]$.*

(ii) *Let $f_1\colon (X_1, \star_1) \to (X_2, \star_2)$ and $f_2\colon (X_2, \star_2) \to (X_3, \star_3)$ be morphisms of punctured spaces. Then it holds $(f_2 \circ f_1)_\star = (f_2)_\star \circ (f_1)_\star$.*

(iii) *For the identity $\mathrm{id}_{(X,\star)}$ it holds $(\mathrm{id}_{(X,\star)})_\star = \mathrm{id}_{\pi_1(X,\star)}$.*

(iv) *If $f, f'\colon (X_1, \star_1) \to (X_2, \star_2)$ are morphisms of punctured spaces homotopic relative to $\{\star_1\}$, then $f_\star = f'_\star$.*

Proof: As for (i): By Exercise 1.2, the induced morphism $f_\star$ is well-defined. For composable paths γ_1 and γ_2 we have $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by definitions. The other assertions are immediate. □

Corollary II.1.5 (Fundamental Group as Topological Invariant): *If the punctured space $(X_1, \star_1)$ is isomorphic to $(X_2, \star_2)$, then their fundamental groups $\pi_1(X_1, \star_1)$ and $\pi_1(X_2, \star_2)$ are isomorphic. Even stronger: If the punctured spaces $(X, \star_1)$ and $(X_2, \star_2)$ are merely homotopic, then $\pi_1(X_1, \star_1)$ is isomorphic to $(\pi_1(X_2, \star_2)$.*

Proposition II.1.6 (Independence of Base Point): *For points x_1 and x_2 of X it holds: If there is a path from x_1 to x_2, then $\pi_1(X, x_1) \cong \pi_1(X, x_2)$. In particular: If X is path-connected, then any two fundamental groups of X at distinct base-points are isomorphic. In this case, we just write $\pi_1(X)$ and call it* the fundamental group of X.

Proof: Let $c\colon [0,1] \to X$ be a path with $c(0) = x_1$ and $c(1) = x_2$. The map $\pi_1(X, x_1) \to \pi_1(X, x_2)$, $[\gamma] \mapsto [c_- \cdot \gamma \cdot c]$ is an isomorphism with inverse map $[\gamma] \mapsto [c \cdot \gamma \cdot c_-]$. □

Definition II.1.7: Let X be a topological space. If X is path-.connected and if $\pi_1(X) = \{\mathrm{id}\}$, then X is called *simply connected*.

Corollary II.1.8: *If X is a contractible topological space, then X is simply connected.*

Proof: As an exercise, you have already shown that if X is contractible, then there is a homotopy $f\colon X \to \{x\}$ to some point x of X, hence there is only one fundamental group $\pi_1(X)$, which is isomorphic to $\pi_1(X, \star)$, which is trivial. □

Example II.1.9: (i) The euclidean space $\mathbb{R}^n$ is simply connected.

(ii) Trees are simply connected.

Import II.1.10 (Theorem of Seifert and van Kampen): *Suppose we have a topological space X with open and path-connected subsets U and V of X such that $X = U \cup V$ and such that $U \cap V$ is non-empty and path-connected. Let $\star$ be a point in $U \cap V$. Consider the fundamental groups $\pi_1(U, \star)$, $\pi_1(V, \star)$ and $\pi_1(U \cap V, \star)$ and the maps $\alpha_1\colon \pi_1(U \cap V, \star) \to \pi_1(U, \star)$ and $\alpha_2\colon \pi_1(U \cap V, \star)$ induced by the embeddings $\iota_1\colon U \cap V \hookrightarrow U$ and $\iota_2\colon U \cap V \hookrightarrow V$. Then $\pi_1(X) \cong \pi_1(U, \star) \star_{\pi_1(U \cap V, \star)} \pi_1(V, \star)$.*

In the following, we will again make use of the fact $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, which we will show again later.

Example II.1.11: Consider a rose with two leaves. Using Seifert and van Kampen, we may compute its fundamental group from its building blocks. For those, we find $\pi_1(U_1) \cong \mathbb{Z} = \langle x \rangle$, $\pi_1(U_2) \cong \mathbb{Z} = \langle y \rangle$ and $\pi_1(U \cap V) = \{1\}$. Hence the fundamental group of the whole space is $\pi_1(X) = \mathbb{Z} \star \mathbb{Z} = F(x, y) = F_2$.

More generally, consider the rose with n petals Γ, i.e. $V = \{\cdot\}$ and $\#E^+ = n$. We obtain with Seifert and van Kampen and via induction that $\pi_1(\Gamma^{\mathrm{geom}}) \cong F_n$.

Corollary II.1.12 (Fundamental Group of Finite Graphs): *Let Γ be a connected finite graph with vertex set V and edge set E. Then $\pi_1(\Gamma^{\mathrm{geom}}) = F_g$, where g is $\#E - \#V + 1$.*

Here, g stands for genus of the graph. Note that there is conflict over the "correct" definition of genus of a graph, i.e the notion used here is non-standard.

Proof: Let T be a spanning tree, i.e. a subtree of Γ, which contains all the vertices of Γ (see Exericse 2 on Exercise Sheet 2), thus $\#V - 1$ edges. Then Γ/T (the graph obtained by contracting T) is a rose with n petals, where $n = \#E - \#V + 1$. From Exercise 1 on Exercise Sheet 5 it is known that $q\colon \Gamma^{\mathrm{geom}} \to (\Gamma/T)^{\mathrm{geom}}$ is a homotopy equivalence and by Corollary II.1.5, also the fundamental groups are isomorphic, which by Example II.1.11 are isomorphic to F_n. □

Import II.1.13 (Oriented Closed Surfaces): *A real two-dimensional manifold X, i.e. a paracompact Hausdorff topological space in which every point has an open neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^2$, is called a* surface*. If X is a compact topological space, the surface X is called* closed*. For any natural number g let Σ_g denote the surface obtained by gluing the edges with the same labels of a polygon with $4g$ edges as follows:*

Sketch missing

For $g = 0$ define Σ_g to be the sphere. The classification of closed oriented surfaces is a well-known result from algebraic topology. It states: Any closed oriented surfaces X is homeomorphic to Σ_g for some non-negative integer g. This g is then called genus of X.

If X is a closed surface of genus g together with a decomposition of X into polygons, and if χ denotes the number

$$\chi = \#vertices - \#edges + \#polygons,$$

then we have the $2g = 2 - \chi$. This number χ is called Euler characteristic *and it is a topological invariant.*

Proposition II.1.14 (Fundamental Group of Closed Oriented Surfaces): *Let X be a closed oriented surface of genus g. Then*

$$\pi_1(X) \cong \left\langle A_1, B_1, \ldots, A_g, B_g \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = 1 \right\rangle =: \pi_g.$$

This group π_g is called the surface group.

Let now D be a closed disk in X and let $X^* := X - D$. Then $\pi_1(X^*)$ is isomorphic to F_{2g}.

Proof: For the statement in (ii), consider the following:

Sketch missing

Since X^* is homotopic to a rose R with $4g$ petals, we have that $\pi_1(X^*) \cong \pi_1(R_{4g}) = F_{4g}$.

For the statement in (i),

Sketch missing

Let D' be a disk included in D and let $U_2 = X - D'$. Then $U_1 \cap U_2$ is an annulus and thus homotopic to a circle. Hence $\pi_1(U_2) \cong F(A_1, B_1, \ldots, A_g, B_g)$, $\pi_1(U_1) = \{1\}$ and $\pi_1(U_1 \cap U_2) \cong \mathbb{Z} = \langle c \rangle$. For Seifert and van Kampen we now still need the morphisms α_1 and α_2. The map $\alpha_1 \colon \pi_1(U_1 \cap U_2) \to \pi_1(U_1) = \{1\}$ is clear. And $\alpha_2 \colon \pi_1(U_1 \cap U_2) \to \pi_1(U_2)$ is declared via $c \mapsto \prod_{i=1}^{g} [A_i, B_i]$. Applying the Theorem of Seifert and van Kampen yields that

$$\pi_1(X) \cong \pi_1(U_1) \star_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \cong \left\langle A_1, B_1, \ldots, A_g, B_g \;\middle|\; 1 = \prod_{i=1}^{g} [A_i, B_i] \right\rangle$$

which concludes the proof. □

2 Covering Theory

Example II.2.1: (i) Let Y be the real line $\mathbb{R}$, let X be the unit circle line $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\|_2 = 1\}$ and consider the map $p \colon Y \to X$, $t \mapsto (\cos(t), \sin(t))$.

(ii) Let Y be the Euclidean plane $\mathbb{R}^2$, let $X = \mathbb{R}^2/\mathbb{Z}^2$ be equipped with the quotient topology and let $p \colon Y \to X$ be simply the quotient map.

(iii) Let $Y := \mathrm{Cay}(F(x_1, \ldots, x_n), \{x_1, \ldots, x_n\})$ be the n-valent tree, take $X := Y/F(x_1, \ldots, x_n)$, which gives the Rose with n petals and let $p \colon Y \to X$ be the quotient map.

Observe that for all those examples, we have the following: For any point x in X there is an open neighbourhood U such that $p^{-1}(U) = \dot\bigcup_{i \in I} (U_i \mid i \in I)$ for disjoint open subsets U_i of X and for any $i \in I$, the restriction $p|_{U_i} \colon U_i \to U$ is a homeomorphism. Furthermore, in all cases, Y is simply connected.

In this section, X and Y will always denote topological spaces and $p \colon Y \to X$ will always be a continuous map.

Definition II.2.2:

(i) Let X be a topological space. If for x in X and any open neighbourhood V of x there is an open connected resp. path-connected neighbourhood U of x contained in V, then X is called *locally connected at* x resp. *locally path-connected at* x. If X is locally connected resp. locally path-connected at any point, then X is called *locally connected* resp. *locally path-connected*.

(ii) If for any point x in X and any open neighbourhood V of x there is an open neighbourhood U of x containted in V such that any closed path in U can be contracted in X to the point x, then X is called *semi-locally simply connected*.

(iii) For x in X, the pre-image $p^{-1}(\{x\})$ is called *fibre of* x. We sometimes denote it $\mathcal{F}_x$.

Definition II.2.3: Let $p\colon Y \to X$ be a continuous map. If for any point x in X there is an open neighbourhood U such that $p^{-1}(U) = \dot\bigcup_{i\in I}(U_i \mid i \in I)$ for disjoint open subsets U_i of X and for any $i \in I$, the restriction $p|_{U_i}\colon U_i \to U$ is a homeomorphism, then p is called a *covering*.

We write $p\colon (Y, y) \to (X, x)$ for a covering with $p(y) = x$ and call it a *covering between marked spaces*. The neighbourhood U as described above is called *elementary neighbourhood*.

Example II.2.4: Let T denote g-holed torus, which is a representative of closed surfaces of genus g, and let $T^* := T - \{\infty\}$. A covering of T^* is called an *origami*.

Import II.2.5 (Basic Properties of Coverings): *Let X be a connected and locally path-connected topological space, let Y be a non-empty and path-connected topological space and let $p\colon Y \to X$ be a covering. Then p is open and surjective and the map*

$$X \longrightarrow \mathbb{N}_0 \cup \{\infty\}, \qquad x \longmapsto \mathcal{F}_x = p^{-1}(\{x\})$$

is constant. The number $\#\mathcal{F}_x$, where x is some element of x, is called the degree of p*, or* number of leaves*, and is denoted* $\deg(p)$*.*

From now on, we assume all coverings to be as described in **??** II.2.5.

Theorem 5 (The Lifting Theorem): *Let $p\colon (Y, y_0) \to (X, x_0)$ be a covering of marked spaces and let $f\colon (Z, z_0) \to (X, x_0)$ be a continuous map between marked spaces. If Z is connected and locally path-connected, then we have the following:*

(i) *There is at most one map* $\hat{f}\colon (Z, z_0) \to (X, x_0)$ *of marked spaces such that* $p \circ \hat{f} = f$. *A map with the property of* $\hat{f}$ *is called a* lift of f to Z.

(ii) *If* $f_*(\pi_1(Z, z_0))$ *is contained in* $p_*(\pi_1(Y, y_0))$ *then there is one and only one lift* $\hat{f}\colon (Z, z_0) \to (Y, y_0)$.

As a direct consequence of this theorem, we obtain the following assertion.

Corollary II.2.6 (Lifting Results): *Suppose* $p\colon (Y, y) \to (X, x)$ *is a covering.*

(i) *For any path* $c\colon [0, 1] \to X$, *and any point* y *in* Y *with* $p(y) = c(0)$ *there is one and only one lift* $\hat{c}\colon [0, 1] \to Y$ *in* y, *i.e.* $\hat{c}(0) = y$ *and* $p \circ \hat{c} = c$.

(ii) *Consider paths* $\hat{c}_1, \hat{c}_2\colon [0, 1] \to Y$ *with* $\hat{c}_1(0) = \hat{c}_2(0)$ *and* $\hat{c}_1(1) = \hat{c}_2(1)$. *The lift* $\hat{c}_1$ *is homotopic to* $\hat{c}_2$ *relative to* $\{0, 1\}$ *if and only if* c_1 *is homotopic to* c_2 *relative to* $\{0, 1\}$.

(iii) *Suppose* $\tilde{X}$ *is simply connected and* $u\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *is a morphism of marked topological spaces. Then there is one and only one covering* $\tilde{p}\colon (\tilde{X}, \tilde{x}_0) \to (Y, y_0)$ *such that* $p \circ \tilde{p} = u$.

Definition II.2.7: Let $u\colon (\tilde{X}, \tilde{x}) \to (X, x)$ be a covering. If for any covering $p\colon (Y, y) \to (X, x)$ there is a unique covering $\tilde{p}\colon (\tilde{X}, \tilde{x}) \to (Y, y)$ such that $p \circ \tilde{p} = u$, then u is called a *universal covering.*

Remark II.2.8: (i) A universal covering $((\tilde{X}, \tilde{x}), u)$ is unique up to unique isomorphy, i.e. for any further universal covering $u'\colon (\tilde{X}', \tilde{x}') \to (X, x)$ there is a unique homeomorphism $h\colon (\tilde{X}', \tilde{x}') \to (\tilde{X}, \tilde{x})$ with $u \circ h = u'$.

(ii) A map $p\colon (Y, y) \to (X, x)$ is a universal cover if and only if Y is simply connected. This follows from (i) and Corollary II.2.6(iii).

Theorem 6 (of the Universal Covering): *Let* X *be a connected, locally path-connected and simply connected and let* x_0 *be a point of* X. *Then there is a universal cover* $u\colon (\tilde{X}, \tilde{x}) \to (X, x_0)$.

This universal covering for X may be constructed. We denote

$$\pi_1(x_0, x) := \{c\colon [0, 1] \to X \mid c(0) = x_0, c(1) = x\}|_{\{0,1\}},$$

define $\tilde{X}$ to be $\{\tilde{X} := \{(x, [c]) \mid x \in X, [c] \in \pi_1(x_0, x)\}$ and define $p\colon \tilde{X} \to X$, $(x, [c]) \mapsto x$. What remains to be done, and what requires some effort, is the construction of a topology on $\tilde{X}$. For a point $\tilde{x} = (x, [c])$ of $\tilde{X}$, and an open neighbourhood U of x, we want to define

$$\tilde{U} := \{(x', [c']) \mid x' \in U, c' = cd\},$$

where d lies in U, which is supposed to become an open neighbourhood of $\tilde{x}$. One can show that these $\tilde{U}$ form the basis for some topology on $\tilde{X}$ and that, with respect to this topology, $\tilde{X}$ is path-connected and $p\colon \tilde{X} \to X$ is a covering.

To finally show that $\tilde{X}$ is simply connected, suppose $w\colon [0,1] \to \tilde{X}$ is a path with $w(0) = w(1) = \tilde{x}_0 = (x_0, \mathbf{x}_0)$ and consider the composition $\bar{w} := p \circ w$. Then w is a lift of $\tilde{w}$ with $w(0) = \tilde{x}_0$. The map $w'\colon [0,1] \to \tilde{X}$, $t \mapsto (\bar{w}(t), \bar{w}|_{[0,1]})$ also is a lift of $\bar{w}$ with $w'(0) = \tilde{x}_0$ and thus $w = w'$ which establishes the claim.

3 From Coverings to Groups and Back

In this section, we assume all coverings to be connected. Recall that a topological space X is said to be Hausdorff, if for any two distinct points x and y of X there are disjoint open neighbourhoods U of x and V of y.

Definition II.3.1 (Deck Group): Let $p\colon Y \to X$ be a covering. Then we denote

$$\mathrm{Deck}(p) := \mathrm{Deck}(Y/X) := \{h\colon Y \to Y \text{ homeomorphism and } p \circ h = p\}.$$

If for any two points y_1 and y_2) with $p(y_1) = p(y_2)$ there is some h in $\mathrm{Deck}(Y/X)$ such that $h(y_1) = y_2$, then the covering p is called *regular* or *normal*.

Remark II.3.2 (Action of Deck Group): Let $p\colon (Y, y_0) \to (X, x_0)$ be a covering. Consider the action $\rho\colon G := \mathrm{Deck}(Y/X) \to \mathrm{Homeo}(Y)$ induced by application of homeomorphisms to points of Y. Then we have the following:

(i) The action is free, i.e. all non-trivial elements have no fixed point.

(ii) The action is *properly discontinuous*. This means that for any point y in Y there is a neighbourhood U such that $\#\{g \in G \mid (\rho(g))(U) \cap U \neq \varnothing\}$ is finite.

(iii) For a point x of X and its fibre $\mathcal{F}_x = p^{-1}(\{x\})$, the action ρ induces an action $\hat{\rho}\colon G = \mathrm{Deck}(Y/X) \to \mathrm{Perm}(\mathcal{F}_x)$, which is also free and thus in particular an embedding of groups.

Assertion (iii) holds because of the uniqueness of liftings described in Import II.2.5(i), Assertion (i) follows from (iii) and for (ii), we proceed as follows: For a point y of Y we choose a neighbourhood U such that $p|_U\colon U \to p(U)$ is a homeomorphism. Hence, for every map h in $\mathrm{Deck}(p)$, we have $h(U) \cap U = \varnothing$.

Proposition II.3.3 (Proper Discontinuity via Coverings): *Let X be a Hausdorff topological space and let $\rho\colon G \to \mathrm{Homeo}(X)$. Consider the projection map $p\colon X \to X/G = \{Gx \mid x \in X\}$, where X/G carries the quotient topology. The covering p is free if and only if ρ is free and properly discontinuous.*

The proof of this assertion will be left as an exercise on an upcoming Exercise Sheet.

Remark II.3.4 (. . . and Back): Let $p\colon Y \to X$ be a normal covering. Then there is one and only one homeomorphism $h\colon Y/\operatorname{Deck}(p) \to X$, such that $h \circ q = p$, where $q\colon Y \to Y/\operatorname{Deck}(p)$ is the quotient map. In this case, we thus have the following commutative diagram:

$$\begin{array}{ccc} Y & & \\ {\scriptstyle p}\big\downarrow & \searrow^{q} & \\ X & \xleftarrow[h]{\sim} & Y/\operatorname{Deck}(p) \end{array}$$

Proof: By the universal property of quotient space we obtain a continuous map $h\colon Y/\operatorname{Deck}(p) \to X$. As p is open and surjective, so is h. If $h(q(y_1)) = h(q(y_2))$, then y_1 and y_2 belong to the same orbit under the action of $\operatorname{Deck}(p)$, i.e. they belong to the same equivalence class, which is to say that $q(y_1) = q(y_2))$. Hence, h is injective. □

Proposition II.3.5 (Deck Group of Universal Covering): *Any universal covering* $u\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *is normal. Moreover,* $\operatorname{Deck}(u) = \pi_1(X, x_0)$.

We give a sketch of a proof. First, we show that the universal covering is normal. Let therefore $\tilde{x}$ and $\tilde{x}'$ be points in the fibre of x_0. Then the existence of a suitable map $h\colon (\tilde{X}, \tilde{x}) \to (\tilde{X}, \tilde{x}')$ in the diagram

$$\begin{array}{ccc} (\tilde{X}, \tilde{x}) & \xrightarrow{\quad h \quad} & (\tilde{X}, \tilde{x}') \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle u} & \\ & (X, x_0) & \end{array}$$

follows more or less directly from the universal property of our universal covering. It can be shown that h is indeed a homeomorphism, as desired.

To compute the Deck group, we make use of the simple connectedness of $\tilde{X}$. We define a group homomorphism

$$\alpha_1\colon \operatorname{Deck}(u) \longrightarrow \pi_1(X, x_0), \qquad h \longmapsto [p(\tilde{c})],$$

where $\tilde{c}$ is any path from $\tilde{x}_0$ to $h(\tilde{x}_0)$. Convince yourself that this yields a well-defined map! To establish an isomorphism, we give an inverse map to α_1, namely

$$\alpha_2\colon \pi_1(X, x_0) \longrightarrow \operatorname{Deck}(\tilde{X}/X), \qquad [c] \longmapsto h,$$

where h is the unique Deck transformation (as the Deck group acts freely) which is determined by mapping the starting point of the unique lift $\tilde{c}_{x_0}$ of c in x_0 to its end point. Verification of the homomorphism properties of α_1 and α_2 and verification of them being inverse to each other will be the content of Exercise 4 on Exercise Sheet 7.

Theorem 7 (Principal Theorem of Covering Theory): *Suppose X is a connected, locally path-connected and semi-locally simply connected and suppose x_0 is a base point of X. Furthermore, let $u\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal covering. Denote*

$$\mathsf{TopCov} := \{p\colon (Y, y_0) \to (X, x_0) \mid p \textit{ covering}, Y \textit{ connected}, Y \neq \varnothing\}/\sim$$

where p_1 and p_2 are related if there is a homeomorphism $h\colon (Y_1, y_0^{(1)}) \to (Y_2, y_0^{(2)})$ with $p_2 \circ h = p_1$, and let $\mathrm{Sg}\pi := \{U \leq \mathrm{Deck}(u)\}$.

(i) *We have the following bijections, which are inverse to each other:*

$$\psi_1\colon \mathsf{TopCov} \longrightarrow \mathrm{Sg}\pi, \qquad [p\colon (Y, y_0) \to (X, x_0)] \longmapsto \mathrm{Deck}(\tilde{p})$$

where $\tilde{p}\colon (\tilde{X}, \tilde{x}_0) \to (Y, y_0)$ with $p \circ \tilde{p} = u$, and

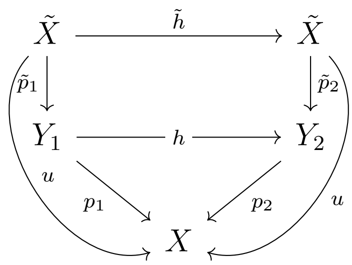
$$\psi_2\colon \mathrm{Sg}\pi \longrightarrow \mathsf{TopCov}, \qquad U \longmapsto [p\colon \tilde{X}/U \to X, [\tilde{x}] \mapsto u(\tilde{x})].$$

(ii) *For two coverings $p_1\colon (Y_1, y_0^{(1)}) \to (X, x_0)$ and $p_2\colon (Y_2, y_0^{(2)}) \to (X, x_0)$ we have the following: There is a unique homeomorphism $h\colon Y_1 \to Y_2$ such that $p_2 \circ h = p_1$ if and only if* $\mathrm{Deck}(\tilde{p}_1)$ *is conjugated to* $\mathrm{Deck}(\tilde{p}_2)$.

(iii) *A covering $p \in \mathsf{TopCov}$ is normal if and only if* $\mathrm{Deck}(p)$ *is a normal subgroup of* $\mathrm{Deck}(u)$. *In this case,* $\mathrm{Deck}(p) \cong \mathrm{Deck}(u)/\mathrm{Deck}(\tilde{p})$.

Proof: (i) Check that ψ_1 is well-defined and that ψ_1 and ψ_2 are inverse to each other.

(ii) "$\Longleftarrow$": Consider the diagram



The map h lifts to $\tilde{h}\colon (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}_0)$, hence we obtain a group homomorphism $h_*\colon \operatorname{Deck}(\tilde{p}_1) \to \operatorname{Deck}(\tilde{p}_2)$ defined by $\sigma_1 \mapsto \tilde{h} \circ \sigma_1 \circ \tilde{h}^{-1}$. As $\tilde{h}$ is a homeomorphism, h_* is an isomorphism.

"$\Longleftarrow$": Consider the same diagram as above. For $\tilde{h} \in \operatorname{Deck}(u)$ is holds $\operatorname{Deck}(\tilde{p}_2) = \tilde{h} \circ \operatorname{Deck}(\tilde{p}_1) \circ \tilde{h}^{-1}$. We claim that $\tilde{h}$ descend to some homeomorphism $h\colon Y_1 \to Y_2$ for which then holds $p_2 \circ h = p_1$.

To show that $\tilde{h}$ descends as desired, let a and b be points of $\tilde{X}$ with $\tilde{p}_1(a) = \tilde{p}_1(b)$. Hence, there is some σ in $\operatorname{Deck}(\tilde{p}_1)$ such that $\sigma(a) = b$. For $\widehat{\sigma} := \tilde{h}\sigma\tilde{h}^{-1}$ in $\operatorname{Deck}(\tilde{p}_2)$ it holds

$$\widehat{\sigma}(\tilde{h}(a)) = \tilde{h}(\sigma(a)) = \tilde{h}(b),$$

which just means that $\tilde{p}_2(h(a)) = \tilde{p}_2(h(b))$. Therefore, $\tilde{h}$ descends. Observe that $\tilde{h}$ descends to a map from Y_1 to Y_2 if and only if $\tilde{h} \circ \operatorname{Deck}(\tilde{p}_1) \circ \tilde{h}^{-1} = \operatorname{Deck}(\tilde{p}_2)$.

(iii) The "if and only if"-part follows directly from (ii). If p is normal, by (ii) we obtain a map $\operatorname{Deck}(u) \to \operatorname{Deck}(p)$ that maps $\tilde{h}$ to its descend h. This map is a group homomorphism and its kernel is precisely $\operatorname{Deck}((\tilde{p})$. Furthermore, it is surjective since any homeomorphism can be lifted. Thus $\operatorname{Deck}(p) \cong \operatorname{Deck}(u)/\operatorname{Deck}(\tilde{p})$. □

Definition II.3.6 (Monodromy Map): Let $p\colon (Y, y_0) \to (X, x_0)$ be a covering of finite degree d and let $\mathcal{F}_{x_0} := p^{-1}(x_0)$. The map

$$m\colon \pi_1(X, x_0) \longrightarrow \operatorname{Sym}(\mathcal{F}_{x_0}) \leftrightarrow S_d, \qquad [c] \longmapsto (y \mapsto y \cdot [c]),$$

where $y \cdot [c]$ is the endpoint of the unique lift $\tilde{c}_y$ of c in y in the fibre of $\mathcal{F}_{x_0}$, is called *monodromy map*. It is an anti-grouphomomorphism, i.e. for paths c_1 and c_2 it holds $m([c_1] \star [c_2]) = m([c_2]) \circ m([c_1])$. The image of m in S_d is called *monodromy group*.

Example II.3.7: Let $G = S_3$ and consider the following cover:

Sketch missing

Observe that p is a normal covering. The monodromy map $m\colon \pi_1(T^*) \cong F_2(x, y) \to S_6$ is determined by $x \mapsto (123)(456)$ and $y \mapsto (14)(26)(35)$. Do be careful; as noted before, m is an anti-homomorphism.

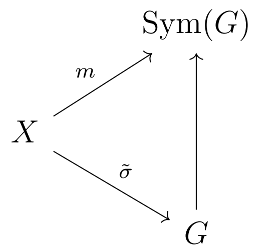
Remark II.3.8 (Normal Coverings): Let $p\colon (Y, y_0) \to (X, x_0)$ be a normal cover and denote $G = \operatorname{Deck}(p)$. We identify elements h of G with elements in the fibre $\mathcal{F}_{x_0}$ by $h \mapsto h(y_0)$.

(i) Firstly, we have the natural action of $\mathrm{Deck}(p)$ on $\mathcal{F}_{x_0} \leftrightarrow G$, namely $\rho\colon G \to \mathrm{Sym}(\mathcal{F}_x)$, which maps h to $(y \mapsto h(y))$. Identifying $\mathrm{Sym}(\mathcal{F}_{x_0})$ with $\mathrm{Sym}(G)$, this idenfitication looks like $h \mapsto h \circ g$, where we use that if $g(y_0) = y$, then $h \circ g(y_0) = h(y)$. Hence, by this identification, the action of G becomes the action of G by G via leftmultiplication.

(ii) Secondly, we have the right-action $m\colon \pi_1(X, x_0) \to \mathrm{Sym}(G)$ by monodromy. By Theorem 6 and Theorem 7(iii), for any universal cover $u\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ we obtain

$$\pi_1(X, x_0) \xrightarrow{\sim} \mathrm{Deck}(u) \xrightarrow{\text{quotientmap}} \mathrm{Deck}(p) \cong \mathrm{Deck}(u)/\,\mathrm{Deck}(\tilde{u}) = G$$

Observe that the following diagram commutes:



where the upwards arrow is the action by right multiplication $h \mapsto (g \mapsto gh)$.

4 The Hyperbolic Plane

We consider the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. On this topological space, we want to declare a metric. Originally this space was cooked up as a counter example to the parallel postulate contained in the Euclidean axioms. It turned out to be useful for other purposes, too. The Hyperbolic Plane is biholomorphic to the universal coverings of “almost all” Riemann surfaces. As a literature suggestion, one might consider “Fuchsian Groups” by Svetlana Katch.

Facts II.4.1 (Hyperbolic Metric): Let $c\colon [a, b] \to \mathbb{H}$ be a piecewise differentiable curve and for $t \in [a, b]$, write $c(t) = x(t) + \mathrm{i}y(t)$ with suitable real numbers $x(t)$ and $y(t)$. Then

$$\ell(c) := \int_a^b \frac{|c'(t)|}{y(t)}\, dt = \int_a^b \frac{1}{y(t)} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

is called the *hyperbolic length of c*.

For points z and w of $\mathbb{H}$, we call

$$d_{\mathbb{H}}(z, w) := \inf\{\ell(c) \mid c\colon [a, b] \to \mathbb{H} \text{ piecewise differentiable}, c(a) = z, c(b) = w\}$$

the *hyperbolic distance of* z *and* w. The corresponding map $(z, w) \mapsto d_{\mathbb{H}}(z, w)$ is called *hyperbolic metric*, and indeed is a metric.

Example II.4.2 (Lengths of Special Curves): (i) The curve $c_b\colon [0, 1] \to \mathbb{H}$, $t \mapsto b\mathrm{i} + t$ has derivative $c_b'\colon t \mapsto 1$. For this curve, the notation from above and any point t in $[0, 1]$, we naturally have $x(t) = t$ and $y(t) = b\mathrm{i}$. The length of this curve is

$$\ell(c_b) = \int_0^1 \frac{|c'(t)|}{y(t)}\, dt = \int_0^1 \frac{1}{b}\, dt = \left[\frac{1}{b}t\right]_0^1 = \frac{1}{b}.$$

Observe that this length tends to zero, when b gets arbitrarily large, and that this length becomes arbitrarily long, when b tends to zero.

(ii) Let a be a real number greater than one and let $c_a\colon [1, a] \to \mathbb{H}$, $t \mapsto t\mathrm{i}$. Then

$$\ell(c_a) = \int_1^a \frac{|\mathrm{i}|}{a}\, dt = \int_1^a \frac{1}{t}\, dt = [\ln(t)]_1^a = \ln(a).$$

(iii) For an arbitrary differentiable path $\widehat{c}_a\colon [0, 1] \to \mathbb{H}$ with $\widehat{c}_a(0) = \mathrm{i}$ and $\widehat{c}_a(1) = \mathrm{i}a$, we have

$$\begin{aligned}\ell(\widehat{c}_a) &= \int_0^1 \frac{1}{y(t)}\sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &\geq \int_0^1 \frac{|y'(t)|}{y(t)}\, dt \geq \int_0^1 \frac{y'(t)}{y(t)}\, dt = [\ln(y(t))]_0^1 = \ln(a) - \ln(1) = \ln(a).\end{aligned}$$

From (ii) and (iii) it follows for $a > 1$ that $d_{\mathbb{H}}(i, ai) = \ln(a)$. In the same way, $d_{\mathbb{H}}(\mathrm{i}, a\mathrm{i}) = \ln(1/a)$ for $a < 1$. In particular, the curve from c_a from (ii) is a geodesic.

Without further explanation, we want to raise awareness that the path from (i) is not geodesic.

Example II.4.3 (The Three Graces): Consider the following maps $\gamma\colon \mathbb{H} \to \mathbb{H}$:

(i) For a real number r, let $\gamma\colon z \mapsto z + r$.

(ii) For a positive real number λ, let $\gamma\colon z \mapsto \lambda z$.

(iii) For an angle θ in $[0, 2\pi]$, let $\gamma\colon z \mapsto (\sin\theta z + \cos\theta)^{-1}(\cos\theta z - \sin\theta)$.

All three of those maps are isometries, which can easily be verified by computation. Observe that all of these maps are of the form $z \mapsto (cz+d)^{-1}(az+b)$ for real numbers a, b, c, d, where $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq 0$. To be explicit, the corresponding matirces are

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Proposition II.4.4 (Möbius Transformations as Isometries): *The matrix group* $\mathrm{Sl}_2(\mathbb{R})$ *acts on* $\mathbb{H}$ *via isometries by*

$$\alpha\colon \mathrm{Sl}_2(\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathbb{H}), \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \gamma_A\colon z \longmapsto \frac{az+b}{cz+d}.$$

For the group $\mathrm{PSl}_2(\mathbb{R}) := \mathrm{Sl}_2(\mathbb{R})/\{\pm I\}$ *the group action* α *induces an action* $\bar{\alpha}\colon \mathrm{PSl}_2(\mathbb{R}) \to \mathrm{Isom}(\mathbb{H})$, $[A] \mapsto \gamma_A$ *and* $\bar{\alpha}$ *is faithful (i.e. injective).*

Proof: Firstly, we have to check that α is indeed a group homomorphism. For this, observe that $\mathrm{Sl}_2(\mathbb{R})$ is generated by the special matrices

$$\left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}.$$

This is essentially seen from Gaussian elimination with extra care for $\mathrm{Sl}_2(\mathbb{R})$. Once one has shown that, the claim follows from II.4.3.

Secondly, as γ_{-I} acts trivially, the induced action $\bar{\alpha}$ is well-defined. A complex number z is a fixed point of $\gamma_A\colon z \mapsto (az+b)/(cz+d)$ if and only if $cz^2 + dz = az + b$. If $a = d = 1$ and $b = c = 0$ or if $a = d = -1$ and $b = c = 0$, the equation degenerates and has the whole of $\mathbb{C}$ as solution set. Otherwise, this equation has at most two solutions. From these considerations, one can draw the conclusion that $\bar{\alpha}$ is faithful.

Observe that the proof of (ii) shows that for any $A \neq \pm I$, the corresponding Möbius transform γ_A has at most two fixed points. $\square$

Facts II.4.5 (Möbius Transforms are Great): The action of $\mathrm{PSl}_2(\mathbb{C})$ acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, which may be identified with $\mathbb{P}^1(\mathbb{C})$.

(i) The group $\mathrm{PSl}_2(\mathbb{C})$ acts 3-transitively on $\mathbb{P}^1(\mathbb{C})$, i.e. for 3 different points z_1, z_2, z_3 in $\mathbb{P}^1(\mathbb{C}) = \widehat{\mathbb{C}}$ there is one and only one σ in $\mathrm{PSl}_2(\mathbb{C})$ such that $\sigma(0) = z_1$, $\sigma(1) = z_2$ and $\sigma(\infty) = z_3$.

(ii) Möbius transformations map generalised circles to generalised circles. By generalised circle, we mean an Euclidean circle in $\mathbb{R}^2$ or a line in $\mathbb{R}^2$.

(iii) Möbius transformations preserve angles between generalised circles.

(iv) For pairwise distinct points z_1, z_2, z_3 and z_4 on the Riemann sphere, we call

$$(z_1, z_2; z_3, z_4) := \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$$

the *cross ratio of* z_1*,* z_2*,* z_3 *and* z_4. For γ in $\mathrm{PSl}_2(\mathbb{C})$ it holds

$$(\gamma(z_1), \gamma(z_2); \gamma(z_3), \gamma(z_4)) = (z_1, z_2; z_3, z_4).$$

Corollary II.4.6 (Conclusions for $\mathbf{PSl_2}(\mathbb{R})$): (i) *If* $[A]$ *belongs to* $\mathrm{PSl}_2(\mathbb{R})$*, then* $\gamma_A(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$*, hence* $\gamma_A(\mathbb{H}) = \mathbb{H}$*.*

(ii) *The group* $\mathrm{PSl}_2(\mathbb{R})$ *acts 3-transitively on* $\mathbb{R} \cup \{\infty\}$*, i.e. for any real numbers* $r_1 \leq r_2 \leq r_3 \leq \infty$ *there is one and only one* $[A]$ *in* $\mathrm{PSl}_2(\mathbb{R})$ *such that* $\gamma_A(0) = r_1$*,* $\gamma_A(1) = r_2$ *and* $\gamma_A(\infty) = \infty$*.*

For (i), check that $\det(A) = 1$ enforces $\gamma_A(\mathrm{i})$ to belong to $\mathbb{H}$.

Proposition II.4.7 (Geodesics): (i) *For distinct points* z_1 *and* z_2 *in* $\mathbb{H}$ *there is a unique geodesic through* z_1 *and* z_2*.*

(ii) *Geodesics are of the following form: Type 1: Vertical line, Type 2: Semi-circle through both points with midpoint on real line.*

(iii) *For any distinct points* z_1 *and* z_2 *of* $\mathbb{H}$*, we can calculate their distance using the cross-ratio using the start point* a *and the end point* b *of the unique geodesic joining them:*

$$d_{\mathbb{H}}(z_1, z_2) = |\ln(a, z_1; b, z_2)|.$$

(iv) *For any geodesic* g *and a point* P *that does not lie on* g*, there are infinitely many geodesics* h *parallel to* g *through* P*. Here, parallel means* $g \cap h = \varnothing$*.*

Using II.4.2 and II.4.5 these assertions can be shown.

Proposition II.4.8 (Return of the Three Graces): (i) *Let* $\gamma_A \colon z \mapsto z + t$*, whose matrix is* $A = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ *and let* $\langle A \rangle$ *act on* $\mathbb{H}$*. The fundamental domain of this action is the rectangle above the line from* $(-t/2, 0)$ *to* $(t/2, 0)$*. The fixed point of* γ_A *is* ∞ *and* $|\mathrm{tr}(A)| = 2$*.*

(ii) *Let* $\gamma_A \colon z \mapsto \lambda^2 z$ *with* $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$*. The fixed points of this map are* $\{0, \infty\}$ *and* $|\mathrm{tr}(A)| = \lambda + \lambda^{-1} > 2$*.*

(iii) *Let* $\gamma_A \colon z \mapsto (\cos\theta z - \sin\theta)/(\sin\theta z + \cos\theta)$*. The fixed points of this map are* $\{\pm \mathrm{i}\}$ *and* $|\mathrm{tr}(A)| = 2|\cos\theta| < 2$*.*

Facts II.4.9: Every A in $\mathrm{PSl}_2(\mathbb{C})$ is conjugated to one of the three graces.

This follows directly from the properties of the Jordan Normal Form over the reals.

Definition II.4.10 (The Three Types): Let $[A] \in \mathrm{PSl}_2(\mathbb{R}) - \{\pm I\}$.

(i) If $[A]$ is conjugated to $[\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)]$ for some $\lambda > 1$, then $[A]$ is called *hyperbolic*. This is the case if and only if $|\mathrm{tr}(A)| > 2$ respectively if and only if there are two fixed points in $\mathbb{R} \cup \{\infty\}$.

(ii) If $[A]$ is conjugated to $[\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)]$ for some real number t, then $[A]$ is called *parabolic*. This is the case if and only if $|\mathrm{tr}(A)| = 2$ respectively if and only if there is one fixed point in $\mathbb{R} \cup \{\infty\}$.

(iii) If $[A]$ is conjugated to $[\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)]$, then $[A]$ is called *elliptic*. This is the case if and only if $|\mathrm{tr}(A)| < 2$ respectively if and only if there is one fixed point in $\mathbb{H}$.

Chapter III

Growth of Groups

Definition III.0.1: Let G be a group and let S be a set of generators. Then the assignment

$$|g|_S := \min\{n \mid g = s_1 \cdots s_r, s_i \in S \cup S^{-1}\}$$

defines a norm on G. Using this norm, $\gamma_{G,S}(n) := \{g \mid |g|_S \leq n\}$ declares the so-called *growth function* and $\Sigma_{G,S}(z) = \sum_{g \in G} z^{|g|} \in \mathbb{Z}[[X]]$ declares the so-called *growth series*.

Obviously, for the norm it hols $|g| = 0$ if and only if $g = 1$ as well as $|g| = |g^{-1}|$ and $|gh| \leq |g| + |h|$. As usual, the norm induces a metric via $d(g,h) = |h^{-1}g|_S$. This metric is invariant under left translation. In fact, this metric is the distance on $\mathrm{Cay}(G,S)$.

For a natural number n, $\gamma_{G,S}(n)$ yields the volume of the radius n-ball in $\mathrm{Cay}(G,S)$.

Using formal manipulations, we may rewrite the growth series as

$$\Sigma(z) = \sum_{n \geq 0} (\gamma(n) - \gamma(n-1)) z^n = (1-z) \sum_{n \geq 0} \gamma(n) z^n.$$

Example III.0.2: (i) Let G be a finite group and let $S = G$. Then firstly $\gamma(0) = 1$ and for any natural number n it holds $\gamma(n) = |G|$.

(ii) Let $G = \mathbb{Z}/r\mathbb{Z}$ and let $S = \{1\}$. Then $\gamma(n) = \min\{r, 2n+1\}$ and we obtain distinct values for $n \in \{0, 1, \ldots, \lfloor r/2 \rfloor\}$.

(iii) Let G be a subgroup of S_{54}, which is the symmetry group of the Rubiks cube, together with the generating permutations that are executable on the Rubiks cube. One knows that for any $n \geq 19$ it holds $\gamma(n) = |G|$ and 19 is optimal.

(iv) Let $G = \mathbb{Z}$ and let $S = \{1\}$. For the natural number n, one has as in (ii) that $\gamma(n) = 2n + 1$. The corresponding growth series is $\Sigma(z) = (1 + z)/(1 - z)$.

(v) Let again $G = \mathbb{Z}$ and let $S = \{2, 3\}$. Intuitively, we should arrive at something like $\gamma(n) \approx 6n$. Drawing an instructive picture, one can read off the beginnings of the growth series to be $\Sigma = 1 + 4z + 8z^2 + 6z^3 + 6z^4 + \dots$ which equals $(1 + 3z + 4z^2 - z^3)/(1 - z)$.

(vi) Let $G = \mathbb{Z}^2$ and $S = \{\pm(1, 0)^t, \pm(0, 1)^t\}$. The corresponding Cayley graph is the standard paper grid. Evaluating the growth function at the natural number n gives $\gamma(n) = \sum_{i=0}^{n} 4i = 1 + 2n + n^2$. This yields the growth series $\Sigma(z) = (1 + z)^2/(1 - z)^2$.

In the following, for a set of generators S that may be embedded into an $\mathbb{R}^d$, we denote by $\mathrm{Conv}(S)$ the convex hull of S in $\mathbb{R}^d$.

Proposition III.0.3: *Let G be $\mathbb{Z}^d$ together with generating set S. Then for the growth function we have $\gamma(n) \sim n^d \operatorname{vol}(\mathrm{Conv}(S))$.*

Proof: Up to order n^{d-1} we have

$$\gamma(n) = \#\{x \in \mathbb{Z}^d \mid x \in nS\} \sim \#\{x \in \tfrac{1}{n}\mathbb{Z}^d \mid x \in \mathrm{Conv}(S)\}$$

which converges to $n^d \operatorname{vol}(\mathrm{Conv}(S))$. □

As an exercise, study the example $G = \mathbb{Z}^2$ and $S = \{-1, 0, 1\}^2$, i.e. compute growth function and series and check the claim of the above proposition. Additionally, study $G = \mathbb{Z}^2$ with the generating set $S = \{(1, 0), (0, 1), (-1, -1)\}$ without inverses!

Theorem 8: *Let G be an abelian group and let S be a finite generating set. Then the growth series $\Sigma_{G,S}$ is a rational function.*

Proof: Consider the group ring $\mathbb{C}G$ consisting of elements of the form $\sum_{g \in G} \alpha_g g$. Then we may declare a grading on $\mathbb{C}G$ by $\deg(g) = |g|_S$, yielding subvectorspaces $F_n = \{g \mid |g|_S \leq n\}$. The direct sum$A = \bigoplus_{n \geq 0} F_n/F_{n-1}$ is again a ring with basis G and in A we have

$$g \cdot h = \begin{cases} gh, & \text{if } |gh| = |g| + |h|, \\ 0, & \text{if } < \end{cases}$$

In summary, A is associative, commutative, graded and generated in degree 1 and finite in degree 1. Opening a book on commutative algebra confronts us with the following fact: "If A is an associative, commutative, graded algebra generated in degree 1 and finite in degree 1, then its Hilbert series $H_A(z) = \sum_{n\geq 0} \dim(A_n)z^n$ is rational and more specifically of the form $p(z)/(1-z)^\alpha$, where α denotes the number of generators of A". □

Lets prove the theorem we used in the above proof.

Assume $A = \langle x_1, \ldots, x_d \rangle$. We may look more generally at a finitely generated A-module M, which due to the grading of A were graded, i.e. we might decompose $M = \bigoplus_{n\geq 0} M_n$. For this module the same claim $H_M(z) = \sum_{n\geq 0} \dim(H_n)z^n$ holds, which we will show. In particular, the claim for A follows.

We show the claim via induction on the number of generators. For zero generators, M is clearly finite-dimensional. For the induction step, we consider the sequence

$$\{y \in M \mid x_d y = 0\} = \ker(\mu_{x_d}) \longrightarrow M \xrightarrow{\mu_{x_d}} M \longrightarrow \operatorname{coker}(\mu_{x_d}) = M/x_d M$$

Then $\ker(\mu_{x_d})$ is a finitely generated $\langle x_1, \ldots, x_{d-1}\rangle$-module and $\operatorname{coker}(\mu_{x_d})$ is a finitely generated $\langle x_1, \ldots, x_{d-1}\rangle$-module, yielding a short exact sequence . . . By the fundamental theorem of homomorphisms, this gives the formula $\sum_{n\geq 0}(\dim K_n - \dim M_n + \dim M_{n+1} - \dim C_{n+1})$ for the involved dimensions, so also

$$\sum_{n\geq 0}(\dim K_n - \dim M_n + \dim M_{n+1} - \dim C_{n+1})z^n = 0,$$

and

$$zH_K - zH_M + H_M - \dim M_0 - H_C - \dim C_0$$

yielding $H_M = (1-z)^{-1}(H_C - zH_K - \dim C_0 + \dim M_0)$, which is of the desired form.

Theorem 9: *If G is virtually abelian, i.e. there is an abelian subgroup of finite index, then for every finite generating set S, the growth series is also rational.*

This statement is particularly interesting to crystallography.

Example III.0.4: Let d be a natural number and let G be the free group on d generators F_d. The corresponding growth function is given by $\gamma(n) = 1 + 2d + 2d(2d-1) + \cdots + 2d(2d-1)^{n-1}$. By calculating differences, one obtains the growth series

$$\Sigma = 1 + 2dz + 2d(2d-1)z^2 + \cdots = \frac{1+z}{1-(2d-1)z}.$$

In fact, if for $G = \mathbb{Z}_d, F_d$ we don't chose a basis as generating set, we still end up with a rational growth function.

1 Amalgamated Products

In the following, let G be the amalgamated product of the groups A and B over C, i.e. we assume C is embedded in A and B. Further, we assume we have sets of generators S_A, S_B and S_C of A, B and C, respectively.

Definition III.1.1: Let C be a subgroup of A. If there is a subset T_A of A that is transversal for C such that for all c in C and t in T_A it holds $|ct| = |c|_{S_C} + |t|_{S_A}$, then the inclusion of C in A is called *admissible*.

Theorem 10: *If the inclusions of C in A and C in B are admissible, then $G = \langle S_G = S_A \cup_{S_C} S_B \rangle$ has growth function*

$$\frac{1}{1/\Sigma_A + 1/\Sigma_B - 1/\Sigma_C}$$

Proof: For every element g of G there is a unique expression $g = ct_1t_2 \cdots t_n$ for suitable t_{2i} in T_A and t_{2i+1} in T_B and c in C, all non-equal to 1, except for the first or the last. Furthermore, this expression is geodesic, i.e. a word of shortest length. Hence

$$\Sigma_G = \sum_{i \geq 1} \Sigma_C \Sigma_{T_B}((\Sigma_{T_A} - 1)(\Sigma_{T_B} - 1))^n \Sigma_{T_A}.$$

Plugging in $\Sigma_A = \Sigma_C \Sigma_{T_A}$ yields the claim. □

Consider a surface of genus $g = 2$ with the three components X, Y and Z obtained from cutting off half doughnuts at the ends, where Z is the remaining middle part and the cut off pieces are $X - Z$ and $Y - Z$. Then $\pi_1(Z) = \pi_1(\mathbb{S}^4 - 4 \text{ discs})$, which is isomorphic to F_3, and which has 4 generators. Furthermore, $\pi_1(X), \pi_1(Y) \cong F_3$ with 5 generators. The growth series is

$$\Sigma_A = \Sigma_B = \frac{(1+z)^2}{1 - 8z + z^2}.$$

We believe that the inclusions of $\pi_1(Z)$ in $\pi_1(X)$ and $\pi_1(Y)$ are admissible.

By induction, one shows that for a surface S_g of genus g with the generators chosen as above, one gets

$$\frac{(1+z)^2}{1 - (8g-6)z + z^2}.$$

Evaluating the analytic continuation of the growth function at 1 often yields the Euler characteristic of the considered space. This is always the case in our zoo of examples up to now.

2 Heisenberg Group

Definition III.2.1: The set

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$$

is called *Heisenberg group.*

A possible generating set of H is given by

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and their commutator is

$$w = [u, v] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We may write $H = \langle u, v \mid [[u, v], v] = [[u, v], u] = 1 \rangle$. Every element of H my uniquely be expressed as $g = u^k v^\ell w^m$. This product produces the matrix

$$u^k v^\ell w^m = \begin{pmatrix} 1 & k & m \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem 11: *For g with entries k, ℓ, m as above it holds* $\max\{|k|+|\ell|, \sqrt{|m|}\} \leq |g| \leq |k| + |\ell| + 6\sqrt{|m|}$.

This can be shown using $[u^k, v^\ell] = w^{k\ell}$, which implies $|w^{m^2}| \leq 4m$. This equality is best validated using the following trick: By definition, $[g, h] = g^{-1}h^{-1}gh$, hence

$$[g, h] - I = g^{-1}h^{-1}gh - I = g^{-1}h^{-1}(gh - hg) = g^{-1}h^{-1}((g-I)(h-I) - (h-I)(g-I))$$

The product on the right is most easily evaluated, especially in our case.

From these simple calculations, we can already see that the growth function γ of H roughly behaves like n^4.

For understanding the growth series, we need to understand geodesics.

Theorem 12 (Stoll): *Let G be the Heisenberg group H with the standard generators. Then Σ_G is rational.*

This result has recently been improved by Duchin and Shapiro: In fact, the growth series Σ_G is rational for any set of generators of the Heisenberg group.

Let $k \geq 2$ be an integer. Then

$$H_k = \begin{pmatrix} 1 & z_1 & \cdots & z_{k-1} & z_k \\ & & & & z_{k+1} \\ & & & & \vdots \\ & & & & z_{2k} \\ & & & & 1 \end{pmatrix} = \langle u_i, v_i \mid [u_i, v_i] = w, [u_i, v_j] = 1, [u_i, u_j] = 1, [v_i, v_j] = 1 \, \forall \, i$$

Theorem 13: *If $k \geq 3$, then the growth series with respect to the generators u_i and v_i is rational and with respect to u_i, v_i and w are transcendental.*

3 Growth Functions

For a given group G with set of generators S, we defined the growth function of this group with respect to S to be the function $\gamma\colon \mathbb{N} \to \mathbb{N}$ declared via

$$\gamma_{G,S}(n) = \#B_{G,S}(1, n) = \#\{g \in G \mid |g|_S \leq n\}.$$

Definition III.3.1: Let $\gamma, \delta\colon \mathbb{N} \to \mathbb{N}$ be functions. If there is a constant $C > 0$ such that for any natural number n it holds $\gamma(n) \leq \delta(Cn)$, we write $\gamma \precsim \delta$ and say γ was *dominated by* δ. If it holds $\gamma \precsim \delta \precsim \gamma$, we say γ and δ were *equivalent*.

Proposition III.3.2: *Let G be a group. If S and S' are generating sets for G, then $\gamma_{G,S} \sim \gamma_{G,S'}$.*

Proof: The finite set S is bounded with respect to the norm generated by S', i.e. there is some constant $C > 0$ such that $S \subseteq B_{G,S'}(1, C)$. Now we have

$$\gamma_{G,S}(n) = \#(S \cup S^{-1})^n) \leq \#(B_{G,S'}(1, C)^n) \leq \#B_{G,S'}(1, Cn) = \gamma_{G,S'}(C, n),$$

which means that $\gamma_{G,S} \precsim \gamma_{G,S'}$. Switching roles gives the other domination. □

Definition III.3.3: Let G be a group. Then we write $\gamma_G = [\gamma_{G,S}]_\sim$ for any generating set S for G and call it *the growth function of G*.

Example III.3.4: We have already seen that $\gamma_{\mathbb{Z}^d} = [n^d]$. If $d \neq e$, then $n^d \not\sim n^e$. Also, we have seen that $\gamma_{T_n} = \gamma_{F_d} = [(2d-1)^n]$, but unfortunately $[(2d-1)^n] = [2^n] = [e^n]$. Hence, we lost a lot of information by passing to the growth function independent of generating set. For the Heisenberg group H_3, we calculated the growth function $\gamma_{H_3} = [n^4]$.

For the remaining weeks, we will concern ourselves with the search of examples of groups with growth function outside of the families constant functions (finite groups), polynomials (powers of $\mathbb{Z}$) and exponential functions (like free groups).

Furthermore, we will try to answer what it means for a group to have a specific growth function.

Similar questions where posed by Milner in Problem 5603 in American Mathematics Monthly. Groups "between" powers of $\mathbb{Z}$ and free groups are now called *groups of intermediate growth*, groups that have growth functions n^d are said to be of *polynomial growth*.

He conjectured that if G has polynomial growth, then there is a finite index subgroup G_0 of G, which is nilpotent. Such groups are called *virtually nilpotent* and this conjecture has since been shown by Gromov in 1985.

Furthermore, the mathematician Grigorchuk constructed groups of intermediate groups in 1983.

Proposition III.3.5: *Let G be a group. If H is a subgroup of G, then $\gamma_H \precsim \gamma_G$. If N is a normal subgroup of G, then $\gamma_{G/N} \precsim \gamma_G$.*

Proof: Let S be a generating set for G and let T be a generating set for H. Without loss of generality, we may assume that T is contained in S, which immediately yields $\gamma_{H,T}(n) \leq \gamma_{G,S}(n)$ for any natural number n. Also, $\gamma_{G/N,S/N}(n) \leq \gamma_{G,S}(n)$. □

If time permits, we will later show that for an infinite group G and a subgroup H of G with finite index, it holds $\gamma_G \sim \gamma_H$.

4 Quasi-Isometries

Definition III.4.1: Let X and Y be metric spaces and let $\varphi\colon X \to Y$ be a map. If there are positive constants λ and C such that for any x, x' in X it hlds

$$\frac{1}{\lambda}|x - x'| - C \leq |\varphi(x) - \varphi(y)| \leq \lambda|x - x'| + C$$

then we call it *quasi-isometric embedding*. If, additionally, there is a positive constant D such that for any y in Y we have $|y - \varphi(X)| \leq D$, then we call φ

an *quasi-isometry*. If there is a quasi-isometry between X and Y, we call them *quasi-isometric* and write $X \sim_{\mathrm{QI}} Y$.

From the definition it is not entirely clear if quasi-isometry is symmetric. As an exercise, one can show that $\varphi\colon X \to Y$ is a quasi-isometry, if there are positive constants λ, C, D and a map $\psi\colon Y \to X$ with

$$|\varphi(x) - \varphi(x')| \leq \lambda|x - x'| + C, \qquad |\psi(y) - \psi(x)| \leq \lambda|y - y'|$$

and $|\psi(\varphi(x)) - x| \leq D$, $|\varphi(\psi(y)) - y| \leq D$. To be clear: This does require the axiom of choice.

Example III.4.2: (i) If X is a metric space with finite diameter, then X is quasi-isometric to a point.

(ii) Let G be a group with generating set S and metric $|\cdot|_S$. We may embed G into its Cayley graph $\mathrm{Cay}(G, S)$ and this embedding is a quasi-isometry.

(iii) The inclusion of the integers into the real numbers is a quasi-isometry.

(iv) Let G be a group with generating sets S and S'. Then $\mathrm{id}\colon (G, |\cdot|_S) \to (G, |\cdot|_{S'})$ is a quasi-isometry. In fact, the identity is bilipschitz.

(v) Let G be a finitely generated group and let $\alpha\colon G \to G$ be an automorphism. Then $\alpha\colon (G, |\cdot|_S) \to (G, |\cdot|_S)$ is a quasi-isometry. This can be seen by the same argument which shows (iv).

(vi) Let H be a subgroup of G with generating sets T for H and S for G. Is the inclusion $H \hookrightarrow G$ a quasi-isometry? If H is finite, then we are golden because a point always embeds.

If H is of finite index in G, then G and H will be quasi-isometric. This can be seen from the characterisation of quasi-isometry given in the exercise above. The first inequality follows for $\lambda = 1$ and $C = 0$, because we may assume that the generating set T is contained in S and thus an S-word is at most as long as the corresponding T-word. For the second inequality, we may find a finite set X in G such that $G = HX$. Then $\psi(g) = \psi(hx) = h$. Now this map ψ does the job.

If G is the Heisenberg group and H is the centre of G, then H does not embed quasi-isometrically. For the element $h := \sum_{i=1}^{3} E_{ii} + E_{13}$ we had in H that $|h|_H = n$, but $|h|_G \sim |n|^{1/2}$.

Proposition III.4.3: *Let G be a finitely generated group. Then G has a unique quasi-isometric class of word metrics.*

This was discussed in example (iii) from above.

As an exercise, one can show the following: If N is a finite normal subgroup of G, then $G \to G/N$ is a quasi-isometry.

Definition III.4.4: Let $\gamma, \delta\colon [0,\infty) \to [0,\infty)$ be non-decreasing functions. If there are constants λ and C such that $\gamma(n) \le \lambda\delta(\lambda n + c) + c$, we say γ is *weakly dominated by* δ, denotes $\gamma \precsim_w \delta$. If it holds $\gamma \precsim_w \delta \precsim_w \gamma$, we call γ and δ *weakly equivalent.*

Proposition III.4.5: *Let $\varphi\colon (H, |\cdot|) \to (G, |\cdot|)$ be a quasi-isometric embedding (which is not necessarily a group homomorphism). Then $\gamma_H \precsim_w \gamma_G$.*

Really: H is a metric space, which is uniformly locally finite. This means there is a function $v\colon [0,\infty) \to [0,\infty)$ with $\#B_H(x, r) \le v(r)$ for any point x in X.

Proof: We do have constants such that

$$\frac{1}{\lambda}|x - x'| - C \le |\varphi(x) - \varphi(x')| \le \lambda|x - x'| + C$$

Denote $D := |1_G - \varphi(1_H)|$. Then $\varphi(B_H(1_H, R))$ will be contained in $B(1_G, \lambda R + D)$ due to the second inequality in the chain above. For points x and x' in H with $\varphi(x) = \varphi(x')$, then $|x - x'| \le \lambda C$ by the first inequality in the chain above. Hence $\#\varphi^{-1}(y) \le \#B_H(x, \lambda C) \le E$ for some constant E. Finally, $\gamma_H(R) = \#B_H(1_H, R) \le E\#\varphi(B_H(1_H, R)) \le \#B_G(1_G, \lambda R + D) = E\gamma_G(\lambda R + D)$. □

Lemma III.4.6: *If $\gamma, \delta\colon [0,\infty) \to [0,\infty)$ are increasing, if there is $t \ge 0$ such that for all $r \ge t$ it holds $\delta(r) \ge 1$, if there is some argument t_0 with $\delta(t_0) > 0$, and if $\gamma \precsim_w \delta$, then there is some ρ such that for all $t \ge t_0$ we have $\gamma(t) \le \rho\delta(\rho t)$.*

Lemma III.4.7: *Let G be a finitely generated group with generating set S. The group G is infinite if and only if there is a quasi-isometry $\mathbb{Z} \to G$.*

Königs Lemma states that if Γ is an infinite graph of finite degree, then there is an infinite ray. It relies on Tychonoffs Theorem.

Proof: Let S be a finite generating set and consider a ray ρ in $\mathrm{Cay}(G, S)$. Translating this ray ρ by $\rho(n)^{-1}$ yields a sequence of rays ρ_n. There is an accumulation point ρ_∞ of ρ_n. This ρ_∞ is a quasi-isometric embedding $\mathbb{Z} \to G$.□

Lemma III.4.8: *If G is infinite and finitely generated by the subset S, then for every $\rho \ge 1$ there is some constant K such that $\rho\gamma_{G,S}(\rho n) \le \gamma(Kn)$.*

Proof: Let $(X_n)_{n\in\mathbb{Z}}$ be geodesic in $\mathrm{Cay}(G,S)$ with $x_0 = 1$ and $|x_n| = n$. The disjoint union $B(x_{-2n}, n) \cup B(x_{2n}, n)$ is contained in $B(1, 3n)$. Hence $2\gamma(n) \leq \gamma(3n)$. Now there is some natural number k with $2^k \geq \rho$, and thus $2^k\gamma(n) \leq \gamma(3^k n)$ yields the claim. □

Corollary III.4.9: *Let G be a finitely generated group. For growth functions γ, δ of G, it holds $\gamma \precsim \delta$ if and onl if $\gamma \precsim_w \delta$.*

Theorem 14 (Efremovich, Svarć): *Let X be a non-empty proper geodesic metric space, let G be a group and let G act properly cocompactly by isometries. Then G and X are quasi-isometric.*

What we are going to show is that for every x in X the map $G \to X$, $g \mapsto gx$ is quasi-isometric. A metric space X is proper, if closed balls are compact. This means that for every x in X the map $X \to [0,\infty)$, $y \mapsto d(x,y)$ is proper. The action of a group of automorphisms is proper, if the map $G \times X \to X \times X$, $(g,x) \mapsto (gx, x)$ is proper. This means for every compact set, the set $\{g \in G \mid gK \cap K \neq \varnothing\}$ is finite. A group G acts cocompactly on X, if $G\backslash X$ is compact and Hausdorff.

Corollary III.4.10: *Let X be a non-empty proper geodesic metric space, let G be a group and let G act properly cocompactly by isometries. If, additionally, X is measured, then $r \mapsto \mathrm{vol}(B_X(x,r)) \precsim_w \gamma_G$.*

Proof: Note that $G\backslash X$ is a metric space by defining $d(Gx, Gy) = \inf\{d(gx, hy) \mid g, h \in G\} = \min\ d(x, gy) \mid g \in G\}$, where the last equality holds due to the properness of X (take any g and let $K = \mathrm{cl}(B(x, d(x,y)+1))$; now there is a finite subset T of G such that $K \cap gK \neq \varnothing$ and for all $g \notin T$ it holds $d(x, gy) \geq d(x,y)+1$. Hence the infimum is realised in T, which yields the well-definedness of the metric on $G\backslash X$).

Let $R := \mathrm{diam}(G\backslash X)$. This R is finite by compactness. Let $B := \mathrm{cl}(B(x_0, R))$ and let $S = \{g \in G \mid gB \cap B \neq \varnothing\} = S^{-1}$, which contains 1_G and is finite.

Finally, let $r = \inf\{d(B, gB) \mid g \notin S\}$ and $\lambda = \max\{d(x_0, sx_0) \mid s \in S\}$.

By choice of g in the definition of r we made sure that B and gB do not intersect, hence for every $g \in S^{\mathsf{c}}$ we have $d(B, gB) > 0$. Now let $g_0 \in S^{\mathsf{c}}$, let $r_0 = d(B, g_0B)$ and now only $T = \{g \in S^{\mathsf{c}} \mid d(B, gB) \leq r_0\}$ need to be considered. By properness, T is finite. This makes sure that r is indeed greater than zero.

Next, we will show that there is some constant λ such that for every g in G it holds $\lambda^{-1}|x_0 - gx_0| \leq |g|_S \leq r^{-1}|x_0 - gx_0| + 1$.

For the first inequality, consider $g = s_1 \cdots s_k$. Then

$$|x_0 - gx_0| \leq |x_0 - s_k x_0| + |s_k x_0 - s_k s_{k-1} x_0| + \cdots + |s_k \cdots s_1 x_0 - gx_0| \leq k\lambda.$$

For the second inequality, we use that X is geodesic. Hence there is a geodesic from x_0 to gx_0. Let k be such that $R + (k-1)r \leq |x_0 - gx_0| < R + kr$. In the following, we will construct points $x_1, \ldots, x_k$ on said geodesic, with $x_k = gx_0$, such that $|x_0 - x_1| < R$ and $|x_{i+1} - x_i| < r$ for $i \geq 1$. Choose group elements g_i such that $x_i \in g_{i-1}B$ and $g_0 = 1$. Let $s_i = g_{i-1}^{-1} g_i$, so $g = s_1 \cdots s_k$. Now

$$|B - s_i B| \leq |g_{i-1}^{-1} x_i - s_i g_i^{-1} x_{i+1}| = |x_i - x_{i+1}| < r.$$

This implies $s_i \in S$. In particular, S generates G. We get that $|g|_S = k \leq r^{-1}|x_0 - gx_0| + 1 - r^{-1}R$. Note that we do need "+1".

Finally, using left invariance yields that the translations are quasi-isometries. Due to the definitions, $|g - h| = |g^{-1}h|_S$ and $|gx_0 - hx_0| = |x_0 - g^{-1}hx_0|$. □

Proof (of the Corollary): Firstly, the balls $B(gx_0, r/3)$ are disjoint. Hence $(\#G_{x_0})^{-1} \gamma_S(k) \operatorname{vol}(r/3) \leq \operatorname{vol}(k\lambda + r/3)$, which immediately shows $\gamma_S \precsim_w \operatorname{vol}$.

Secondly, choose x in $B(x_0, K)$. There is some g such that x belongs to gB and such that $|g|_S \leq r^{-1}|x_0 - gx_0| + 1$. This tells us that $B(x_0, K) \subseteq \bigcup_{|g| \leq r^{-1}(K+R)+1} gB$ and thus $\operatorname{vol}_X(K) \leq \operatorname{vol}_X(R) \gamma_S(K/r + 1 - R/r)$, which means that $\operatorname{vol}_X \precsim_w \gamma_S$. □

5 Solvable Groups

Let G be a group. If there are subgroups $G_1, \ldots, G_{s-1}, G_s = \{1_G\}$ such that G_{i+1} is normal in G_i and G_i/G_{i+1} is normal for any legal indices, the group G is called *solvable*.

For example, groups of upper triangular matrices are solvable, or the wreath product $\mathbb{Z} \wr \mathbb{Z}$. Let A be an invertible 2×2-matrix over $\mathbb{Z}$. Then the group

$$\mathbb{Z}^2 \rtimes_A \mathbb{Z} = \{f(v) = A^i v + b \mid i \in \mathbb{Z}, b \in \mathbb{Z}^2\}$$

is solvable.

Definition III.5.1: Let G be a group. If every subgroup of G is finitely generated, then the group G is called *Noetherian*.

A group G is Noetherian if and only if every non-empty collection of subgroups has a maximal element. This, in turn, is equivalent to every ascending chain of subgroups stabilising.

Proposition III.5.2: *Let G be a solvable group. The group G is Noetherian if and only if it is polycyclic, i.e. there is a chain $G = G_0 \triangleright G_1 \triangleright \triangleright \cdots \triangleright G_s = \{1\}$, such that G_i/G_{i+1} is cyclic.*

This is essentially shown using Jordan-Hölders-Theorem and the Structure Theorem for Abelian Groups.

Definition III.5.3: Let G be a polycyclic group. The well-defined number of quotients isomorphic to $\mathbb{Z}$ is called the *Hirsch length of G*.

Theorem 15: *Let G be a polycyclic group. Then there is a torsion-free normal subgroup of finite index.*

Having a group of finite index induces a permutation of cosets, which gives a map to a finite symmetric group, whose kernel is normal and has finite index again. Hence, adding normal does not cost anything.

Proof: This statement is shown via induction on the Hirsch length. If G_0/G_1 is finite, repeat with G_1. If the quotient $G_0/G_1 = \langle t \rangle$ is infinite, there is a torsion-free subgroup K of G_i, which has finite index, and $\langle t, K \rangle = H = K \rtimes_t \mathbb{Z}$ does the trick. □

6 Nilpotent Groups

Let G be a group. If there is a series of subgroups $G_0 \triangleright G_1 \triangleright \cdots \triangleright G_s = \{1\}$ such that G_i/G_{i+1} is central in G/G_{i+1}, then G is *nilpotent*.

Alternatively, one may characterise nilpotency via the lower central series. For $i = 1$ let $\gamma_1 = G$ and for $i \geq 2$, let $\gamma_{i+1} := [\gamma_i, G]$. The minimal index c with $\gamma_{c+1} = \{1\}$ is called the *class of G*.

For the commutator $[x, y] = x^{-1}y^{-1}xy$, there are some useful (but hard to rememeber) formulae: $[xy, z] = [x, z]^y[y, z]$ and $[x, yz] = [x, z][x, y]^z$. This has the following consequence:

Corollary III.6.1: *Let G be a group and let $H = \langle X \rangle_G$, $K = \langle Y \rangle_G$ be normal subgroups of G. Then $[H, K] = \langle [x, y] \mid x \in X, y \in Y \rangle_G$.*

For a natural number n, one defines $[x_1, \dots, x_n] := [[x_1, \dots, x_{n-1}], x_n]$. Note that it absolutely matters where the brackets are put in the right side, since taking the commutator is not associative. Expressions like this are called elementary left-normed commutators. For those, one has the identity

$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y^{-1}] = 1$. As before, one has for normal subgroups H, K and L of G that $[H, K, L] \leq [K, L, H][L, H, K]$.

Furthermore, we can now verify for the terms of the lower central series that $[\gamma_i, \gamma_j] \leq \gamma_{i+j}$.

Corollary III.6.2: *Let $G = \langle X \rangle$ be a group and for $i \geq 2$ let $\gamma_i = [\gamma_i, G]$. Then γ_i is normally generated by $[x_1, \ldots, x_i]$ for $x_j \in X$.*

Corollary III.6.3: *If G is a finitely generated nilpotent group, then γ_i/γ_{i+1} is finitely generated abelian.*

This means that finitely generated nilpotent groups are polycyclic.

Theorem 16: *Let G be a nilpotent group. Then its set of torsion elements constitute a subgroup.*

This statement is shown by induction on the class of G. Groups of class 1 are abelian, for which this statement is clearly true.

Consider a mimimal counterexample G. Let x be an element of finite order of G and let $G' := \gamma_2$. The group $\langle x, G' \rangle$ has the property that the set of its torsion elements forms a subgroup. For any $i \geq 2$, we have $\gamma_i(\langle x, G' \rangle) \subseteq \gamma_{i+1}(G)$ and thus $xy \in \langle x, G' \rangle \langle y, G' \rangle$.

Definition III.6.4: Let G be a group. We write $(\delta_i)_{i \in I}$ for a series of subgroups of G such that δ_i/γ_i is the torsion subgroup of G/γ_i.

We have the characterisation $\delta_i = \{x \in G \mid x^n \in \gamma_i \text{ for some } n \neq 0\}$. Note that δ_i/γ_i has to be interpreted correctly using preimages.

Proposition III.6.5: *Let G be a group. The series $(\delta_i)_{i \in I}$ as defined before is in fact a central series with $[\delta_i, \delta_j] \subseteq \delta_{i+j}$ and δ_i/δ_{i+1} is torsion-free abelian.*

This can be shown using the following lemma:

Lemma III.6.6: *If G is torsion-free nilpotent and if in G it holds $x^n = y^n$ for some $n \neq 0$, then in fact $x = y$.*

For abelian groups, this is clear. Otherweise $1 = [y, y^n] = [y, x^n]$, so $x^n = y^{-1}x^n y = (x^y)^n$. By induction $x = x^y$ and we conclude using the abelian case.

Proof (of the Proposition): Let x in δ_i and y in δ_j. Consider the torsion-free quotient G/δ_{i+j}. We have that $x^m \in \gamma_i$ and $y^n \in \gamma_j$, such that $[x^m, y^n] \in \gamma_{i+j}$. This element is trivial in G/δ_{i+j}, hence $x^m \equiv (x^m)^{y^n} \equiv (x^{y^n})^m$, which by the lemma implies that $x \equiv x^{y^n}$ and thus $y^n \equiv (y^n)^x \equiv (y^x)^n$, i.e. $y \equiv y^x$ which means that $[x, y] \in \delta_{i+j}$. $\square$

Theorem 17 (Dixmier): *Let G be finitely generated virtually nilpotent group. Then there is some d such that $\gamma_G \precsim n^d$.*

A refinement was later given by Guivarćh and Bass:

Theorem 18: *If G is a finitely generated virtually nilpotent group, then $\gamma_G \sim n^d$ for $d = \sum_{i\geq 1} i \operatorname{rank}_{\mathbb{Q}}(\gamma_i/\gamma_{i+1})$.*

Proof: Thanks to previous recitements, we may forget about "virtually" and "torsion". Let $G = \langle x_{i,j} \rangle$. For $i \geq 1$ we have $\delta_i/\delta_{i+1} = \langle x_{i,1}, \dots, x_{i,r_i} \rangle \cong \mathbb{Z}^{r_i}$. Consider a word w of length n in $\{x_{i,j}\}$ and put it in the normal form $w = x_{i,1}^{e_{i,1}} x_{i,2}^{e_{i,2}} \cdots x_{i,r_i}^{e_{i,r_i}} \cdots$ Bringing $x_{i,j}^{e_{i,j}}$ to the left costs at most n^2 commutators $[x_{i,k}, x_{i,j}]$.

Sometimes switch x_{ij} with $[x_{k,\ell}, x_{m,n}, x_{r,q} \dots]$, …at most $A \cdot n^s$weight s commutators.

In $\delta_s/\delta_{s+1} \cong \mathbb{Z}^{r_s}$, which has growth n^{r_s}, we see at most $A \cdot n^s$ generators. Hence the growth is bounded by $(An^s)^{r_s} = n^{sr_s}$. Now w is determined by its value v_1 in δ_1/δ_2, v_2 in δ_2/δ_3, …Hence the total number of values is $B_1 n^{1r_1} B_2 n^{2r_2} \cdots = Bn^{1r_1+2r_2+\cdots} = Bn^d$.

For the lower bound: For every $z \in \gamma_\ell$, there is some constant A such that $|z^n| \leq A|n|^{1/c}$. This can be seen as follows: Without loss of generality, we may assume that $z = [x, y]$ for x in γ_{c-1} and y in G. Let $m = \rceil n^{1/c} \lceil$ and $n = qm^{c-1} + r$. By induction, $x^{m^{c-1}}$ is a word u of length at most B and x^r is a word v of length at most B modulo γ_c. Now

$$z^n = [x^n, y] = [x^{qm^{r-1}+1,y]=[n,y^q][v,g] which has length at most 8m+8}$$

modulo γ_c. Therefore, we can produce all words $x_{1,1}^{e_{1,1}} \cdots x_c^{e_{c,1}} \cdots x_{c,r_c}^{e_{c,r_c}}$ with $|e_{i,j}| \leq n^i$ in a ball of radius C, which gives $n^d \precsim \gamma_G$ $\square$