What are operator spaces?

Operator algebra group
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Contents

1 Short History

The theory of operator spaces grew out of the analysis of completely positive and completely bounded mappings. These maps were first studied on C^*-algebras, and later on suitable subspaces of C^*-algebras. For such maps taking values in $B(\mathcal{H})$ representation and extension theorems were proved [Sti55], [Arv69], [Haa80], [Wit81], [Pau82]. Many of the properties shared by completely positive mappings can be taken over to the framework of operator systems [CE77]. Operator systems provide an abstract description of the order structure of selfadjoint unital subspaces of C^*-algebras. Paulsen's monograph [Pau86] presents many applications of the theory of completely bounded maps to operator theory. The extension and representation theorems for completely bounded maps show that subspaces of C^*-algebras carry an intrinsic metric structure which is preserved by complete isometries. This structure has been characterized by Ruan in terms of the axioms of an operator space [Rua88]. Just as the theory of C^*-algebras can be viewed as noncommutative topology and the theory of von Neumann algebras as noncommutative measure theory, one can think of the theory of operator spaces as noncommutative functional analysis.

This program has been presented to the mathematical community by E.G. Effros [Eff87] in his address to the ICM in 1986. The following survey articles give a fairly complete account of the development of the theory: [CS89], [MP94], [Pis97].

2 Operator Spaces and Completely Bounded Maps

2.1 Basic facts

The spaces

Let X be a complex vector space. A **matrix seminorm** [EW97b] is a family of mappings $\|\cdot\| : M_n(X) \to \mathbb{R}$, one on each **matrix level**[1] $M_n(X) = M_n \otimes X$ for $n \in \mathbb{N}$, such that

$$\text{(R1)}\ \|\alpha x \beta\| \leq \|\alpha\|\|x\|\|\beta\| \text{ for all } x \in M_n(X),\ \alpha \in M_{m,n},\ \beta \in M_{n,m}$$
$$\text{(R2)}\ \|x \oplus y\| = \max\{\|x\|, \|y\|\} \text{ for all } x \in M_n(X),\ y \in M_m(X).$$[2]

Then every one of these mappings $\|\cdot\| : M_n(X) \to \mathbb{R}$ is a seminorm. If one (and then every one) of them is definite, the operator space seminorm is called a **matrix norm**.

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[1] The term *matrix level* is to be found for instance in []

[2] It suffices to show one of the following two weaker conditions:

$$\text{(R1}'\text{)}\ \|\alpha x \beta\| \leq \|\alpha\|\|x\|\|\beta\| \text{ for all } x \in M_n(X),\ \alpha \in M_n,\ \beta \in M_n,$$
$$\text{(R2)}\ \|x \oplus y\| = \max\{\|x\|, \|y\|\} \text{ for all } x \in M_n(X),\ y \in M_m(X),$$

which is often found in the literature, or

$$\text{(R1)}\ \|\alpha x \beta\| \leq \|\alpha\|\|x\|\|\beta\| \text{ for all } x \in M_n(X),\ \alpha \in M_{m,n},\ \beta \in M_{n,m},$$
$$\text{(R2}'\text{)}\ \|x \oplus y\| \leq \max\{\|x\|, \|y\|\} \text{ for all } x \in M_n(X),\ y \in M_m(X),$$

which seems to be appropriate in convexity theory.

A **matricially normed space** is a complex vector space with a matrix norm. It can be defined equivalently, and is usually defined in the literature, as a complex vector space with a family of norms with (R1) and (R2) on its matrix levels.

If $M_n(X)$ with this norm is complete for one n (and then for all n), then X is called an **operator space**[3]([Rua88], cf. [Wit84a]).

For a matricially normed space (operator space) X the spaces $M_n(X)$ are normed spaces (Banach spaces).[4] These are called the **matrix levels** of X (**first matrix level**, **second level** ...).

The operator space norms on a fixed vector space X are partially ordered by the pointwise order on each matrix level $M_n(X)$. One says that a greater operator space norm **dominates** a smaller one.

The mappings

A linear mapping Φ between vector spaces X and Y induces a linear mapping $\Phi^{(n)} = \mathrm{id}_{M_n} \otimes \Phi$,

$$\begin{aligned} \Phi^{(n)} : M_n(X) &\to M_n(Y) \\ [x_{ij}] &\mapsto [\Phi(x_{ij})] , \end{aligned}$$

the nth **amplification** of Φ.

For matricially normed X and Y, one defines

$$\|\Phi\|_{\mathrm{cb}} := \sup \left\{ \|\Phi^{(n)}\| \;\middle|\; n \in \mathbb{N} \right\} .$$

Φ is called **completely bounded** if $\|\Phi\|_{\mathrm{cb}} < \infty$ and **completely contractive** if $\|\Phi\|_{\mathrm{cb}} \leq 1$.

Among the complete contractions, the complete isometries and the complete quotient mappings play a special role. Φ is called **completely isometric** if all $\Phi^{(n)}$ are isometric,[5] and a **complete quotient mapping** if all $\Phi^{(n)}$ are quotient mappings.[6]

The set of all completely bounded mappings from X to Y is denoted by $CB(X, Y)$ [Pau86, Chap. 7].

An operator space X is called **homogeneous** if each bounded operator $\Phi \in B(M_1(X))$ is completely bounded with the same norm: $\Phi \in CB(X)$, and $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|$ [Pis96].

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[3] In the literature, the terminology is not conseqent. We propose this distinction between *matricially normed space* and *operator space* in analogy with *normed space* and *Banach space*.

[4] In the literature, the normed space $M_1(X)$ usually is denoted also by X. We found that a more distinctive notation is sometimes usefull.

[5] I. e.: $\|x\| = \|\Phi^{(n)}(x)\|$ for all $n \in \mathbb{N}$, $x \in M_n(X)$.

[6] I. e.: $\|y\| = \inf\{\|x\| \mid x \in {\Phi^{(n)}}^{-1}(y)\}$ for alle $n \in \mathbb{N}$ and $y \in M_n(Y)$, or equivalently $\Phi^{(n)}(\mathrm{Ball}^\circ M_n(X)) = \mathrm{Ball}^\circ M_n(Y)$ for all $n \in \mathbb{N}$, where $\mathrm{Ball}^\circ M_n(X) = \{x \in M_n(X) \mid \|x\| < 1\}$.

Notations

Using the disjoint union

$$M(X) := \dot{\bigcup_{n\in\mathbb{N}}} M_n(X),$$

the notation becomes simpler.[7]

Examples

$B(\mathcal{H})$ is an operator space by the identification $M_n(B(\mathcal{H})) = B(\mathcal{H}^n)$. Generally, each C^*-algebra A is an operator space if $M_n(A)$ is equipped with its unique C^*-norm. Closed subspaces of C^*-algebras are called **concrete operator spaces**. Each concrete operator space is an operator space. Conversely, by the theorem of Ruan, each operator space is completely isometrically isomorphic to a concrete operator space.

Commutative C^*-algebras are homogeneous operator spaces.

The transposition Φ on $l_2(I)$ has norm $\|\Phi\| = 1$, but $\|\Phi\|_{\text{cb}} = \dim l_2(I)$. If I is infinite, then Φ is bounded, but not completely bounded.

If $\dim \mathcal{H} \geq 2$, then $B(\mathcal{H})$ is not homogeneous [Pau86, p. 6].

Smith's lemma

For a matricially normed space X and a linear operator $\Phi : X \to M_n$, we have $\|\Phi\|_{\text{cb}} = \|\Phi^{(n)}\|$. In particular, Φ is completely bounded if and only if $\Phi^{(n)}$ is bounded [Smi83, Thm. 2.10].

Rectangular matrices

For a matricially normed space X, the spaces $M_{n,m}(X) = M_{n,m} \otimes X$ of $n \times m$-matrices over X are normed by adding zeros so that one obtains a square matrix, no matter of which size.

Then

$$M_{n,m}(B(\mathcal{H})) = B(\mathcal{H}^m, \mathcal{H}^n)$$

holds isometrically.

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[7] The norms on the matrix levels $M_n(X)$ are then one mapping $M(X) \to \mathbb{R}$. The amplifications of $\Phi : X \to Y$ can be described as one mapping $\Phi : M(X) \to M(Y)$. We have

$$\|\Phi\|_{\text{cb}} = \sup\{\|\Phi(x)\| \mid x \in M(X),\ \|x\| \leq 1\}.$$

Φ is completely isometric if $\|x\| = \|\Phi(x)\|$ for all $x \in M(X)$, and Φ is a complete quotient mapping if $\|y\| = \inf\{\|x\| \mid x \in \Phi^{-1}(y)\}$ for all $y \in M(Y)$ or $\Phi(\text{Ball}^\circ X) = \text{Ball}^\circ Y$, where $\text{Ball}^\circ X = \{x \in M(X) \mid \|x\| < 1\}$.

2.2 Ruan's theorem

Each concrete operator space is an operator space. The converse is given by
Ruan's theorem: *Each (abstract) operator space is completely isometrically isomorphic to a concrete operator space [Rua88].*
More concretely, for a matricially normed space X let S_n be the set of all complete contractions from X to M_n. Then the mapping

$$\begin{array}{rcl} \Phi : X & \rightarrow & \bigoplus_{n\in\mathbb{N}} \bigoplus_{\Phi\in S_n} M_n \\ x & \mapsto & (\Phi(x))_\Phi \end{array}$$

is a completely isometric embedding of X into a C^*-algebra [ER93].

A proof relies on the separation theorem for absolutely matrix convex sets.

This theorem can be used to show that many constructions with concrete operator spaces yield again concrete operator spaces (up to complete isometry).

2.3 Elementary constructions

Subspaces and quotients

Let X be a matricially normed space and $X_0 \subset X$ a linear subspace. Then $M_n(X_0) \subset M_n(X)$, and X_0 together with the restriction of the operator space norm again is a matricially normed space. The embedding $X_0 \hookrightarrow X$ is completely isometric. If X is an operator space and $X_0 \subset X$ is a closed subspace, then $M_n(X_0) \subset M_n(X)$ is closed and X_0 is an operator space.

Algebraically we have $M_n(X/X_0) = M_n(X)/M_n(X_0)$. If X_0 is closed, then X/X_0 together with the quotient norm on each matrix level is matricially normed (an operator space if X is one). The quotient mapping $X \rightarrow X/X_0$ is a complete quotient mapping.

More generally, a **subspace** of a matricially normed space (operator space) X is a matricially normed space (operator space) Y together with a completely isometric operator $Y \rightarrow X$. A **quotient** of X is a matricially normed space (operator space) Y together with a complete quotient mapping $X \rightarrow Y$.

Matrices over an operator space

The vector space $M_p(X)$ of matrices over a matricially normed space X itself is matricially normed in a natural manner: The norm on the nth level $M_n(M_p(X))$ is given by the identification

$$M_n(M_p(X)) = M_{np}(X)$$

[BP91, p. 265]. We[8] write

$$\mathbb{M}_p(X)$$

[8] In the literature, the symbol $M_p(X)$ stands for both the operator space with first matrix level $M_p(X)$ and for the pth level of the operator space X. We found that the distinction between $\mathbb{M}_p(X)$ and $M_p(X)$ clarifies for instance the definition of the operator space structure of $CB(X, Y)$.

for $M_p(X)$ with this operator space structure. In particular,

$$M_1(\mathbb{M}_p(X)) = M_p(X)$$

holds isometrically. Analogously $M_{p,q}(X)$ becomes a matricially normed space $\mathbb{M}_{p,q}(X)$ by the identification

$$M_n(M_{p,q}(X)) = M_{np,nq}(X).$$

By adding zeros it is a subspace of $\mathbb{M}_r(X)$ for $r \geq p,\ q$.

Examples: For a C^*-algebra A, $\mathbb{M}_p(A)$ is the C^*-Algebra of $p \times p$-matrices over A with its natural operator space structure.

The Banach space $M_p(A)$ is the first matrix level of the operator space $\mathbb{M}_p(A)$.

The complex numbers have a unique operator space structure which on the first matrix level is isometric to $\mathbb{C}$, and for this $M_p(\mathbb{C}) = M_p$ holds isometrically. We write

$$\mathbb{M}_p := \mathbb{M}_p(\mathbb{C}).$$

Then $\mathbb{M}_p$ always stands for the C^*-algebra of $p \times p$-matrices with its operator space structure. The Banach space M_p is the first matrix level of the operator space $\mathbb{M}_p$.

Columns and rows of an operator space

The space X^p of p-tupels over an operator space X can be made into an operator space for instance by reading the p-tupels as $p \times 1$- or as $1 \times p$-matrices. This leads to the frequently used **columns** and **rows** of an operator space X:

$$C_p(X) := \mathbb{M}_{p,1}(X) \quad \text{and} \quad R_p(X) := \mathbb{M}_{1,p}(X).$$

The first matrix level of these spaces are

$$M_1(C_p(X)) = M_{p,1}(X) \quad \text{and} \quad M_1(R_p(X)) = M_{1,p}(X), \text{ respectively.}$$

If $X \neq \{0\}$, the spaces $C_p(X)$ and $R_p(X)$ are not completely isometric. In general even the first matrix levels $M_{p,1}(X)$ and $M_{1,p}(X)$ are not isometric.

$\mathcal{C}_p := C_p(\mathbb{C})$ is called the **p-dimensional column space** and $\mathcal{R}_p := R_p(\mathbb{C})$ the **p-dimensional row space**.

The first matrix levels of $\mathcal{C}_p$ and $\mathcal{R}_p$ are isometric to l_2^p, but $\mathcal{C}_p$ and $\mathcal{R}_p$ are not completely isometric.

2.4 The space $CB(X, Y)$

Let X and Y be matricially normed spaces. A matrix $[T_{ij}] \in M_n(CB(X, Y))$ determines a completely bounded operator

$$\begin{aligned} T : X &\rightarrow \mathbb{M}_n(Y) \\ x &\mapsto [T_{ij}(x)]\,. \end{aligned}$$

Defining $\|[T_{ij}]\| = \|T\|_{\mathrm{cb}}$, $CB(X,Y)$ becomes a matricially normed space. It is an operator space, if Y is one. The equation

$$\mathbb{M}_p(CB(X,Y)) \stackrel{\mathrm{cb}}{=} CB(X,\mathbb{M}_p(Y))$$

holds completely isometrically.

2.5 The dual

The **dual**[9] of a matricially normed space X is defined as $X^* = CB(X,\mathbb{C})$ [Ble92a].[10] Its first matrix level is the dual of the first matrix level of X: $M_1(X^*) = (M_1(X))^*$.

The canonical embedding $X \hookrightarrow X^{**}$ is completely isometric [BP91, Thm. 2.11].

Some formulae

For matricially normed spaces X and Y, $m \in \mathbb{N}$, $y \in M_m(Y)$ and $T \in CB(X,Y)$ we have

$$\|y\| = \sup\{\|\Phi(y)\| \mid n \in \mathbb{N},\ \Phi \in CB(Y,\mathbb{M}_n),\ \|\Phi\|_{\mathrm{cb}} \le 1\}$$

and

$$\|T\|_{\mathrm{cb}} = \sup\{\|\Phi^{(n)} \circ T\|_{\mathrm{cb}} \mid n \in \mathbb{N},\ \Phi \in CB(Y,\mathbb{M}_n),\ \|\Phi\|_{\mathrm{cb}} \le 1\}.$$

A matrix $[T_{ij}] \in M_n(X^*)$ defines an operator

$$\begin{aligned} T : M_n(X) &\rightarrow \mathbb{C} \\ [x_{ij}] &\mapsto \sum_{i,j} T_{ij}x_{ij}. \end{aligned}$$

Thus we have an algebraic identification of $M_n(X^*)$ and $M_n(X)^*$ and further of $M_n(X^{**})$ and $M_n(X)^{**}$. The latter even is a complete isometry ([Ble92b, Cor. 2.14]):

$$\mathbb{M}_n(X^{**}) \stackrel{\mathrm{cb}}{=} \mathbb{M}_n(X)^{**}.$$

The isometry on the first matrix level is shown in [Ble92a, Thm. 2.5]. This already implies[11] the complete isometry. More generally we have[12]

$$CB(X^*,\mathbb{M}_n(Y)) \stackrel{\mathrm{cb}}{=} CB(\mathbb{M}_n(X)^*,Y).$$

[9] In the literature, this dual was originally called *standard dual* [Ble92a].

[10] The norm of a matricially normed space X is given by the unit ball $\mathrm{Ball}X \subset M(X)$. Here, $\mathrm{Ball}X^* = \{\Phi : X \to M_n \mid n \in \mathbb{N},\ \Phi \text{ completely contractive}\}$.

[11]

$$M_k(\mathbb{M}_n(X)^{**}) = M_k(M_n(X))^{**} = M_{kn}(X)^{**} = M_{kn}(X^{**}) = M_k(\mathbb{M}_n(X^{**})).$$

[12] This follows from $\mathbb{M}_n(X^{**}) \stackrel{\mathrm{cb}}{=} \mathbb{M}_n(X)^{**}$ and the above mentioned formula

$$\|T\|_{\mathrm{cb}} = \sup\{\|\Phi^{(n)} \circ T\|_{\mathrm{cb}} \mid n \in \mathbb{N},\ \Phi \in CB(Y,\mathbb{M}_n),\ \|\Phi\|_{\mathrm{cb}} \le 1\}.$$

X is called **reflexive**, if $X \stackrel{\text{cb}}{=} X^{**}$. An operator space X is reflexive if and only if its first matrix level $M_1(X)$ is a reflexive Banach space.

The adjoint operator

For $T \in CB(X,Y)$, the **adjoint operator** T^* is defined as usual. We have: $T^* \in CB(Y^*, X^*)$, and $\|T\|_{\text{cb}} = \|T^*\|_{\text{cb}}$. The mapping

$$\begin{array}{rcl} {}^* : CB(X,Y) & \to & CB(Y^*, X^*) \\ T & \mapsto & T^* \end{array}$$

even is completely isometric [Ble92b, Lemma 1.1].[13]

T^* is a complete quotient mapping if and only if T is completely isometric; T^* is completely isometric if T is a complete quotient mapping. Especially for a subspace $X_0 \subset X$ we have [Ble92a]:

$$X_0^* \stackrel{\text{cb}}{=} X^*/X_0^\perp$$

and, if X_0 is closed,

$$(X/X_0)^* \stackrel{\text{cb}}{=} X_0^\perp.$$

2.6 Direct sums

∞-direct sums

Let I be an index set and X_i for each $i \in I$ an operator space. Then there are an operator space X and complete contractions $\pi_i : X \to X_i$ with the following *universal mapping property*: For each family of complete contractions $\varphi_i : Z \to X_i$ there is exactly one complete contraction $\varphi : Z \to X$ such that $\varphi_i = \pi_i \circ \varphi$ for all i. X is called ∞-direct sum of the X_i and is denoted by $\bigoplus_\infty (X_i \mid i \in I)$. The π_i are complete quotient mappings.

One can construct a ∞-direct sum for instance as the linear subspace $X = \{(x_i) \in \prod_{i\in I} X_i \mid \sup\{\|x_i\| \mid i \in I\} < \infty\}$ of the cartesian product of the X_i, the π_i being the projections on the components. We have $M_n(X) = \{(x_i) \in \prod_{i\in I} M_n(X_i) \mid \sup\{\|x_i\| \mid i \in I\} < \infty\}$, and the operator space norm is given by $\|(x_i)\| = \sup\{\|x_i\| \mid i \in I\}$.

[13] The isometry on the matrix levels follows from the isometry on the first matrix level using the above mentioned formula $CB(\mathbb{M}_n(X)^*, Y) \stackrel{\text{cb}}{=} CB(X^*, \mathbb{M}_n(Y))$:

$$\begin{array}{rcl} M_n(CB(X,Y)) & = & M_1(CB(X, \mathbb{M}_n(Y))) \\ & \hookrightarrow & M_1(CB(\mathbb{M}_n(Y)^*, X^*)) \\ & = & M_1(CB(Y^*, \mathbb{M}_n(X^*))) \\ & = & M_n(CB(Y^*, X^*)). \end{array}$$

1-direct sums

Let I be an index set and X_i for each $i \in I$ an operator space. Then there are an operator space X and complete contractions $\iota_i : X_i \to X$ with the following *universal mapping property*: For each family of complete contractions $\varphi_i : X_i \to Z$ there is exactly one complete contraction $\varphi : X \to Z$ such that $\varphi_i = \varphi \circ \iota_i$ for all i. X is called 1-direct sum of the X_i and is denoted by $\bigoplus_1 (X_i \mid i \in I)$.The ι_i are completely isometric.

One can construct a 1-direct sum for instance as the closure of the sums of the images of the mappings $X_i \hookrightarrow X_i^{**} \xrightarrow{\pi_i^*} (\bigoplus_\infty (X_i^* \mid i \in I))^*$, where π_i is the projection from $\bigoplus_\infty (X_i^* \mid i \in I)$ onto X_i^*.

The equation

$$(\bigoplus_1 (X_i \mid i \in I))^* = \bigoplus_\infty (X_i^* \mid i \in I)$$

holds isometrically.

p-direct sums

p-direct sums of operator spaces for $1 < p < \infty$ can be obtained by interpolation between the ∞- and the 1-direct sum.

2.7 *MIN* and *MAX*

Let E be a normed space. Among all operator space norms on E which coincide on the first matrix level with the given norm, there is a greatest and a smallest. The matricially normed spaces given by these are called $MAX(E)$ and $MIN(E)$. They are characterized by the following *universal mapping property*:[14] For a matricially normed space X,

$$M_1(CB(MAX(E), X)) = B(E, M_1(X))$$

and

$$M_1(CB(X, MIN(E))) = B(M_1(X), E)$$

holds isometrically.

We have [Ble92a]

$$MIN(E)^* \overset{\text{cb}}{=} MAX(E^*),$$
$$MAX(E)^* \overset{\text{cb}}{=} MIN(E^*).$$

For $\dim(E) = \infty$,

$$\mathrm{id}_E : MIN(E) \to MAX(E)$$

[14] MAX is the left adjoint and MIN the right adjoint of the forgetfull functor which maps an operator space X to the Banach space $M_1(X)$.

is not completely bounded[Pau92, Cor. 2.13].[15]

Subspaces of *MIN*-spaces are *MIN*-spaces: For each isometric mapping $\varphi : E_0 \to E$, the mapping $\varphi : MIN(E_0) \to MIN(E)$ is completely isometric.

Quotients of *MAX*-spaces are *MAX*-spaces: For each quotient mapping $\varphi : E \to E_0$, the mapping $\varphi : MAX(E) \to MAX(E_0)$ is a complete quotient mapping.

Construction of *MIN*:

For a commutative C^*-algebra $A = C(K)$, each bounded linear mapping $\Phi : M_1(X) \to A$ is automatically completely bounded with $\|\Phi\|_{\text{cb}} = \|\Phi\|$ [Loe75].[16]

Each normed space E is isometric to a subspace of the commutative C^*-algebra $l_\infty(\text{Ball}(E^*))$. Thus the operator space $MIN(E)$ is given as a subspace of $l_\infty(\text{Ball}(E^*))$.

For $x \in M_n(MIN(E))$ we have

$$\|x\| = \sup\left\{ \|f^{(n)}(x)\| \,\middle|\, f \in \text{Ball}(E^*) \right\}.$$

The unit ball of $MIN(E)$ is given as the absolute matrix polar of $\text{Ball}(E^*)$.

Construction of *MAX*:

For a index set I, $l_1(I) = c_0(I)^*$. $l_1(I)$ is an operator space as dual of the commutative C^*-algebra $c_0(I)$, and each bounded linear mapping $\Phi : l_1(I) \to M_1(X)$ is automatically completely bounded with $\|\Phi\|_{\text{cb}} = \|\Phi\|$.[17]

Each Banach space[18] E is isometric to a quotient of $l_1(\text{Ball}(E))$. Thus the operator space $MAX(E)$ is given as a quotient of $l_1(\text{Ball}(E))$.

For $x \in M_n(MAX(E))$ we have

$$\|x\| = \sup\{\|\varphi^{(n)}(x)\| \mid n \in \mathbb{N},\ \varphi : E \to M_n,\ \|\varphi\| \leq 1\}.$$

The unit ball of $MAX(E)$ is given as the absolute matrix bipolar of $\text{Ball}(E^)$.

2.8 Injective operator spaces

2.8.1 Definition

A matricially normed space X is called **injective** if completely bounded mappings into X can be extended with the same norm. More exactly:

For all matricially normed spaces Y_0 and Y, each complete contraction $\varphi : Y_0 \to X$ and each complete isometry $\iota : Y_0 \to Y$ there is a complete contraction $\tilde{\varphi} : Y \to X$ such that $\tilde{\varphi}\iota = \varphi$.

[15] Paulsen uses in his proof a false estimation for the projection constant of the finite dimensional Hilbert spaces; the converse estimation is correct [Woj91, p. 120], but here useless. The gap can be filled [Lam97, Thm. 2.2.15] using the famous theorem of Kadets-Snobar: *The projection constant of an n-dimensional Banach space is less or equal than $\sqrt{n}$* [KS71].

[16] I. e. A is a *MIN*-space.

[17] I. e. $l_1(I)$ is a *MAX*-space.

[18] A similar construction is possible for normed spaces.

It suffices to consider only operator spaces Y_0 and Y. Injective matricially normed spaces are automatically comlete, so they are also called **injective operator spaces**.

2.8.2 Examples and elementary constructions

$B(\mathcal{H})$ is injective [].

Completely contractively projectable subspaces of injective operator spaces are injective.

L^∞-direct sums of injective operator spaces are injective.

Injective operator systems, injective C^*-algebras and injective von Neumann-algebras are injective operator spaces.

2.8.3 Characterizations

For a matricially normed space X the following conditions are equivalent:

a) X is injective.

b) For each complete isometry $\iota : X \to Z$ there is a complete contraction $\pi : Z \to X$ such that $\pi\iota = \mathrm{id}_X$. I. e. X is completely contractively projectable in each space containing it as a subspace.

c) For each complete isometry $\iota : X \to Z$ and each complete contraction $\varphi : X \to Y$ there is a complete contraction $\tilde{\varphi} : Z \to Y$ such that $\tilde{\varphi}\iota = \varphi$. I. e. Complete contractions from X can be extended completely contractively to any space conaining X as a subspace.[19]

d) X is completely isometric to a completely contractively projectable subspace of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

e) X is completely isometric to pAq, where A is an injective C^*-algebra and p and q are projections in A.[]

Robertson[] characterized the infinite dimensional injective subspaces of $B(l_2)$ up to isometry (not complete isometry!). They are $B(l_2)$, l_∞, l_2, $l_\infty \oplus l_2$ and $\bigoplus_{n\in\mathbb{N}}^{L^\infty} l_2$. (Countable L^∞-direct sums of such are again comletely isometric to one of these.) If an injective subspace of $B(l_2)$ is isometric to l_2, it is completely isometric to $\mathcal{R}_{l_2}$ or $\mathcal{C}_{l_2}$.[]

Injective envelopes

Let X be a matricially normed space.

An operator space Z together with a completely isometric mapping $\iota : X \to Z$ is called an **injective envelope** of X if Z is injective, and if id_Z is the *unique* extension of ι onto Z.

This is the case if and only if Z is the only injective subspace of Z which contains the image of X.[]

Each matricially normed space has an injective envelope. It is unique up to a canonical isomorphism.

[19] Equivalently: Completely bounded mappings from X can be extended with the same norm.

A matricially normed space Z together with a completely isometric mapping $\iota : X \to Z$ is called an **essential extension** of X if a complete contraction $\varphi : Z \to Y$ is completely isometric if only $\varphi \circ \iota$ is completely isometric.

$\iota : X \to Z$ is an injective envelope if and only if Z is injective and $\iota : X \to Z$ is an essential extension.

Every injective envelope $\iota : X \to Z$ is a maximal essential extension, i. e. for each essential extension $\tilde{\iota} : X \to \tilde{Z}$, there is a completely isometric mapping $\varphi : \tilde{Z} \to Z$ such that $\varphi \circ \tilde{\iota} = \iota$.

3 Operator Systems and Completely Positive Maps

3.1 Definitions

Let V be a complex vector space. An **involution** on V is a conjugate linear map $* : V \to V$, $v \mapsto v^*$, such that $v^{**} = v$. A complex vector space is an involutive vector space if there is an involution on V. Let V be an involutive vector space. Then V_{sa} is the real vector space of selfadjoint elements of V, i.e. those elements of V, such that $v^* = v$. An involutive vector space is an **ordered** vector space if there is a proper cone[20] $V^+ \subset V_{\mathrm{sa}}$. The elements of V^+ are called positive and there is an order on V_{sa} defined by $v \leq w$ if $w - v \in V^+$ for $v, w \in V_{\mathrm{sa}}$.

An element $\mathbb{1} \in V^+$ is an **order unit** if for any $v \in V_{\mathrm{sa}}$ there is a real number $t > 0$, such that $-t\mathbb{1} \leq v \leq t\mathbb{1}$. If V_+ has an order unit then $V_{\mathrm{sa}} = V_+ - V_+$.

The cone V^+ is called **Archimedian** if $w \in -V^+$ whenever there exists $v \in V_{\mathrm{sa}}$ such that $tw \leq v$ for all $t > 0$. If there is an order unit $\mathbb{1}$ then V_+ is Archimedian if $w \in -V^+$ whenever $tw \leq \mathbb{1}$ for all $t > 0$.

Let V be an ordered vector space. If V_+ is Archimedian and contains a distinguished order unit $\mathbb{1}$ then $(V, \mathbb{1})$ is called an **ordered unit space**.

Let V, W be involutive vector spaces. We define an involution $\star$ on the space $L(V, W)$ of all linear mappings from $V \to W$ by $\varphi^\star(v) = \varphi(v^*)^*$, $\varphi \in L(V, W)$. If moreover V, W are ordered vector spaces, then φ is **positive** if $\varphi^\star = \varphi$ and $\varphi(V^+) \subset W^+$. If $(V, \mathbb{1})$ and $(W, \mathbb{1}')$ are ordered unit spaces a positve map $\varphi : V \to W$ is called **unital** if $\varphi(\mathbb{1}) = \mathbb{1}'$.

Let V be an involutive vector space. Then $M_n(V)$ is also an involutive vector space by $[v_{ij}]^* = [v_{ji}^*]$. V is a **matrix ordered** vector space if there are proper cones $M_n(V)^+ \subset M_n(V)_{\mathrm{sa}}$ for all $n \in \mathbb{N}$, such that $\alpha^* M_p(V)^+ \alpha \subset M_q(V)^+$ for all $\alpha \in M_{pq}$ and $p, q \in \mathbb{N}$ holds[21]. This means that $(M_n(V)_+)_{n \in \mathbb{N}}$ is a matrix cone.

Let V, W be matrix ordered vector spaces. A linear mapping $\phi : V \to W$ is **completely positive** if $\phi^{(n)} : M_n(V) \to M_n(W)$ is positive for all $n \in \mathbb{N}$. A **complete order isomorphism**[22] from V to W is a completely positive map from $V \to W$ that is bijective, such that the inverse map is completely positive.

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[20] A cone K is a subset of a vector space, such that $K + K \subset K$ and $\mathbb{R}_+ K \subset K$. If moreover $(-K) \cap K = \{0\}$ holds, K is a proper cone.

[21] If $V^+ = M_1(V)^+$ is a proper cone, by the matrix condition all the cones $M_n(V)^+$ will be proper.

[22] Note that some authors don't include surjectivity in the definition.

The well-known **Stinespring theorem** for completely positive maps reads [Pau86, Theorem 4.1]:

Let A be a unital C^*-algebra and let H be a Hilbert space. If $\psi : A \to B(H)$ is completely positive then there are a Hilbert space H_π, a unital $*$-homomorphism $\pi : A \to B(H_\pi)$ and a linear mapping $V : H \to H_\pi$, such that $\psi(a) = V^*\pi(a)V$ for all $a \in A$.

Let V be an involutive vector space. Then V is called an **operator system** if it is a matrix ordered ordered unit space, such that $M_n(V)^+$ is Archimedian for all $n \in \mathbb{N}$. In this case $M_n(V)$ is an ordered unit space with order unit $\mathbb{1}_n = \mathbb{1} \otimes \mathrm{id}_{M_n}$ for all $n \in \mathbb{N}$, where $\mathbb{1} \in V_+$ is the distinguished order unit of V and id_{M_n} is the unit of $\mathbb{M}_n$.

Example

Let H be a Hilbert space. Then, obviously, $B(H)$ is an ordered unit space with order unit the identity operator. Using the identification $M_n(B(H)) = B(H^n)$ we let $M_n(B(H))_+ = B(H^n)_+$. So we see that $B(H)$ is an operator system.

Let L be an operator system. Then any subspace $S \subset L$ that is selfadjoint, i.e. $S^* \subset S$, and contains the order unit of L is again an operator system with the induced matrix order. So unital C^*-algebras and selfadjoint subspaces of unital C^*-algebras containig the identity are operator systems.

Note that a unital complete order isomorphism between unital C^*-algebras must be a $*$-isomorphism [Cho74, Corollary 3.2]. So unital C^*-algebras are completely characterized by their matrix order. They are not characterized by their order. For instance take the opposite algebra A^{op} of a unital C^*-algebra A. Then $A_+ = A^{op}_+$ but A and A^{op} are not $*$-isomorphic. Obviously $M_2(A)_+ \neq M_2(A^{op})_+$.

3.2 Characterization

Choi and Effros [CE77, Theorem 4.4] showed the following characterization theorem:

Let V be an operator system. Then there are a Hilbert space H and a unital complete order isomorphism from V to a selfadjoint subspace of $B(H)$.

A unital complete order isomorphism is obtained by

$$\begin{aligned} \Phi : V &\to \bigoplus_{n\in\mathbb{N}} \bigoplus_{\varphi\in S_n} \mathbb{M}_n \\ x &\mapsto (\varphi(x))_\varphi, \end{aligned}$$

where S_n is the set of all unital completely positive maps $\varphi : V \to \mathbb{M}_n$.

3.3 Matrix order unit norm

Let L be an operator system. We define norms by

$$\|x\|_n := \inf \left\{ r \in \mathbb{R} | \begin{pmatrix} r\mathbb{1}_n & x \\ x^* & r\mathbb{1}_n \end{pmatrix} \in M_{2n}(L)_+ \right\} \tag{1}$$

for all $n \in \mathbb{N}$ and $x \in M_n(L)$. With these norms L becomes an operator space.

If Φ is any unital completely positve embedding from L into some $B(H)$ (cf. section 3.2) then $\|\Phi^{(n)}(x)\| = \|x\|_n$ for all $n \in \mathbb{N}$ and $x \in M_n(L)$. This holds because $\|y\| \leq 1$ if and only if

$$0 \leq \begin{pmatrix} \mathbb{1} & y \\ y^* & \mathbb{1} \end{pmatrix}$$

for all $y \in B(H)$ and all Hilbert spaces H.

Let L and S be operator systems and let $\psi : L \to S$ be completely positive. We supply L and S with the norms from equation (1). Then ψ is completely bounded and $\|\psi(\mathbb{1})\| = \|\psi\| = \|\psi\|_{cb}$ (cf. [Pau86, Proposition 3.5]).

3.4 Injective operator systems

An operator system R is called **injectiv** if given operator systems $N \subset M$ each completely positive map $\varphi : N \to R$ has a completely positive extension $\psi : M \to R$.

If an operator system R is injective then there is a unital complete order isomorphism from R onto a unital C^*-algebra. The latter is conditionally complete[23]. (cf. [CE77, Theorem 3.1])

4 Hilbertian Operator Spaces

4.1 The spaces

An operator space X is called **hilbertian**, if $M_1(X)$ is a Hilbert space $\mathcal{H}$. An operator space X is called **homogeneous**, if each bounded operator $T : M_1(X) \to M_1(X)$ is completely bounded and $\|T\|_{\mathrm{cb}} = \|T\|$ [Pis96].

Examples The minimal hilbertian operator space $MIN_{\mathcal{H}} := MIN(\mathcal{H})$ and the maximal hilbertian operator space $MAX_{\mathcal{H}} := MAX(\mathcal{H})$, the **column Hilbert space** $\mathcal{C}_{\mathcal{H}} := B(\mathbb{C}, \mathcal{H})$ and the **row Hilbert space** $\mathcal{R}_{\mathcal{H}} := B(\overline{\mathcal{H}}, \mathbb{C})$ are homogeneous hilbertian operator spaces on the Hilbert space $\mathcal{H}$.

Furthermore, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we have completely isometric isomorphisms [ER91, Thm. 4.1] [Ble92b, Prop. 2.2]

$$CB(\mathcal{C}_{\mathcal{H}}, \mathcal{C}_{\mathcal{K}}) \stackrel{\mathrm{cb}}{=} B(\mathcal{H}, \mathcal{K}) \text{ and } CB(\mathcal{R}_{\mathcal{H}}, \mathcal{R}_{\mathcal{K}}) \stackrel{\mathrm{cb}}{=} B(\overline{\mathcal{K}}, \overline{\mathcal{H}}).$$

These spaces satisfy the following dualities [Ble92b, Prop. 2.2] [Ble92a, Cor. 2.8]:

$$\begin{aligned} \mathcal{C}_{\mathcal{H}}^* &\stackrel{\mathrm{cb}}{=} \mathcal{R}_{\overline{\mathcal{H}}} \\ \mathcal{R}_{\mathcal{H}}^* &\stackrel{\mathrm{cb}}{=} \mathcal{C}_{\overline{\mathcal{H}}} \\ MIN_{\mathcal{H}}^* &\stackrel{\mathrm{cb}}{=} MAX_{\overline{\mathcal{H}}} \\ MAX_{\mathcal{H}}^* &\stackrel{\mathrm{cb}}{=} MIN_{\overline{\mathcal{H}}}. \end{aligned}$$

[23] An ordered vector space V is conditionally complete if any upward directed subset of V_{sa} that is bounded above has a supremum in V_{sa}.

For each Hilbert space $\mathcal{H}$ there is a unique completely self dual homogeneous operator space, the **operator Hilbert space** $OH_{\mathcal{H}}$ [Pis96, §1]:

$$OH^*_{\mathcal{H}} \overset{\text{cb}}{=} OH_{\overline{\mathcal{H}}}.$$

The intersection and the sum of two homogeneous hilbertian operator spaces are again homogeneous hilbertian operator spaces [Pis96].

4.2 The morphisms

The space $CB(X, Y)$ of completely bounded mappings between two homogeneous hilbertian operator spaces X, Y enjoys the following properties (cf. [MP95, Prop. 1.2]):

1. $(CB(X, Y), \|\cdot\|_{\text{cb}})$ is a Banach space.
2. $\|ATB\|_{\text{cb}} \le \|A\|\|T\|_{\text{cb}}\|B\|$ for all A, $B \in B(\mathcal{H})$, $T \in CB(X, Y)$.
3. $\|T\|_{\text{cb}} = \|T\|$ for all T with $\text{rank}(T) = 1$.

Consequently, $CB(X, Y)$ is a symmetrically normed ideal (s.n. ideal) in the sense of Calkin, Schatten [Sch70] and Gohberg [GK69].

The classical examples for s.n. ideals are the famous Schatten ideals:

$$S_p := \{T \in B(\mathcal{H}) \mid \text{the sequence of singular values of } T \text{ is in } \ell_p\} \quad (1 \le p < \infty).$$

Many, but not all s.n. ideals can be represented as spaces of completely bounded maps between suitable homogeneous hilbertian operator spaces.

The first result in this direction was

$$CB(\mathcal{R}_{\mathcal{H}}, \mathcal{C}_{\mathcal{H}}) = S_2(\mathcal{H}) = HS(\mathcal{H})$$

isometrically [ER91, Cor. 4.5].

We have the following characterizations isometrically or only isomorphically ($\simeq$) [Mat94], [MP95], [Lam97]:

$CB(\downarrow, \rightarrow)$	$MIN_{\mathcal{H}}$	$\mathcal{C}_{\mathcal{H}}$	$OH_{\mathcal{H}}$	$\mathcal{R}_{\mathcal{H}}$	$MAX_{\mathcal{H}}$
$MIN_{\mathcal{H}}$	$B(\mathcal{H})$	$HS(\mathcal{H})$	$\simeq HS(\mathcal{H})$	$HS(\mathcal{H})$	$\simeq N(\mathcal{H})$
$\mathcal{C}_{\mathcal{H}}$	$B(\mathcal{H})$	$B(\mathcal{H})$	$S_4(\mathcal{H})$	$HS(\mathcal{H})$	$HS(\mathcal{H})$
$OH_{\mathcal{H}}$	$B(\mathcal{H})$	$S_4(\mathcal{H})$	$B(\mathcal{H})$	$S_4(\mathcal{H})$	$\simeq HS(\mathcal{H})$
$\mathcal{R}_{\mathcal{H}}$	$B(\mathcal{H})$	$HS(\mathcal{H})$	$S_4(\mathcal{H})$	$B(\mathcal{H})$	$HS(\mathcal{H})$
$MAX_{\mathcal{H}}$	$B(\mathcal{H})$	$B(\mathcal{H})$	$B(\mathcal{H})$	$B(\mathcal{H})$	$B(\mathcal{H})$

As a unique completely isometric isomorphism, we get $CB(\mathcal{C}_{\mathcal{H}}) \overset{\text{cb}}{=} B(\mathcal{H})$ (cf. [Ble95, Thm. 3.4]). The result

$$CB(MIN_{\mathcal{H}}, MAX_{\mathcal{H}}) \simeq N(\mathcal{H})$$

is of special interest. Here, we have a new quite natural norm on the nuclear operators, which is not equal to the canonical one.

Even in the finite dimensional case, we only know

$$\frac{n}{2} \leq \|\mathrm{id} : MIN(\ell_2^n) \to MAX(\ell_2^n)\|_{\mathrm{cb}} \leq \frac{n}{\sqrt{2}}$$

[Pau92, Thm. 2.16]. To compute the exact constant is still an open problem. Paulsen conjectured that the upper bound is sharp [Pau92, p. 121].

Let E be a Banach space and $\mathcal{H}$ a Hilbert space. An operator $T \in B(E, \mathcal{H})$ is completely bounded from $MIN(E)$ to $\mathcal{C}_{\mathcal{H}}$, if and only if T is 2-summing [Pie67] from E to $\mathcal{H}$ (cf. [ER91, Thm. 5.7]):

$$M_1(CB(MIN(E), \mathcal{C}_{\mathcal{H}})) = \Pi_2(E, \mathcal{H})$$

with $\|T\|_{\mathrm{cb}} = \pi_2(T)$.

4.3 The column Hilbert space $\mathcal{C}_{\mathcal{H}}$

For the Hilbert space $\mathcal{H} = \ell_2$, we can realize the column Hilbert space $\mathcal{C}_{\mathcal{H}}$ as a column, by the embedding

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & B(\mathcal{H}), \\ \begin{pmatrix} \vdots \\ \xi_i \\ \vdots \\ \vdots \end{pmatrix} & \mapsto & \begin{bmatrix} \vdots & 0 & \cdots & 0 & \cdots \\ \xi_i & \vdots & & \vdots & \\ \vdots & 0 & \cdots & 0 & \\ \vdots & \vdots & & & \ddots \end{bmatrix}. \end{array}$$

Via this identification, $\mathcal{C}_{\ell_2^n}$ is the n-dimensional column space $\mathcal{C}_n$.

$\mathcal{C}_{\mathcal{H}}$ is a homogeneous hilbertian operator space: All bounded maps on $\mathcal{H}$ are completely bounded with the same norm on $\mathcal{C}_{\mathcal{H}}$. Actually we have $CB(\mathcal{C}_{\mathcal{H}}) \overset{\mathrm{cb}}{=} B(\mathcal{H})$ completely isometrically [ER91, Thm. 4.1].

$\mathcal{C}_{\mathcal{H}}$ is a injective operator space (cf. [Rob91]).

Tensor products

Let X be an operator space. We have complete isometries [ER91, Thm. 4.3 (a)(c)] [Ble92b, Prop. 2.3 (i)(ii)]:

$$\mathcal{C}_{\mathcal{H}} \otimes_h X \overset{\mathrm{cb}}{=} \mathcal{C}_{\mathcal{H}} \overset{\vee}{\otimes} X$$

and

$$X \otimes_h \mathcal{C}_{\mathcal{H}} \overset{\mathrm{cb}}{=} X \overset{\wedge}{\otimes} \mathcal{C}_{\mathcal{H}}.$$

Herein, $\otimes_h$ is the Haagerup tensor product, $\overset{\vee}{\otimes}$ the injective tensor product and $\overset{\wedge}{\otimes}$ the projective tensor product.

For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we have complete isometries [ER91, Cor. 4.4.(a)] [Ble92b, Prop. 2.3(iv)]

$$\mathcal{C}_{\mathcal{H}} \otimes_h \mathcal{C}_{\mathcal{K}} \overset{\mathrm{cb}}{=} \mathcal{C}_{\mathcal{H}} \overset{\vee}{\otimes} \mathcal{C}_{\mathcal{K}} \overset{\mathrm{cb}}{=} \mathcal{C}_{\mathcal{H}} \overset{\wedge}{\otimes} \mathcal{C}_{\mathcal{K}} \overset{\mathrm{cb}}{=} \mathcal{C}_{\mathcal{H} \otimes_2 \mathcal{K}}.$$

4.3.1 Characterizations

In connection with the column Hilbert space, it is enough to calculate the row norm

$$\|T\|_{\text{row}} := \sup_{n\in\mathbb{N}} \sup_{\|[x_1\dots x_n]\|_{M_{1,n}(X)}\le 1} \|[Tx_1 \dots Tx_n]\|_{M_{1,n}(Y)}$$

or the column norm

$$\|T\|_{\text{col}} := \sup_{n\in\mathbb{N}} \sup_{\left\|\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right\|_{M_{n,1}(Y)}\le 1} \left\|\begin{bmatrix} Tx_1 \\ \vdots \\ Tx_n \end{bmatrix}\right\|_{M_{n,1}(X)}$$

of an operator T, instead of the cb-norm, to ascertain the complete boundedness.

Let X be an operator space and $S : \mathcal{C}_{\mathcal{H}} \to X$ bzw. $T : X \to \mathcal{C}_{\mathcal{H}}$. Then we have $\|S\|_{\text{cb}} = \|S\|_{\text{row}}$ resp. $\|T\|_{\text{cb}} = \|T\|_{\text{col}}$ ([Mat94, Prop. 4] resp. [Mat94, Prop. 2]).

The column Hilbert space is characterized as follows,

(A) as a hilbertian operator space [Mat94, Thm. 8]:

For an operator space X on an Hilbert space $\mathcal{H}$, we have the following equivalences:

1. *X is completely isometric to $\mathcal{C}_{\mathcal{H}}$.*
2. *For all operator spaces Y and all $T : X \to Y$ we have $\|T\|_{\text{cb}} = \|T\|_{\text{row}}$, and for all $S : Y \to X$ we have $\|S\| = \|S\|_{\text{row}}$. For all operator spaces Y and all $T : Y \to X$ we have $\|T\|_{\text{cb}} = \|T\|_{\text{col}}$, and for all $S : X \to Y$ we have $\|S\| = \|S\|_{\text{col}}$.*
3. *X coincides with the maximal hilbertian operator space on columns and with the minimal hilbertian operator space on rows. That means isometrically*

$$\begin{aligned} M_{n,1}(X) &= M_{n,1}(MAX(\mathcal{H})) \\ M_{1,n}(X) &= M_{1,n}(MIN(\mathcal{H})) \end{aligned}$$

(B) as an operator space: *For an operator space X TFAE:*

1. *There is a Hilbert space $\mathcal{H}$, such that $X \overset{\text{cb}}{=} \mathcal{C}_{\mathcal{H}}$ completely isometrically.*
2. *We have*

$$M_{n,1}(X) = \oplus_2 M_1(X)$$

and

$$M_{1,n}(X) = M_{1,n}(MIN(M_1(X)))$$

isometrically. [Mat94, Thm. 10].

3. *$CB(X)$ with the composition as multiplication is an operator algebra* [Ble95, Thm. 3.4].

4.4 Column Hilbert space factorization

Let X, Y be operator spaces. We say that a linear map $T : M_1(X) \to M_1(Y)$ factors through a column Hilbert space, if there is a Hilbert space $\mathcal{H}$ and completely bounded maps $T_2 : X \to \mathcal{C}_\mathcal{H}$, $T_1 : \mathcal{C}_\mathcal{H} \to Y$ with $T = T_1 \circ T_2$. We define

$$\gamma_2(T) := \inf \|T_1\|_{\mathrm{cb}} \|T_2\|_{\mathrm{cb}},$$

where the infimum runs over all possible factorizations. If no such factorisation exists we say $\gamma_2(T) := \infty$. $\Gamma_2(X, Y)$ is the Banach space of all linear maps $T : X \to Y$ with $\gamma_2(T) < \infty$ [ER91, Chap. 5],[Ble92b, p. 83].

Let X_1 and Y_1 be operator spaces and $T \in \Gamma_2(X, Y)$, $S \in CB(X_1, X)$, $R \in CB(Y, Y_1)$. Then we have the CB ideal property

$$\gamma_2(RTS) \le \|R\|_{\mathrm{cb}} \gamma_2(T) \|S\|_{\mathrm{cb}}.$$

We interpret a matrix $T = [T_{ij}] \in M_n(\Gamma_2(X, Y))$ as a mapping from X to $M_n(Y)$: $[T_{ij}](x) := [T_{ij}(x)]$. T has a factorization in completely bounded mappings

$$X \xrightarrow{T_2} M_{1,n}(\mathcal{C}_\mathcal{H}) \xrightarrow{T_1} M_n(Y).$$

Again, we define

$$\gamma_2(T) := \inf \|T_1\|_{\mathrm{cb}} \|T_2\|_{\mathrm{cb}},$$

where the infimum is taken over all factorizations. So we get an operator space structure on $\Gamma_2(X, Y)$ [ER91, Cor. 5.4].

Let X, Y be operator spaces and Y_0 an operator subspace of Y. Then the inclusion $\Gamma_2(X, Y_0) \hookrightarrow \Gamma_2(X, Y)$ is completely isometric [ER91, Prop. 5.2].

Let X, Y be operator spaces. It is well known that every linear map

$$T : X \to Y^*$$

defines a linear functional

$$f_T : Y \otimes X \to \mathbb{C}$$

via

$$\langle f_T, y \otimes x \rangle := \langle T(x), y \rangle.$$

This identification determines the complete isometry [ER91, Thm. 5.3] [Ble92b, Thm. 2.11]

$$\Gamma_2(X, Y^*) \overset{\mathrm{cb}}{=} (Y \otimes_h X)^*.$$

Let X, Y, Z be operator spaces. We get a complete isometry

$$\Gamma_2(Y \otimes_h X, Z) \overset{\mathrm{cb}}{=} \Gamma_2(X, \Gamma_2(Y, Z))$$

via the mapping

$$\begin{array}{rcl} T & \mapsto & \widetilde{T} \\ \widetilde{T}(x)(y) & := & T(y \otimes x) \end{array}$$

[ER91, Cor. 5.5].

5 Multiplicative Structures

For an abstract C^*-algebra the GNS construction provides a concrete representation of its elements as bounded operators on a Hilbert space. For non-selfadjoint algebras there is, hitherto, no analogue in the framework of classical functional analysis. But endowed with an operator space structure (which is compatible with the multiplicative structure), these non-selfadjoint algebras do have a representation in some $B(\mathcal{H})$ (theorem of Ruan type for operator algebras).

The so-called operator modules (over algebras) are also characterized by Axioms of Ruan type; here, matrices whose entries are algebra elements take the place of the scalar ones. The corresponding morphisms are the completely bounded module homomorphisms, the most important properties of which make their appearance in Representation, Decomposition and Extension Theorems.

5.1 Operator modules

Let $A_1, A_2 \subset B(\mathcal{H})$ be C^*-algebras with $\mathbb{1}_{\mathcal{H}} \in A_1, A_2$. A closed subspace X of $B(\mathcal{H})$ is called a concrete (A_1, A_2)-**operator module**, if $A_1 X \subset X$ and $X A_2 \subset X$. Whenever $A_1 = A_2 = A$, we call X a concrete A-**operator bimodule** (cf. [ER88, p. 137]).

In analogy to the operator space or the operator algebra situation, there is an abstract characterization of operator modules (cf. [Pop00, Déf. 4.1]):
Consider, as above, two unital C^*-algebras $A_1, A_2 \subset B(\mathcal{H})$ with $\mathbb{1}_{\mathcal{H}} \in A_1, A_2$, and an (algebraic) (A_1, A_2)-module X. We call X an abstract (A_1, A_2)-operator module, if it carries an operator space structure satisfying the following axioms (of Ruan type):

$$\text{(R1)} \quad \|axb\|_m \leq \|a\|\|x\|_m\|b\|$$

$$\text{(R2)} \quad \left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\},$$

where $m, n \in \mathbb{N}$, $a \in M_m(A_1)$, $x \in M_m(X)$, $y \in M_n(Y)$, $b \in M_m(A_2)$.

For abstract operator modules holds a **representation theorem of Ruan type** (cf. [Pop00, Thm. 4.7]):
Let V be an abstract (A_1, A_2)-operator module. Then there exist a Hilbert space $\mathcal{K}$, a complete isometry $\Theta : X \hookrightarrow B(\mathcal{K})$ and *-representations π_1, π_2 of A_1 resp. A_2 in $B(\mathcal{K})$ such that:

$$\Theta(axb) = \pi_1(a)\Theta(x)\pi_2(b),$$

where $x \in X$, $a \in A_1$, $b \in A_2$. In case $A_1 = A_2$, one can even choose $\pi_1 = \pi_2$.

Basic examples of operator modules

Let A be a unital C^*-algebra, X a normed space, and Y an operator space. Then $B(X, A)$ resp. $CB(Y, A)$ are operator spaces via the identifications $M_n(B(X, A)) =$

$B(X, M_n(A))$ resp. $M_n(CB(Y,A)) = CB(Y, M_n(A))$. These become A-operator bimodules, when endowed with the natural module operations as follows ([ER88, p. 140]):

$$(a \cdot \varphi \cdot b)(x) = a\varphi(x)b$$

for all $a, b \in A$, $\varphi \in B(X,A)$ resp. $CB(Y,A)$, $x \in X$ resp. $x \in Y$.

In the category of operator modules, the morphisms are the completely bounded module homomorphisms. For these we have a representation and an extension theorem.

Representation theorem (cf. [Hof95, Kor. 1.4]):
Let $\mathcal{H}$ be a Hilbert space, M a C^*-algebra in $B(\mathcal{H})$, and A, B C^*-subalgebras of M. Then the following hold true:

(a) (cf. [Pau86, Thm. 7.4]) For each completely bounded (A,B)-module homomorphism $\Phi : M \to B(\mathcal{H})$, there exist a Hilbert space $\mathcal{K}$, a *-representation $\pi : M \to B(\mathcal{K})$ and linear operators $v, w \in B(\mathcal{H}, \mathcal{K})$ sharing the following properties:

  (a1) $\Phi(x) = v^*\pi(x)w$ for all $x \in M$, i.e. $(\mathcal{K}; \pi; v^*; w)$ is a representation of Φ

  (a2) $\|\Phi\|_{\text{cb}} = \|v\|\|w\|$

  (a3) $\overline{\text{lin}}(\pi(M)v\mathcal{H}) = \overline{\text{lin}}(\pi(M)w\mathcal{H}) = \mathcal{K}$

  (a4) $v^*\pi(a) = av^*$ for all $a \in A$ and $\pi(b)w = wb$ for all $b \in B$.

(b) (cf. also [Smi91, Thm.3.1]) If, in addition, $M \subset B(\mathcal{H})$ is a von Neumann algebra and $\Phi : M \to B(\mathcal{H})$ is a normal completely bounded (A,B)-module homomorphism, one can require the *-representation π of part (a) to be normal. There exist families $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ in the commutant of A and B, respectively, with the following properties (the sums are to be taken in the WOT topology):

  (b1) $\Phi(x) = \sum_{i\in I} a_i x b_i$ for all $x \in M$

  (b2) $\sum_{i\in I} a_i a_i^* \in B(\mathcal{H})$, $\sum_{i\in I} b_i^* b_i \in B(\mathcal{H})$ and $\|\Phi\|_{\text{cb}} = \|\sum_{i\in I} a_i a_i^*\|^{\frac{1}{2}} \|\sum_{i\in I} b_i^* b_i\|^{\frac{1}{2}}$.

Extension theorem ([Wit84a, Thm. 3.1], cf. also [MN94, Thm. 3.4] and [Pau86, Thm. 7.2]):
Let F be an injective C^*-algebra, and let $A, B \subset F$ be two unital C^*-subalgebras. Consider furthermore two (A,B)-operator modules E_0 and E with $E_0 \subset E$. Then for each $\phi_0 \in CB_{(A,B)}(E_0, F)$, there exists an extension $\phi \in CB_{(A,B)}(E, F)$ with $\phi|_{E_0} = \phi_0$ and $\|\phi\|_{\text{cb}} = \|\phi_0\|_{\text{cb}}$.

The decomposition theorem for completely bounded module homomorphisms can be found in the corresponding chapter.

5.2 Completely bounded module homomorphisms

Let $A, B \subset B(\mathcal{H})$ be C^*-algebras with $\mathbb{1}_\mathcal{H} \in A, B$, and let E and F be two (A, B)-operator modules, i.e. (algebraic) A-left-B-right-modules. A mapping $\phi \in L(E, F)$ is called (A, B)**-module homomorphism** (in case $A = B$ A**-bimodule homomorphism**) if

$$\phi(axb) = a\phi(x)b$$

for all $a \in A$, $b \in B$, $x \in E$.
Furthermore we will write $CB_{(A,B)}(E, F)$ for the set of all completely bounded (A, B)-module homomorphisms between E and F. The space $CB_{(A,B)}(E)$ with the composition of operators as multiplication is a Banach algebra.

Let $A_1, A_2 \subset B(\mathcal{H})$ be C^*-algebras such that $\mathbb{1}_\mathcal{H} \in A_1, A_2$. Let further $A \subset A_1 \cap A_2$ be a unital *-subalgebra of A_1 and A_2 with $\mathbb{1}_\mathcal{H} \in A$. An A-bimodule homomorphism

$$\Phi : A_1 \to A_2$$

is called **self-adjoint** if

$$\Phi(x)^* = \Phi(x^*)$$

for all $x \in A_1$.

Dealing with completely bounded module homomorphisms, we have at our disposal a representation theorem, an extension theorem and the following **decomposition theorem of Wittstock** ([Wit81, Satz 4.5] and cf. [Pau86, Thm. 7.5]):

Let A, E and F be unital C^*-algebras. Let moreover F be injective, and A be a subalgebra of E and F with $\mathbb{1}_E = \mathbb{1}_F = \mathbb{1}_A$. Then for each self-adjoint completely bounded A-bimodule homomorphism $\phi : E \to F$, there exist two completely positive A-bimodule homomorphisms ϕ_1 and ϕ_2 sharing the properties $\phi = \phi_1 - \phi_2$ and $\|\phi\|_{\mathrm{cb}} = \|\phi_1 + \phi_2\|_{\mathrm{cb}}$.

Consider two von Neumann algebras M and N, and two C^*-algebras $A_1,\ A_2 \subset B(H)$, where $\mathbb{1}_\mathcal{H} \in A_1$, A_2 and $A_1 \cup A_2 \subset M \cap N$. We then have the **decomposition theorem of Tomiyama-Takesaki** (cf. [Tak79, Def. 2.15]): Each operator $\phi \in CB_{(A_1,A_2)}(M, N)$ has a unique decomposition $\phi = \phi^\sigma + \phi^s$, $\phi^\sigma,\ \phi^s \in CB_{(A_1,A_2)}(M, N)$ normal resp. singular, where $\|\phi^\sigma\|_{\mathrm{cb}},\ \|\phi^s\|_{\mathrm{cb}} \le \|\phi\|_{\mathrm{cb}}$. We thus obtain the algebraically direct sum decomposition:

$$CB_{(A_1,A_2)}(M, N) \quad = \quad CB^\sigma_{(A_1,A_2)}(M, N) \oplus CB^s_{(A_1,A_2)}(M, N). \qquad (2)$$

Here, the notions "normal" and "singular", repectively, are built in analogy to the framework of linear functionals on a von Neumann algebra M.[24]

We list some basic facts about the spaces and mappings mentioned in (2):

[24] Let M_* denote the (unique) predual of M. Then we have the ℓ_1-direct sum decomposition

$$M^* = M_* \oplus_{\ell_1} (M^*)^s$$

of M^* into normal (i.e. w^*-continuous) and singular functionals. [In the literature, one usually writes $M_*^\perp$ instead of M^{*s}, corresponding to $M_*(= M^{*\sigma})$.] Analogously, an operator $\phi \in B(M, N)$, M, N von Neumann algebras, is called **normal** (i.e. w^*-w^*-continuous), if $\phi^*(N_*) \subset M_*$, and it is called **singular**, if $\phi^*(N_*) \subset M^{*s}$.

(a) In case $M = N$, all the spaces in (2) are Banach algebras.

(b) The following properties of ϕ are hereditary for the normal part ϕ^σ and the singular part ϕ^s : completely positive, homomorphism, *-homomorphism.

(c) If $\alpha \in \mathrm{Aut}(M)$ and $\beta \in \mathrm{Aut}(N)$ are *-automorphisms, we have $(\beta\phi\alpha)^\sigma = \beta\phi^\sigma\alpha$ and $(\beta\phi\alpha)^s = \beta\phi^s\alpha$.

(d) For $\phi \in CB(B(\mathcal{H}))$, $\mathcal{H}$ a Hilbert space, we have: $\phi \in CB^s(B(\mathcal{H})) \Leftrightarrow \phi|_{K(\mathcal{H})} \equiv 0$.

Let $\mathcal{H}$ be a Hilbert space, and let $A_1, A_2 \subset B(\mathcal{H})$ be two C^*-algebras with $\mathbb{1}_{\mathcal{H}} \in A_1, A_2$. Then we obtain [Pet97, Prop. 4.2.5]:

$$CB^\sigma_{(A_1,A_2)}(B(\mathcal{H})) \overset{\text{cb}}{=} CB_{(A_1,A_2)}(K(\mathcal{H}), B(\mathcal{H})) \tag{3}$$

$$CB^s_{(A_1,A_2)}(B(\mathcal{H})) \overset{\text{cb}}{=} CB_{(A_1,A_2)}(Q(\mathcal{H}), B(\mathcal{H})) \tag{4}$$

completely isometrically, where $Q(\mathcal{H}) = B(\mathcal{H})/K(\mathcal{H})$ denotes the Calkin algebra.

Let X be an arbitrary operator space. Then the space of all completely bounded (A_1, A_2)-module homomorphisms between X and $B(\mathcal{H})$ can be identified with the dual of a module Haagerup tensor product in the following way ([Pet97, p. 67], cf. also [ER91, Cor. 4.6], [Ble92b, Prop. 2.3]):

$$CB_{(A_1,A_2)}(X, B(\mathcal{H})) \overset{\text{cb}}{=} (R_{\overline{\mathcal{H}}} \otimes_{hA_1} X \otimes_{hA_2} C_{\mathcal{H}})^*$$

completely isometrically. Hence we see that $CB(B(\mathcal{H}))$ itself and (looking at (3), (4)), just so, $CB^\sigma_{(A_1,A_2)}(B(\mathcal{H}))$ and $CB^s_{(A_1,A_2)}(B(\mathcal{H}))$ are dual operator spaces [Pet97, p. 70].

5.3 Operator algebras

In analogy to concrete operator spaces we define (cf. [BRS90, Def. 1.1]): An **operator algebra** is a closed, not necessarily self-adjoint subalgebra X of $B(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space).

Example: For selfadjoint X we have the theory of C^*-algebras.

As in the operator space situation, one can also adopt an abstract point of view: here, this leads to considering Banach algebras which are operator spaces and are equipped with a multiplication compatible with the operator space structure. These provide an abstract characterization of the (concrete) operator algebras (cf. below: analogue of **Ruan's theorem** for operator algebras).

If X, Y, Z are operator spaces, and if $\Phi : X \times Y \to Z$ is bilinear, we can define another bilinear map in the following way (cf.: Amplification of bilinear mappings):

$$\begin{aligned} \Phi^{(n,l)} : M_{n,l}(X) \times M_{l,n}(Y) &\to M_n(Z) \\ ([x_{ij}], [y_{jk}]) &\mapsto \left[\sum_{j=1}^{l} \Phi(x_{ij}, y_{jk})\right] \quad (l, n \in \mathbb{N}). \end{aligned}$$

This is called the **bilinear amplification**[25] of Φ.

Φ is called **completely bounded** if $\|\Phi\|_{\mathrm{cb}} := \sup_n \|\Phi^{(n,n)}\| < \infty$, and **completely contractive** if $\|\Phi\|_{\mathrm{cb}} \le 1$.[26] Compare this definition with the approach presented in Completely bounded bilinear Mappings.

[In the sequel, for Banach algebras with unit e, we will require $\|e\| = 1$.] An operator space $(X, \|\cdot\|_n)$ with a bilinear, associative and completely contractive map $m: X \times X \to X$, the multiplication, is called an abstract operator algebra (cf. [BRS90, Def. 1.4]). Here, the multiplication on $M_n(X)$ is just the matrix multiplication m_n.
In the unital case m is automatically associative [BRS90, Cor. 2.4].

We have an analogue of Ruan's theorem ([Ble95, Thm. 2.1], cf. also [BRS90, Thm. 3.1]): Let A be a unital Banach algebra and an operator space. Then A is completely isometrically isomorphic to an operator algebra if and only if the multiplication on A is completely contractive.
This yields the following stability result:
1.) The quotient of an operator algebra with a closed ideal is again an operator algebra [BRS90, Cor. 3.2].
With this at hand, one deduces another important theorem on hereditary properties of operator algebras:
2.) The class of operator algebras is stable under complex interpolation [BLM95, (1.12), p. 320].

Adopting a more general point of view than in the Ruan type Representation Theorem above, one obtains the following [Ble95, Thm. 2.2]: Let A be a Banach algebra and an operator space. Then A is completely isomorphic to an operator algebra if and only if the multiplication on A is completely bounded. (cf. the chapter Examples!)

Basic examples of operator algebras are provided by the completely bounded maps on some suitable operator spaces. More precisely, for an operator space X, one obtains the following [Ble95, Thm. 3.4]: $CB(X)$ with the composition as multiplication, is completely isomorphic to an operator algebra if and only if X is completely isomorphic to a column Hilbert space. – An analogue statement holds for the isometric case.

In the following result, for operator algebras A and B, the assumption that A and B be (norm-) closed, is essential (in contrast to the whole rest) [ER90b, Prop. 3.1]:
A unital complete isometry φ between $A \subset B(\mathcal{H})$ and $B \subset B(\mathcal{K})$ ($\mathcal{H}$, $\mathcal{K}$ Hilbert spaces), where $\mathbb{1}_{B(\mathcal{H})} \in A$, $\mathbb{1}_{B(\mathcal{K})} \in B$, is already an algebra homomorphism.

Examples

In the sequel we will equip the spaces ℓ_p $(1 \le p \le \infty)$ with the pointwise product and consider them as Banach algebras. We will further consider the Schatten classes S_p $(1 \le p \le \infty)$ endowed with either the usual multiplication or the Schur product.

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[25] In the literature, e.g. in [BRS90, p. 190], the bilinear amplification $\Phi^{(n,n)}$ is often referred to as *the* amplification and is noted by $\Phi^{(n)}$.

[26] In order to define the notion of complete boundedness of bilinear mappings, it suffices to consider only the $\Phi^{(n,n)}$ instead of all the $\Phi^{(n,l)}$; this definition is usually chosen in the literature about completely bounded bi- and, analogously, multilinear maps [BRS90, p. 190], [CES87, p. 281].

1. *The space* ℓ_2 [BLM95, Thm. 2.1]

   With the following operator space structures, ℓ_2 is completely isometrically isomorphic to an operator algebra: R_{ℓ_2}, C_{ℓ_2}, OH_{ℓ_2}, $R_{\ell_2} \cap C_{\ell_2}$, MAX_{ℓ_2}.
   More generally: The space ℓ_2 is completely isometrically isomorphic to an operator algebra, if endowed with an operator space structure which dominate dominates both R_{ℓ_2} and C_{ℓ_2}.

   With the following operator space structures, ℓ_2 is not completely isomorphic to an operator algebra: $R_{\ell_2} + C_{\ell_2}$, MIN_{ℓ_2}.
   More generally: The space ℓ_2 is not completely isomorphic to an operator algebra, if endowed with an operator space structure which is dominate dominated by both R_{ℓ_2} and C_{ℓ_2}.

2. *The spaces*[27] $MIN(\ell_p)$, $MAX(\ell_p)$ *and* $O\ell_p = (MIN(\ell_\infty), MAX(\ell_1))_{\frac{1}{p}}$

   In the extreme cases $p = 1$ resp. $p = \infty$, we have two opposite results [BLM95, Thm. 3.1]:

   (a) Equipped with any operator space structure, ℓ_1 is completely isometrically isomorphic to an operator algebra.
   (b) $MIN(\ell_\infty)$ is, up to complete isomorphy, the only operator algebra structure on ℓ_∞.

   For $1 \leq p \leq \infty$ the following holds true (cf. [BLM95, Thm. 3.4]):

   (a) $MIN(\ell_p)$ is completely isomorphic to an operator algebra if and only if $p = 1$ or $p = \infty$.
   (b) $MAX(\ell_p)$, in case $1 \leq p \leq 2$, is completely isometrically isomorphic to an operator algebra. In all the other cases $MAX(\ell_p)$ is not completely isomorphic to an operator algebra.

   On the contrary, the operator space structure on the ℓ_p spaces obtained via complex interpolation always defines an operator algebra structure. More precisely [BLM95, Cor. 3.3]:
   For each $1 \leq p \leq \infty$, $O\ell_p$ is completely isometrically isomorphic to an operator algebra.

3. *The Schatten classes* S_p

   We write OS_p for the operator space structure defined on S_p by G. Pisier. This operator space structure is obtained by complex interpolation between $S_\infty = K(\ell_2)$ and $S_1 = K(\ell_2)^*$.

[27] For the construction of the operator spaces $O\ell_p$ compare the chapter on complex interpolation.

(a) Let us first consider the usual product on the Schatten classes S_p. Here we have the following negative result [BLM95, Thm. 6.3]: For each $1 \leq p < \infty$, the operator space OS_p with the usual product is not completely isomorphic to an operator algebra.

(b) Consider now the Schur product on the Schatten classes S_p. Here we obtain positive results, even for different operator space structures:

(b1) $MAX(S_p)$ with the Schur product is, in case $1 \leq p \leq 2$, completely isometrically isomorphic to an operator algebra [BLM95, Thm. 6.1].

(b2) OS_p with the Schur product is, in case $2 \leq p \leq \infty$, completely isometrically isomorphic to an operator algebra [BLM95, Cor. 6.4].

Caution is advised: The space OS_1 (and likewise OS_1^{op}), whether endowed with the usual or the Schur product, is not completely isomorphic to an operator algebra [BLM95, Thm. 6.3].

6 Tensor Products

An operator space tensor product is an operator space whose structure is deduced from the operator space structure of the factors. Operator space tensor products are defined for all operator spaces and have functorial properties. On the tensor product of two fixed operator spaces one usually considers operator space norms which are cross norms.

A lot of spaces, especially spaces of mappings, may be considered as operator space tensor products of simpler ones. The theory of operator space tensor products follows the lines of the theory of tensor products of Banach spaces. But at some points tensor products of operator spaces have new properties not found for tensor products of Banach spaces or even better properties as their counterparts. So in some cases the theory of operator space tensor products gives solutions to problems not solvable within the theory of Banach spaces (cp. [ER90a, Thm. 3.2]).

The Haagerup tensor product $\otimes_h$ has a variety of applications in the theory of operator spaces and completely bounded operators.

The injective operator space tensor product $\overset{\vee}{\otimes}$ is the least[28] and the projective operator space tensor product $\overset{\wedge}{\otimes}$ is the greatest [29]among all operator space tensor products. [BP91, Prop. 5.10].

On the algebraic tensor product $X \otimes Y$ of operator spaces X, Y one can compare an operator space tensor norm $\| \cdot \|_\alpha$ with the injective tensor norm $\| \cdot \|_\lambda$ and the projective tensor norm $\| \cdot \|_\gamma$ of normed spaces:

$$\| \cdot \|_\lambda \leq \| \cdot \|_{\vee,1} \leq \| \cdot \|_{\alpha,1} \leq \| \cdot \|_{\wedge,1} \leq \| \cdot \|_\gamma .$$

[28] i.e. the injective operator space tensor norm is minimal among all operator space tensor norms.

[29] i.e. the projective operator space tensor norm is maximal among all operator space tensor norms.

6.1 Operator space tensor products

An operator space tensor product is the completion of the algebraic tensor product with respect to an operator space tensor norm.

An **operator space tensor norm** $\|\cdot\|_\alpha$ is defined for each pair (X, Y) of operator spaces and endows their algebraic tensor product $X \otimes Y$ with the structure of an matrix normed space $(X \otimes Y, \|\cdot\|_\alpha)$ thuch that the following two properties 1 and 2 [BP91, Def. 5.9]. hold.

The completion is called the α-**operator space tensor product** of X and Y and is denoted by $X \otimes_\alpha Y$.

1. *For the complex numbers holds*

$$\mathbb{C} \otimes_\alpha \mathbb{C} = \mathbb{C}.$$

2. *For all $S \in CB(X_1, X_2)$ and $T \in CB(Y_1, Y_2)$ the operator $S \otimes T : X_1 \otimes Y_1 \to X_2 \otimes Y_2$ has a continuous extension*

$$S \otimes_\alpha T \in CB(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2).$$

The bilinear mapping

$$\begin{aligned} \otimes_\alpha : CB(X_1, X_2) \times CB(Y_1, Y_2) &\to CB(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2) \\ (S, T) &\mapsto S \otimes_\alpha T \end{aligned}$$

is jointly completely contractive.[30]

Property 2 may be replaced by the assumptions 3 and 4.[31]

[30] i.e., let $[S_{ij}] \in M_p(CB(X_1, X_2))$, $[T_{kl}] \in M_q(CB(Y_1, Y_2))$, $p, q \in \mathbb{N}$, then the norm of the linear operator

$$[S_{ij} \otimes_\alpha T_{kl}] \in M_{pq}(CB(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2))$$

is estimated by

$$\|[S_{ij} \otimes T_{kl}]\|_{\mathrm{cb}} \le \|[S_{ij}]\|_{\mathrm{cb}} \|[T_{kl}]\|_{\mathrm{cb}}.$$

Remark: This is indeed an equality.

[31] (2) $\Rightarrow$(3),(4): Condition 3 is a special case of 2. Let $I \in M_p(CB(\mathbb{M}_p(X), X))$, $J \in M_q(CB(\mathbb{M}_q(Y), Y))$ be matrices, which are algebraically the the identical mappings of the vector spaces $M_p(X)$ respectivly $M_q(Y)$. By assumption 2 we have

$$\begin{aligned} I \otimes_\alpha J &\in M_{pq}(CB(\mathbb{M}_p(X) \otimes_\alpha \mathbb{M}_q(Y), X \otimes_\alpha Y)) \\ &= M_1(CB(\mathbb{M}_p(X) \otimes_\alpha \mathbb{M}_q(Y), \mathbb{M}_{pq}(X \otimes_\alpha Y))), \\ &\|I \otimes_\alpha J\|_{\mathrm{cb}} \le \|I\|_{\mathrm{cb}} \|J\|_{\mathrm{cb}} = 1. \end{aligned}$$

Now $I \otimes_\alpha J : \mathbb{M}_p(X) \otimes_\alpha \mathbb{M}_q(Y) \to \mathbb{M}_{pq}(X \otimes_\alpha Y)$ is the *shuffle*-map in 4.

(3),(4)$\Rightarrow$(2): Let $[S_{ij}] \in M_p(CB(X_1, X_2))$, $[T_{kl}] \in M_q(CB(Y_1, Y_2))$, $p, q \in \mathbb{N}$, and $S \in$

3. *An operator space tensor product $\otimes_\alpha$ is functorial: For all $S \in CB(X_1, X_2)$ and $T \in CB(Y_1, Y_2)$ the operator $S \otimes T : X_1 \otimes Y_1 \to X_2 \otimes Y_2$ has a continuous extension*

$$S \otimes_\alpha T \in CB(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2),$$

and

$$\|S \otimes_\alpha T\|_{\mathrm{cb}} \leq \|S\|_{\mathrm{cb}} \|T\|_{\mathrm{cb}}.$$

Remark: This is indeed an equality $\|S \otimes_\alpha T\|_{\mathrm{cb}} = \|S\|_{\mathrm{cb}} \|T\|_{\mathrm{cb}}$.

4. *The algebraic* shuffle *isomorphism $M_p(X) \otimes M_q(Y) \cong M_{pq}(X \otimes Y)$ has a continuous extension to a complete contraction:*

$$\mathbb{M}_p(X) \otimes_\alpha \mathbb{M}_q(Y) \to \mathbb{M}_{pq}(X \otimes_\alpha Y).$$

This complete contraction is called the *shuffle* map of the α-operator space tensor product.

Condition 4 is equivalent to the following two conditions: The *shuffle* mappings

$$\begin{array}{rcl} \mathbb{M}_p(X) \otimes_\alpha Y & \to & \mathbb{M}_p(X \otimes_\alpha Y), \\ X \otimes_\alpha \mathbb{M}_q(Y) & \to & \mathbb{M}_q(X \otimes_\alpha Y) \end{array}$$

are completely contractive.

Operator space tensor producte may have further special properties:

An operator space tensor product $\otimes_\alpha$ is called

symmetric, *if $X \otimes_\alpha Y \overset{\mathrm{cb}}{=} Y \otimes_\alpha X$ is a complete isometry;*

associative, *if $(X \otimes_\alpha Y) \otimes_\alpha Z \overset{\mathrm{cb}}{=} X \otimes_\alpha (Y \otimes_\alpha Z)$ is acomplete isometry;*

injective, *if for all subspaces $X_1 \subset X$, $Y_1 \subset Y$ the map $X_1 \otimes_\alpha Y_1 \hookrightarrow X \otimes_\alpha Y$ is acomplete isometry;*

projective, *if for all subspaces $X_1 \subset X$, $Y_1 \subset Y$ the map $X \otimes_\alpha Y \to X/X_1 \otimes_\alpha Y/Y_1$ is a complete quotient map;*

$CB(X_1, \mathbb{M}_p(X_2))$, $T \in CB(Y_1, \mathbb{M}_q(Y_2))$ the corresponding operators. By 3 holds

$$\begin{array}{rcl} S \otimes_\alpha T & \in & CB(X_1 \otimes_\alpha Y_1, \mathbb{M}_p(X_2) \otimes_\alpha \mathbb{M}_q(Y_2)), \\ \|S \otimes_\alpha T\|_{\mathrm{cb}} & \leq & \|S\|_{\mathrm{cb}} \|T\|_{\mathrm{cb}} = \|[S_{ij}]\|_{\mathrm{cb}} \|[T_{kl}]\|_{\mathrm{cb}}. \end{array}$$

We apply the *shuffle* map $A : \mathbb{M}_p(X_2) \otimes_\alpha \mathbb{M}_q(Y_2) \to \mathbb{M}_{pq}(X_2 \otimes_\alpha Y_2)$ and obtain from 4

$$\begin{array}{rcl} [S_{ij} \otimes_\alpha T_{kl}] = A(S \otimes_\alpha T) & \in & M_1(CB(X_1 \otimes_\alpha Y_1, \mathbb{M}_{pq}(X_2 \otimes_\alpha Y_2))) = M_{pq}(CB(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2), \\ \|[S_{ij} \otimes_\alpha T_{kl}]\|_{\mathrm{cb}} & \leq & \|S \otimes_\alpha T\|_{\mathrm{cb}} \leq \|[S_{ij}]\|_{\mathrm{cb}} \|[T_{kl}]\|_{\mathrm{cb}}. \end{array}$$

Hence $\otimes_\alpha$ is jointly completely bounded.

self dual, *if the algebraic embedding* $X^* \otimes Y^* \subset (X \otimes_\alpha Y)^*$ *has a completely isometric extension* $X^* \otimes_\alpha Y^* \subset (X \otimes_\alpha Y)^*$.

In many applications one finds the Haagerup-tensor product. It is not symmetric, but associative, injective, projective and self dual.

cross norms

Sometimes one considers an operator space norm on the algebraic tensor product of two fixed operator spaces. Then one usually demands that this norm and its dual norm are at least cross norms. Operator space tensor norms always have these properties.

An operator space norm $\|\cdot\|_\alpha$ on the algebraic tensor product $X \otimes Y$ of two operator spaces X and Y is said to be a **cross norm**, if

$$\|x \otimes y\|_{\alpha,pq} = \|x\|_p \, \|y\|_q$$

for all $p, q \in \mathbb{N}$, $x \in M_p(X)$, $y \in M_q(Y)$ holds.

For cross norms $\mathbb{C} \otimes_\alpha X \stackrel{\text{cb}}{=} X$ is completely isometric.

For an operator space norm $\|\cdot\|_\alpha$ on the algebraic tensor product of two fixed operator spaces X and Y one usually asks for the following three properties (i)–(iii).[32]

(i) $\|\cdot\|_\alpha$ *is a cross norm.*

(ii) *Let* $\varphi \in X^*$, $\psi \in Y^*$ *be linear functionals and*

$$\begin{aligned} \varphi \otimes \psi : X \otimes Y &\rightarrow \mathbb{C} \\ \langle x \otimes y, \varphi \otimes \psi \rangle &:= \langle x, \varphi \rangle \langle y, \psi \rangle \end{aligned}$$

where $x \in X$, $y \in Y$. *their tensorproduct. The tensor product* $\varphi \otimes \psi$ *has a continuous linear extension to* $X \otimes_\alpha Y$.

Then the **dual** operator space norm $\|\cdot\|_{\alpha^*}$ is defined on the algebraic tensor product $X^* \otimes Y^*$ by the algebraic embedding

$$X \otimes Y \subset (X^* \otimes_\alpha Y^*)^*.$$

(iii) *The dual operator space norm* $\|\cdot\|_{\alpha^*}$ *is a cross norm.*

There is a smallest operator space norms among the operator space norms on $X \otimes Y$, for which $\|\cdot\|_\alpha$ and the dual norm $\|\cdot\|_{\alpha^*}$ are cross norms. This is the injective operator space tensor norm $\|\cdot\|_\vee$. [BP91, Prop. 5.10].

[32]The conditions (i)-(iii) are equivalent to the following: The bilinear maps

$$\begin{aligned} X \times Y \rightarrow X \otimes_\alpha Y, &\qquad (x, y) \mapsto x \otimes y \\ X^* \times Y^* \rightarrow (X \otimes_\alpha Y)^*, &\qquad (\varphi, \psi) \mapsto \varphi \otimes \psi \end{aligned}$$

are jointly completely bounded.

There is a greatest operator space norm among the operator space norms on $X \otimes Y$, for which $\|\cdot\|_\alpha$ and the dual norm $\|\cdot\|_{\alpha^*}$ are cross norms. This is the projective operator space tensor norm $\|\cdot\|_\wedge$ [BP91, Prop. 5.10].

On the algebraic tensor product $X \otimes Y$ one can compare the operator space norms $\|\cdot\|_\alpha$ for which $\|\cdot\|_\alpha$ and the dual norm $\|\cdot\|_{\alpha^*}$ are cross norms with the injective tensor norm $\|\cdot\|_\lambda$ and the projective tensor norm $\|\cdot\|_\gamma$ of normed spaces:

$$\|\cdot\|_\lambda \leq \|\cdot\|_{\vee,1} \leq \|\cdot\|_{\alpha,1} \leq \|\cdot\|_{\wedge,1} \leq \|\cdot\|_\gamma.$$

6.2 Injective operator space tensor product

The representations of two operator spaces X in $B(\mathcal{H})$ and Y in $B(\mathcal{K})$ yield a representation of the algebraic tensor product of X and Y in $B(\mathcal{H} \otimes_2 \mathcal{K})$. The operator space structure obtained in this way turns out to be independent of the representations chosen. It is called the **injectiveoperator space tensor product** of X and Y and is denoted by $X \overset{\vee}{\otimes} Y$ [BP91, p. 285]. Hence, in the case of C^*–algebras, the injective operator space tensor product and the minimal C^*–tensor product coincide.[33]

By means of the duality of tensor products we obtain a formula [BP91, Thm. 5.1] for the injective operator space tensor norm of an element $u \in M_n(X \otimes Y)$ which is representation free:

$$\|u\|_\vee = \sup \|\langle u, \varphi \otimes \psi \rangle\|_{M_{nkl}},$$

where $k, l \in \mathbb{N}$, $\varphi \in \mathrm{Ball}(M_k(X^*))$ and $\psi \in \mathrm{Ball}(M_l(Y^*))$.

Interpreting, as is usual, the elements of the algebraic tensor product as finite rank operators we have the completely isometric embeddings [BP91, Cor. 5.2]

$$X \overset{\vee}{\otimes} Y \hookrightarrow CB(X^*, Y) \quad \text{resp.} \quad X \overset{\vee}{\otimes} Y \hookrightarrow CB(Y^*, X).$$

The injective operator space tensor norm is the least cross norm whose dual norm again is a cross norm.

The injective operator space tensor product is symmetric, associative and injective. But it is not projective [BP91, Cor. 5.2].

The injective norm is the dual norm of the projective operator space tensor norm [BP91, Thm. 5.6]; but the projective operator space tensor norm is not in general the dual of the injective operator space tensor norm even if one of the two spaces involved is finite dimensional [ER90a, p. 168], [ER91, p. 264].

[33] For this reason the injective operator space tensor product is also called *spatial* tensor product of operator spaces and denoted by $X \otimes_{\min} Y$.

Some formulae for the injective operator space tensor product

If E and F are normed spaces, then [BP91, Prop. 4.1]

$$MIN(E) \overset{\vee}{\otimes} MIN(F) \overset{\text{cb}}{=} MIN(E \otimes_\lambda F)$$

holds completely isometrically.

6.2.1 Exact operator spaces

We consider now the exact sequence

$$0 \hookrightarrow K(\ell_2) \hookrightarrow B(\ell_2) \to Q(\ell_2) \to 0.$$

An operator space X is said to be **exact** [Pis95, §1], if the short sequence of injective tensor products

$$0 \hookrightarrow X \overset{\vee}{\otimes} K(\ell_2) \hookrightarrow X \overset{\vee}{\otimes} B(\ell_2) \to X \overset{\vee}{\otimes} Q(\ell_2) \to 0$$

is again exact. Then tensorizing with such an operator space preserves the exactness of arbitrary exact sequences of C^*-algebras (for C^*-algebras cf [Kir83]). Obviously, all finite dimensional operator spaces are exact.

Exactness is inherited by arbitrary subspaces. The injective tensor product of two exact operator spaces is again exact. For an exact space X we are given a degree of exactness by the quantity

$$\text{ex}(X) = \|X \overset{\vee}{\otimes} Q(\mathcal{H}) \to (X \overset{\vee}{\otimes} B(\mathcal{H}))/(X \overset{\vee}{\otimes} K(\mathcal{H}))\|.$$

We have $1 \leq \text{ex}(X) < \infty$ [Pis95, §1], because the mapping

$$(X \overset{\vee}{\otimes} B(\mathcal{H}))/(X \overset{\vee}{\otimes} K(\mathcal{H})) \to X \overset{\vee}{\otimes} Q(\mathcal{H})$$

is a complete contraction. For non-exact operator spaces X we put $\text{ex}(X) = \infty$.

For an exact C^*-algebra [34] A we have $\text{ex}(A) = 1$.

For an operator space X we have:

$$\text{ex}(X) = \sup\{\text{ex}(L) : L \subset X, \dim L < \infty\}.$$

so we can confine our examinations to finite dimensional spaces. From this it is also immediate that: $\text{ex}(X_0) \leq \text{ex}(X)$ if $X_0 \subset X$. One has for finite dimensional operator spaces X_1 and X_2 the complete variant of the Banach-Mazur distance

$$d_{CB}(X_1, X_2) = \inf\{\|\varphi\|_{\text{cb}}\|\varphi^{-1}\|_{\text{cb}}\}$$

(the infimum is taken over all isomorphisms φ from X_1 to X_2). Via this Banach-Mazur distance we can define the quantity

$$d_{SK}(X) := \inf\{d_{CB}(X, L), \dim(L) = \dim(X), L \subset M_n, n \in \mathbb{N}\}$$

According to [Pis95, Thm. 1] $\text{ex}(X) = d_{SK}(X)$ holds, and $\text{ex}(X) \leq \sqrt{\dim(X)}$.

[34] A characterization of exact C^*-algebras is given in [Kir94] and [Kir95].

6.3 Projective operator space tensor product

The **projective operator space tensor product** $X \overset{\wedge}{\otimes} Y$ of two operator spaces X and Y is characterized by the following complete isometry:

$$(X \overset{\wedge}{\otimes} Y)^* \overset{\text{cb}}{=} CB(X, Y^*) \overset{\text{cb}}{=} CB(Y, X^*).$$

where linear mappings are identified in the usual way with bilinear forms.

One can also characterize the projective operator space tensor product by the following universal property [BP91, Def. 5.3]

$$CB(X \overset{\wedge}{\otimes} Y, Z) \overset{\text{cb}}{=} JCB(X \times Y; Z),$$

where Z is an operator space.

Here, $JCB(X \times Y; Z)$ denotes the operator space of jointly completely bounded bilinear mappings.

One also has an explicit expression for the projective operator space tensor norm of an element $u \in M_n(X \otimes Y)$: (cf. [ER91, Formel (2.10)])

$$\|u\|_\wedge = \inf \{\|\alpha\|\|x\|_p\|y\|_q\|\beta\| \ : \ u = \alpha(x \otimes y)\beta\},$$

where $p, q \in \mathbb{N}$, $x \in M_p(X)$, $y \in M_q(Y)$ and $\alpha \in M_{n,pq}$, $\beta \in M_{pq,n}$.

The projective operator space tensor norm is symmetric, associative and projective [ER91, p. 262]. But it is not injective.

The projective operator space tensor norm is the greatest operator space tensor norm which is a cross norm [BP91, Thm. 5.5].

Its dual norm is the injective operator space tensor norm [BP91, Thm. 5.6]; but the projective operator space tensor norm is not in general the dual of the injective operator space tensor norm even if one of the two spaces involved is finite dimensional [ER90a, p. 168], [ER91, p. 264].

Some formulae for the projective operator space tensor product

1. If E and F are normed spaces, we have [BP91, Prop. 4.1]

   $$MAX(E) \overset{\wedge}{\otimes} MAX(F) \quad \overset{\text{cb}}{=} \quad MAX(E \otimes_\gamma F)$$

   completely isometrically.

2. Taking the projective operator space tensor product of various combinations of the column Hilbert spaces $\mathcal{C}$, the row Hilbert spaces $\mathcal{R}$ and an arbitrary operator space X one obtains the following completely isometric identifications: [ER91], [Ble92b, Prop. 2.3]

   (i) $X \overset{\wedge}{\otimes} \mathcal{C}_\mathcal{H} \overset{\text{cb}}{=} X \otimes_h \mathcal{C}_\mathcal{H}$

   (ii) $\mathcal{R}_\mathcal{H} \overset{\wedge}{\otimes} X \overset{\text{cb}}{=} \mathcal{R}_\mathcal{H} \otimes_h X$

(iii) $\mathcal{C}_{\mathcal{H}} \overset{\wedge}{\otimes} \mathcal{C}_{\mathcal{K}} \overset{\text{cb}}{=} \mathcal{C}_{\mathcal{H}} \overset{\vee}{\otimes} \mathcal{C}_{\mathcal{K}} \overset{\text{cb}}{=} \mathcal{C}_{\mathcal{H}} \otimes_h \mathcal{C}_{\mathcal{K}} \overset{\text{cb}}{=} \mathcal{C}_{\mathcal{H}\otimes_2\mathcal{K}}$

(iv) $\mathcal{R}_{\mathcal{H}} \overset{\wedge}{\otimes} \mathcal{R}_{\mathcal{K}} \overset{\text{cb}}{=} \mathcal{R}_{\mathcal{H}} \overset{\vee}{\otimes} \mathcal{R}_{\mathcal{K}} \overset{\text{cb}}{=} \mathcal{R}_{\mathcal{H}} \otimes_h \mathcal{R}_{\mathcal{K}} \overset{\text{cb}}{=} \mathcal{R}_{\mathcal{H}\otimes_2\mathcal{K}}$

(v) $\mathcal{R}_{\overline{\mathcal{H}}} \overset{\wedge}{\otimes} \mathcal{C}_{\mathcal{K}} \overset{\text{cb}}{=} \mathcal{R}_{\overline{\mathcal{H}}} \otimes_h \mathcal{C}_{\mathcal{K}} \overset{\text{cb}}{=} T(\mathcal{H},\mathcal{K})$

(vi) $X \overset{\wedge}{\otimes} T(\mathcal{H},\mathcal{K}) \overset{\text{cb}}{=} \mathcal{R}_{\overline{\mathcal{H}}} \overset{\wedge}{\otimes} X \overset{\wedge}{\otimes} \mathcal{C}_{\mathcal{K}} \overset{\text{cb}}{=} \mathcal{R}_{\overline{\mathcal{H}}} \otimes_h X \otimes_h \mathcal{C}_{\mathcal{K}}$

(vii) $CB(X, B(\mathcal{K},\mathcal{H})) \overset{\text{cb}}{=} (\mathcal{R}_{\overline{\mathcal{H}}} \overset{\wedge}{\otimes} X \overset{\wedge}{\otimes} \mathcal{C}_{\mathcal{K}})^* \overset{\text{cb}}{=} (\mathcal{R}_{\overline{\mathcal{H}}} \otimes_h X \otimes_h \mathcal{C}_{\mathcal{K}})^*$,

where the space of trace class operators $T(\mathcal{H},\mathcal{K})$ is endowed with its natural operator space structure: $T(\mathcal{H},\mathcal{K}) :\overset{\text{cb}}{=} K(\mathcal{K},\mathcal{H})^*$.

3. Let M and N be von Neumann algebras and denote by $M\overline{\otimes}N$ the von Neumann tensor product[35]. For the preduals one has

$$M_* \overset{\wedge}{\otimes} N_* \overset{\text{cb}}{=} (M\overline{\otimes}N)_*$$

completely isometrically [ER90a].

Let G and H be locally compact topological groups and denote by $VN(G)$, $VN(H)$ the corresponding group von Neumann algebras.[36] It is well-known that $VN(G)\,\overline{\otimes}\,VN(H) = VN(G \times H)$. Since the Fourier algebra[37] $A(G)$ can be identified with the predual of the group von Neumann algebra $VN(G)$ [Eym64], this implies that

$$A(G) \overset{\wedge}{\otimes} A(H) \overset{\text{cb}}{=} A(G \times H)$$

holds completely isometrically[38] [ER90a].

6.4 The Haagerup tensor product

The Haagerup tensor product was first introduced by Effros and Kishimoto [EK87] for C*-algebras generalizing the original work of U. Haagerup [Haa80].

The Haagerup tensor product $X \otimes_h Y$ of two operator spaces X and Y is characterized by the complete isometry

$$(X \otimes_h Y)^* \overset{\text{cb}}{=} CB(X \times Y; \mathbb{C}),$$

[35] If $M \subset B(\mathcal{H})$, $N \subset B(\mathcal{K})$ are von Neumann algebras, $M\overline{\otimes}N$ is defined to be the closure in the weak operator topology of the algebraic tensor product $M \otimes N \subset B(\mathcal{H} \otimes_2 \mathcal{K})$.

[36] The group von Neumann algebra $VN(G)$ of a locally compact group G is defined to be the von Neumann algebra generated by the left regular representation of G in $B(L_2(G))$.

[37] For a locally compact group G, the set $\{f * \check{g} | f, g \in L_2(G)\} \subset C_0(G)$ turns out to be a linear space and even an algebra (with pointwise multiplication). Its completion with respect to the norm $\|u\| = \inf\{\|f\|_2\|g\|_2 \mid u = f * \check{g}\}$ is a Banach algebra and is called the Fourier algebra of G.

[38] If G is a locally compact *abelian* group, then $A(G)$ is identified – via the Fourier transform – with $L_1(\widehat{G})$, where $\widehat{G}$ denotes the dual group of G. Thus, for locally compact groups G and H, the identification $A(G) \overset{\wedge}{\otimes} A(H) \overset{\text{cb}}{=} A(G \times H)$ can be thought of as a non commutative analogue of the well-known classical identification $L_1(G) \otimes_\gamma L_1(H) = L_1(G \times H)$.

where bilinear forms are identified with linear maps in the usual fashion.

We can also characterize the Haagerup tensor product by the following universal property: For an operator space Z we have

$$CB(X \otimes_h Y; Z) \overset{\text{cb}}{=} CB(X \times Y; Z)$$

completely isometrically.

Here, $CB(X \times Y; Z)$ denotes the operator space of completely bounded bilinear mappings.

For operator spaces X and Y the **Haagerup operator space tensor norm** of $u \in M_n(X \otimes Y)$ is explicitly given by (cf. [ER91, Formel (2.11)], [BP91, Lemma 3.2])

$$\|u\|_h = \inf \|x\|_{n,l} \, \|y\|_{l,n},$$

where $l \in \mathbb{N}$, $x \in M_{n,l}(X)$, $y \in M_{l,n}(Y)$ and u is the tensor matrix product $u = x \odot y$. The Haagerup tensor product $X \otimes_h Y$ then of course is the completion of the algebraic tensor product $X \otimes Y$ with respect to this operator space tensor norm. There are several other useful formulae 2 3, 4 for the Haagerup norm.

The Haagerup tensor product is not symmetric as shown by concrete examples. But it is associative, injective [PS87, p. 272; Thm. 4.4], [BP91, Thm. 3.6], projektiv [ER91, Thm. 3.1] and selfdual [ER91, Thm. 3.2]. Thus the embedding

$$X^* \otimes_h Y^* \hookrightarrow (X \otimes_h Y)^*$$

is a complete isometry.

The extension of the identity mapping on the algebraic tensor product of two operator spaces X, Y from the Haagerup tensor product into the injective tensor product is injective. One therefore obtains a canonical embedding

$$X \otimes_h Y \subset X \overset{\vee}{\otimes} Y.$$

The complex interpolation of operator spaces and the Haagerup tensor product commute [Pis96, Thm. 2.3]. Let (X_0, X_1) and (Y_0, Y_1) be compatible pairs of operator spaces. Then $(X_0 \otimes_h Y_0, X_1 \otimes_h Y_1)$ is a compatible pair of operator spaces and we have completely isometrically

$$(X_0 \otimes_h Y_0, X_1 \otimes_h Y_1)_\vartheta \overset{\text{cb}}{=} (X_0, X_1)_\vartheta \otimes_h (Y_0, Y_1)_\vartheta$$

for $0 \leq \vartheta \leq 1$.

On normed spaces there is no tensor norm which at the same time is associative, injective, projective and selfdual. The Haagerup tensor product can be interpreted as a generalization of the ***H*-tensor product** introduced by Grothendieck[39] for normed

[39] This tensor norm also is known as γ_2 [Pis86].

spaces E and F [BP91, pp. 277-279, Prop. 4.1]. In fact, on the first matrix level we have:

$$\begin{aligned} MIN(E) \otimes_h MIN(F) &= E \otimes_H F, \\ MAX(E) \otimes_h MAX(F) &= E \otimes_{H^*} F \end{aligned}$$

isometrically. The non-associativity of the H-tensor product is reflected by the fact that in general $MIN(E) \otimes_h MIN(F)$ and $MIN(E \otimes_H F)$ are not completely isometric.

Some formulae for the Haagerup tensor product

1. Let A and B be C*-algebras in $B(\mathcal{H})$. Then on the algebraic tensor product $A \otimes B$ the *Haagerup tensor norm* is explicitly given by

   $$\|u\|_h := \inf\left\{ \|\textstyle\sum_{\nu=1}^n a_\nu a_\nu^*\|^{\frac{1}{2}} \, \|\sum_{\nu=1}^n b_\nu^* b_\nu\|^{\frac{1}{2}} \quad : \; u = \sum_{\nu=1}^n a_\nu \otimes b_\nu \right\},$$

   where $n \in \mathbb{N}$, $a_\nu \in A$, $b_\nu \in B$.

   The Haagerup norm of $\sum_{\nu=1}^n a_\nu \otimes b_\nu \in A \otimes B$ equals the cb–norm of the elementary operator $B(\mathcal{H}) \ni x \mapsto \sum_{\nu=1}^n a_\nu x b_\nu$. The Haagerup tensor product $A \otimes_h B$ is the completion of the algebraic tensor product $A \otimes B$ with respect to the above norm. The following more general definition in particular yields a completely isometric embedding $A \otimes_h B \hookrightarrow CB(B(\mathcal{H}))$.

2. We have

   $$\|u\|_h = \inf\{\textstyle\sum_{\kappa=1}^k \|x_\kappa\| \|y_\kappa\| \quad : \; u = \sum_{\kappa=1}^k x_\kappa \odot y_\kappa\}$$

   where $k, l \in \mathbb{N}$, $x_\kappa \in M_{n,l}(X)$, $y_\kappa \in M_{l,n}(Y)$. In fact, one summand suffices [BP91, Lemma 3.2].

3. For elements $u \in M_n(X \otimes Y)$ in the algebraic tensor product there is an $l \in \mathbb{N}$ and elements $x \in M_{n,l}(X)$, $y \in M_{l,n}(Y)$ such that

   $$\begin{aligned} u &= x \odot y \\ \|u\|_h &= \|x\| \|y\|. \end{aligned}$$

   The infimum occuring in the formula describing the norm in this case is actually a minimum [ER91, Prop. 3.5].

4. The Haagerup norm of an element $u \in M_n(X \otimes_h Y)$ can also be expressed using a supremum:[40]

   $$\|u\|_h = \sup \|\langle u, \varphi \odot \psi \rangle\|_{M_{n^2}},$$

[40] For $x \in X$, $y \in Y$, we have:

$$(\varphi \odot \psi)(x \otimes y) = \langle x \otimes y, \varphi \odot \psi \rangle = \left[\sum_{j=1}^{l} \langle x \otimes y, \varphi_{ij} \otimes \psi_{jk} \rangle \right] = \varphi(x)\psi(y) \in M_n.$$

Here, we used the definitions of two fundamental notions in operator space theory: the tensor matrix multiplication $\varphi \odot \psi$ of mappings φ, ψ and the joint amplification of the duality of tensor products.

where $l \in \mathbb{N}$, $\varphi \in M_{n,l}(X^*)$, $\psi \in M_{l,n}(Y^*)$, $\|\varphi\| = \|\psi\| = 1$ [ER91, Prop. 3.4].

5. From the definition of the Haagerup norm one easily deduces that the *shuffle*-map

$$\mathbb{M}_p(X) \otimes_h \mathbb{M}_q(Y) \to \mathbb{M}_{pq}(X \otimes_h Y)$$

is a complete contraction. Hence the Haagerup tensor product enjoys property 2 of an operator space tensor product.

But the *shuffle*-map is not an isometry in general as shown by the following example:[41]

$$\mathbb{M}_n(C_l) \otimes_h \mathbb{M}_n(R_l) \stackrel{\text{cb}}{=} \mathbb{M}_l(\mathbb{M}_n((T_n)) \neq \mathbb{M}_l(\mathbb{M}_n(\mathbb{M}_n)) \stackrel{\text{cb}}{=} \mathbb{M}_n(\mathbb{M}_n(C_l \otimes_h R_l)).$$

Since the bilinear mapping

$$\begin{array}{rcl} \otimes_h : CB(X_1, X_2) \times CB(Y_1, Y_2) & \to & CB(X_1 \otimes_h X_2, Y_1 \otimes_h Y_2) \\ (S, T) & \mapsto & S \otimes_h T \end{array}$$

is contractive, it is jointly completely contractive.[42]

In fact, using 8, we see that it is even completely contractive.

6. For the row and column structure the *shuffle*-map even is a complete isometry. We have the **Lemma of Blecher and Paulsen** [BP91, Prop. 3.5]:

$$C_n(X) \otimes_h R_n(Y) \stackrel{\text{cb}}{=} \mathbb{M}_n(X \otimes_h Y).$$

In many cases it suffices to prove a statement about the Haagerup tensor product on the first matrix level and then to deduce it for all matrix levels using the above formula.[43]

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[41] Algebraically, we have on both sides the same spaces of matrices. But on the left side one obtains the finer operator space structure $T_n := \mathbb{M}_n^*$ of the trace class:

$$\begin{array}{rl} & \mathbb{M}_n(C_l) \otimes_h \mathbb{M}_n(R_l) \stackrel{\text{cb}}{=} C_l(\mathbb{M}_n) \otimes_h R_l(\mathbb{M}_n) \stackrel{\text{cb}}{=} \mathbb{M}_l(\mathbb{M}_n \otimes_h \mathbb{M}_n) \\ = & \mathbb{M}_l(C_n \otimes_h R_n \otimes_h C_n \otimes_h R_n) = \mathbb{M}_l(C_n \otimes_h T_n \otimes_h R_n) \stackrel{\text{cb}}{=} \mathbb{M}_l(M_n(T_n)). \end{array}$$

On the right side, we get the coarser operator space structure of the matrices M_n:

$$\mathbb{M}_n(\mathbb{M}_n(C_l \otimes_h R_l) = \mathbb{M}_n(\mathbb{M}_n(\mathbb{M}_l)) \stackrel{\text{cb}}{=} \mathbb{M}_l(\mathbb{M}_n(\mathbb{M}_n)).$$

[42] This follows from the equivalence of the property 2 and the properties 3 and 4 of operator space tensor products.

[43] This method can be applied to obtain this complete isometry itself. It is easy to see that on the first matrix level we have

$$M_1(C_n(X) \otimes_h R_n(Y)) = M_n(X \otimes_h Y)$$

isometrically. From this the complete isometry follows – for all $p \in \mathbb{N}$ we have:

$$\begin{array}{rcl} M_p(C_n(X) \otimes_h R_n(Y)) & = & M_1(C_p(C_n(X)) \otimes_h R_p(R_n(Y))) = M_1(C_{pn}(X) \otimes_h R_{pn}(Y)) \\ & = & M_{pn}(X \otimes_h Y) = M_p(\mathbb{M}_n(X \otimes_h Y)) \end{array}$$

isometrically .

Here we list some special cases of the Lemma of Blecher and Paulsen:

$$\begin{aligned} C_n \otimes_h R_n &\overset{\text{cb}}{=} \mathbb{M}_n, \\ C_n \otimes_h X &\overset{\text{cb}}{=} C_n(X), \\ X \otimes_h R_n &\overset{\text{cb}}{=} R_n(X), \\ C_n \otimes_h X \otimes_h R_n &\overset{\text{cb}}{=} \mathbb{M}_n(X). \end{aligned}$$

7. In contrast to 6, for $R_n \otimes_h C_n$ one obtains the finer operator space structure of the trace class

$$T_n := \mathbb{M}_n^* \overset{\text{cb}}{=} R_n \overset{\wedge}{\otimes} C_n \overset{\text{cb}}{=} R_n \otimes_h C_n.$$

   For an operator space X we have [Ble92b, Prop. 2.3]:

$$\begin{aligned} R_n \otimes_h X &\overset{\text{cb}}{=} R_n \overset{\wedge}{\otimes} X, \\ X \otimes_h C_n &\overset{\text{cb}}{=} C_n \overset{\wedge}{\otimes} X, \\ R_n \otimes_h X \otimes_h C_n &\overset{\text{cb}}{=} T_n \overset{\wedge}{\otimes} X, \\ R_n \otimes_h X^* \otimes_h C_n &\overset{\text{cb}}{=} \mathbb{M}_n(X)^*. \end{aligned}$$

8. By the very construction the bilinear mapping $X \times Y \to X \otimes_h Y$, $(x, y) \mapsto x \otimes y$ is a complete contraction.

   Hence its amplification, the tensor matrix product

$$\begin{aligned} \odot_h : \mathbb{M}_{n,l}(X) \times \mathbb{M}_{l,n}(Y) &\rightarrow \mathbb{M}_n(X \otimes_h Y) \\ (x, y) &\mapsto x \odot y, \end{aligned}$$

   also is a complete contraction. The linearization of the tensor matrix product gives the complete contraction

$$\mathbb{M}_{n,l}(X) \otimes_h \mathbb{M}_{l,n}(Y) \to \mathbb{M}_n(X \otimes_h Y).$$

9. The bilinear mapping[44]

$$\begin{aligned} \otimes_h : CB(X_1, X_2) \times CB(Y_1, Y_2) &\rightarrow CB(X_1 \otimes_h Y_1, X_2 \otimes_h Y_2) \\ (S, T) &\mapsto S \otimes_h T \end{aligned}$$

[44] The amplification of $\otimes_h$ is nothing but the tensor matrix multiplication $\odot_h$ of operator matrices.

is completely contractive[45] and gives rise to a complete contraction

$$CB(X_1, X_2) \otimes_h CB(Y_1, Y_2) \to CB(X_1 \otimes_h Y_1, X_2 \otimes_h Y_2).$$

10. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Taking the Haagerup tensor product of the column Hilbert space $\mathcal{C}$ and the row Hilbert space $\mathcal{R}$ one obtains completely isometrically the space of compact operators K resp. of trace class[46] operators T [ER91, Cor. 4.4]:

$$\begin{aligned} \mathcal{R}_{\overline{\mathcal{H}}} \otimes_h \mathcal{C}_{\mathcal{K}} &\overset{\text{cb}}{=} T(\mathcal{H}, \mathcal{K}), \\ \mathcal{C}_{\mathcal{K}} \otimes_h \mathcal{R}_{\overline{\mathcal{H}}} &\overset{\text{cb}}{=} K(\mathcal{H}, \mathcal{K}). \end{aligned}$$

This example also shows that the Haagerup tensor product is not symmetric.

6.5 Completely bounded bilinear mappings

In the case of bilinear mappings between operator spaces one has to distinguish between two different notions of *complete boundedness*: on one hand we have the **jointly completely bounded** [BP91, Def. 5.3 (**jointly completely bounded**)] and, on the other hand, the **completely bounded** bilinear mappings [CS87, Def. 1.1].

The class of *completely bounded* bilinear maps is is contained in the first one. These notions are in perfect analogy to those of bounded bilinear forms on normed spaces. For *completely bounded* bilinear mappings, we have at our disposal similar representation and extension theorems[47] as in the case of completely bounded linear maps. There are two tensor products corresponding to the above two classes of bilinear mappings, namely the projective and the Haagerup tensor product. Depending on the class of bilinear maps, one uses different methods to define the amplification of a bilinear mapping $\Phi : X \times Y \to Z$.

[45] Let $S \in M_n(CB(X_1, X_2))$, $T \in M_n(CB(Y_1, Y_2))$. For $x \in M_{p,q}(X_1)$, $y \in M_{q,p}(Y_1)$ we have

$$\begin{aligned} (S \odot T)^{(p)}(x \odot y) &= (S^{(p,q)}(x)) \odot (T^{(q,p)}(y)), \\ \|(S \odot T)^{(p)}(x \odot y)\|_{M_{pn}(X_2 \otimes_h Y_2)} &\leq \|S^{(p,q)}(x)\|\|T^{(q,p)}(y)\| \leq \|S\|_{\text{cb}}\|T\|_{\text{cb}}\|x\|\|y\|. \end{aligned}$$

By the definition of the Haagerup norm (in $X_1 \otimes_h Y_1$) we obtain

$$\begin{aligned} \|(S \odot T)^{(p)}(x \odot y)\|_{M_{pn}(X_2 \otimes Y_2)} &\leq \|S\|_{\text{cb}}\|T\|_{\text{cb}}\|x \odot y\|_{M_p(X_1 \otimes_h Y_1)}, \\ \|S \odot_h T\|_{\text{cb}} &\leq \|S\|_{\text{cb}}\|T\|_{\text{cb}}. \end{aligned}$$

This means that $\otimes_h$ is completely contractive.

[46] T is endowed with its natural operator space structure $T(\mathcal{H}, \mathcal{K}) :\overset{\text{cb}}{=} K(\mathcal{K}, \mathcal{H})^*$.

[47] Extension theorems for completely bounded bilinear (and, more generally, multilinear) maps can be derived from the injectivity of the Haagerup tensor product.

Amplification

In the literature, there are two different notions of an amplification of a bilinear mapping.

We shall call the first kind 1 of amplification the joint amplification. This joint amplification is needed to obtain a matrix duality – which is fundamental in the duality theory of operator spaces –, starting from an ordinary duality $\langle X, X^* \rangle$.

The notion of joint amplification leads to the jointly completely bounded bilinear maps as well as the projective operator space tensor product.

We will speak of the second kind 2 of an amplification as *the* amplification of a bilinear mapping. This notion leads to the completely bounded bilinear maps and the Haagerup tensor product.

In the sequel, we will use the notation $\Phi : X \times Y \to Z$ for a bilinear mapping and $\tilde{\Phi} : X \otimes Y \to Z$ for its linearization.

Both notions of an amplification of a bilinear map Φ are formulated in terms of the **amplification of its linearization**:

$$\tilde{\Phi}^{(n)} : M_n(X \otimes Y) \to M_n(Z).$$

1. The **joint amplification** of Φ produces the bilinear mapping

$$\Phi^{(p \times q)} : (x, y) \mapsto \tilde{\Phi}^{(pq)}(x \otimes y) = [\Phi(x_{ij}, y_{kl})] \in M_p(M_q(Z)) = M_{pq}(Z).$$

   of the operator matrices $x = [x_{ij}] \in M_p(X)$ and $y = [y_{kl}] \in M_q(Y)$. Here, the tensor product of operator matrices is defined via

$$x \otimes y = [x_{ij}] \otimes [y_{kl}] := [x_{ij} \otimes y_{kl}] \in M_{pq}(X \otimes Y) = M_p(M_q(X \otimes Y)).$$

2. In the case of completely bounded bilinear maps one deals with the **tensor matrix multiplication** [Eff87]

$$x \odot y = [x_{ij}] \odot [y_{jk}] := \left[\sum_{j=1}^{l} x_{ij} \otimes y_{jk} \right] \in M_n(X \otimes Y)$$

   of operator matrices $x = [x_{ij}] \in M_{n,l}(X)$ and $y = [y_{jk}] \in M_{l,n}(Y)$.

   For more formulae, see: tensor matrix multiplication.

   The (n,l)-th **amplification** of a bilinear map $\Phi : X \times Y \to Z$ is defined by

$$\begin{aligned} \Phi^{(n,l)} : M_{n,l}(X) \times M_{l,n}(Y) &\to M_n(Z) \\ (x, y) &\mapsto \tilde{\Phi}^{(n)}(x \odot y) = \left[\sum_{j=1}^{l} \Phi(x_{ij}, y_{jk}) \right] \in M_n(Z) \end{aligned}$$

   for $l, n \in \mathbb{N}$, $x = [x_{ij}] \in M_{n,l}(X)$, $y = [y_{jk}] \in M_{l,n}(Y)$. In case $n = l$, we shortly write[48]

$$\begin{aligned} \Phi^{(n)} := \Phi^{(n,n)} : M_n(x) \otimes M_n(Y) &\to M_n(Z) \\ (x, y) &\mapsto \Phi^{(n)}(x, y) := \tilde{\Phi}^{(n)}(x \odot y). \end{aligned}$$

[48] In the literature, the amplification of a bilinear map often is defined only for quadratic operator matrices and is called *the* amplification $\Phi^{(n)}$.

Nevertheless, we will also be dealing with the amplification for rectangular matrices since this permits the formulation of statements where for fixed $n \in \mathbb{N}$ all amplifications $\Phi^{(n,l)}$, $l \in \mathbb{N}$, are considered.

Jointly complete boundedness

Let X, Y, Z be operator spaces. A bilinear mapping $\Phi : X \times Y \to Z$ is called **jointly completely bounded** [BP91, Def. 5.3 (**jointly completely bounded**)] if the norms of the joint amplifications of Φ are uniformly bounded:

$$\|\Phi\|_{\mathrm{jcb}} := \sup \|\Phi^{(p\times q)}(x \otimes y)\| < \infty,$$

where $p, q \in \mathbb{N}$, $x \in \mathrm{Ball}(M_p(X))$, $y \in \mathrm{Ball}(M_q(Y))$ [BP91, Def. 5.3]. The norm $\|\Phi\|_{\mathrm{jcb}}$ equals the norm $\|\tilde{\Phi}\|_{\mathrm{cb}}$ of the linearization

$$\tilde{\Phi} : X \overset{\wedge}{\otimes} Y \to Z$$

on the projective operator space tensor product.

$JCB(X \times Y; Z)$ denotes the operator space consisting of the jointly completely bounded bilinear maps. One obtains a norm on each matrix level by the identification

$$M_n(JCB(X \times Y; Z)) = JCB(X \times Y; \mathbb{M}_n(Z)).$$

We have

$$JCB(X \times Y; Z) \overset{\mathrm{cb}}{=} CB(X, CB(Y, Z)) \overset{\mathrm{cb}}{=} CB(Y, CB(X, Z)).$$

completely isometrically. By taking the transposition $\Phi^t(y, x) := \Phi(x, y)$ we obtain a complete isometry

$$JCB(X \times Y; Z) \overset{\mathrm{cb}}{=} JCB(Y \times X; Z).$$

Complete boundedness

For the definition of the **completely bounded** bilinear maps we need the amplification $\Phi^{(n)}$, the linearization $\tilde{\Phi} : X \otimes Y \to Z$. and the tensor matrix multiplication $x \odot y$ of operator matrices x, y.

A bilinear mapping $\Phi : X \times Y \to Z$, $n \in \mathbb{N}$ is called **completely bounded** if

$$\|\Phi\|_{\mathrm{cb}} := \sup \|\Phi^{(n)}(x, y)\| < \infty$$

where $n \in \mathbb{N}$, $x \in \mathrm{Ball}(M_n(X))$, $y \in \mathrm{Ball}(M_n(Y))$.

The norm $\|\Phi\|_{\mathrm{cb}}$ equals the norm $\|\tilde{\Phi}\|_{\mathrm{cb}}$ of the linearization

$$\tilde{\Phi} : X \otimes_h Y \to Z$$

on the Haagerup tensor product.

Furthermore, the norms $\|\Phi\|_n$ are obtained using the tensor matrix products of all [49] rectangular matrices of n rows resp. n columns:

$$\|\Phi\|_n := \sup \|\Phi^{(n,l)}(x, y)\|$$

[49] Note that the norm of the bilinear map

$$\Phi^{(n)} : M_n(x) \otimes M_n(Y) \to M_n(Z)$$

in general is smaller than the norm $\|\Phi\|_n$.

where $l \in \mathbb{N}$, $x \in \mathrm{Ball}(M_{n,l}(X))$, $y \in \mathrm{Ball}(M_{l,n}(Y))$.

We have

$$\|\Phi\|_{\mathrm{cb}} = \sup\{\|\Phi\|_n \ : \ n \in \mathbb{N}\}.$$

The norm $\|\Phi\|_n$ equals the norm $\|\tilde{\Phi}^{(n)}\|$ of the amplification of the linearization

$$\tilde{\Phi}^{(n)} : M_n(X \otimes_h Y) \to M_n(Z)$$

on the Haagerup tensor product.

A bilinear form Φ is already seen to be completely bounded if $\|\Phi\|_1 < \infty$. Then we have $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|_1$.[50]

$CB(X \times Y; Z)$ denotes the operator space consisting of completely bounded bilinear maps. One obtains a norm on each matrix level using the identification

$$M_n(CB(X \times Y; Z)) = CB(X \times Y; \mathbb{M}_n(Z)).$$

Corresponding to the completely bounded bilinear maps we have the linear maps which are completely bounded on the Haagerup tensor product. The identification

$$CB(X \times Y; Z) \stackrel{\mathrm{cb}}{=} CB(X \otimes_h Y; Z)$$

holds completely isometrically.

Completely bounded bilinear mappings are in particular jointly completely bounded. The embedding $CB(X \times Y; Z) \subset JCB(X \times Y; Z)$ is a complete contraction.

The transpose $\Phi^t(y, x) := \Phi(x, y)$ of a completely bounded bilinear mapping Φ in general is not completely bounded.[51]

For completely bounded bilinear (and, more generally, multilinear) maps $\Phi \in CB(A \times B; B(\mathcal{H}))$ we have some generalizations of Stinespring's representation theorem.

Representation

Completely bounded bilinear forms were first studied on C*-algebras A, B [EK87]. For a bilinear form $\Phi : A \times B \to \mathbb{C}$ the following properties are equivalent:

(1) Φ is completely bounded.

(2) There is a constant c such that

$$|\sum_{j=1}^{l} \Phi(a_j, b_j)| \le c \,\|\sum_{j=1}^{l} a_j a_j^*\|^{\frac{1}{2}} \|\sum_{j=1}^{l} b_j^* b_j\|^{\frac{1}{2}}$$

for all $l \in \mathbb{N}$, $a_j \in A$, $b_j \in B$.

[50] More generally, the equation $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|_1$ holds for bilinear maps with values in a commutative C*-algebra A since every bounded linear map taking values in A is automatically completely bounded and $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|$ [Loe75, Lemma 1]. For bilinear maps $\Phi : X \times Y \to M_n(A)$ we have $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|_n$.

[51] To this corresponds the fact that the Haagerup tensor product is not symmetric.

(3) There is a constant c and states $\omega \in S(A)$, $\rho \in S(B)$ such that

$$|\Phi(a,b)| \leq c\,\omega(aa^*)^{\frac{1}{2}}\rho(b^*b)^{\frac{1}{2}}$$

for all $a \in A$, $b \in B$.

(4) There exist *-representations $\pi_\omega : A \to B(\mathcal{H}_\omega)$ and $\pi_\rho : A \to B(\mathcal{H}_\rho)$ with cyclic vectors $\xi_\omega \in \mathcal{H}_\omega$, $\xi_\rho \in \mathcal{H}_\rho$ and an operator $T \in B(\mathcal{H}_\omega, \mathcal{H}_\rho)$ such that $\Phi(a,b) = \langle T\pi_\omega(a)\xi_\omega, \pi_\rho(b)\xi_\rho\rangle$ for all $a \in A$, $b \in B$.

One can choose $c = \|T\| = \|\Phi\|_{\text{cb}}$ and $\|\xi_\omega\| = \|\xi_\rho\| = 1$.[52]

6.6 Module tensor products

So far the only module tensor product of operator modules that has been studied is the module Haagerup tensor product [Rua89] [BMP].

6.6.1 Module Haagerup tensor product

Let X be a right operator module over a C^*-algebra A, Y a left A-operator module, and W an operator space. A bilinear mapping $\Psi : X \times Y \to W$ is called **balanced**, if the equation

$$\Psi(x \cdot a, y) = \Psi(x, a \cdot y)$$

obtains for all $x \in X, y \in Y, a \in A$.

The **module Haagerup tensor product** is defined to be the operator space $X \otimes_{hA} Y$ (which is unique up to complete isometry) together with a bilinear, completely contractive, balanced mapping

$$\otimes_{hA} : X \times Y \to X \otimes_{hA} Y,$$

such that the following holds true: For each bilinear, completely bounded balanced map

$$\Psi : X \times Y \to W$$

there is a unique linear completely bounded map

$$\widetilde{\Psi} : X \otimes_{hA} Y \to W$$

satisfying $\widetilde{\Psi} \circ \otimes_{hA} = \Psi$ and $\|\widetilde{\Psi}\|_{\text{cb}} = \|\Psi\|_{\text{cb}}$.

The module Haagerup tensor product can be realized in different ways:

1. Let
$$N := \text{lin}\{(x \cdot a) \otimes y - x \otimes (a \cdot y) \mid a \in A, x \in X, y \in Y\}.$$
The quotient space $(X \otimes_h Y)\big/\overline{N}$ with its canonical matrix norms is an operator space which satisfies the defintion of $X \otimes_{hA} Y$ [BMP].

[52] For further references see [CS89, Sec. 4].

2. Let us denote by $X \otimes_A Y$ the algebraic module tensor product, i.e. the quotient space $(X \otimes_{\mathrm{alg}} Y)/N$. For $n \in \mathbb{N}$ and $u \in M_n(X \otimes_A Y)$, by

$$p_n(u) := \inf \left\{ \|S\|\|T\| \,\middle|\, u = \left[\sum_{k=1}^{l} S_{ik} \otimes_A T_{kj}\right], l \in \mathbb{N}, S \in M_{nl}(X), T \in M_{ln}(Y) \right\}$$

we define a semi-norm on $M_n(X \otimes_A Y)$. We obtain

$$\mathrm{Kern}(p_n) = M_n(\mathrm{Kern}(p_1)),$$

and the semi-norms p_n give an operator space norm on $(X \otimes_A Y)/\mathrm{Kern}(p_1)$ [Rua89]. The completion of this space satisfies the definition of $X \otimes_{hA} Y$ [BMP].

Examples

Let X be an operator space. Then the space of completely bounded (A_1, A_2)-module homomorphisms between X and $B(\mathcal{H})$ can be identified with the dual of a module Haagerup tensor product in the following way ([Pet97, p. 67], cf. also [ER91, Cor. 4.6], [Ble92b, Prop. 2.3]):

$$CB_{(A_1,A_2)}(X, B(\mathcal{H})) \overset{\mathrm{cb}}{=} (R_{\overline{\mathcal{H}}} \otimes_{hA_1} X \otimes_{hA_2} C_{\mathcal{H}})^*$$

completely isometrically.

7 Complete Local Reflexivity

An operator space X is called **completely locally reflexive** [EJR98, §1], if to each finite dimensional subspace there is $L \subset X^{**}$ a net of completely contractive mappings $\varphi_\alpha : L \to X$ that converges to the embedding $L \to X^{**}$ in the point weak* topology.[53]

This property is inherited by arbitrary subspaces. In general it is not preserved by quotients. In the case that for example the kernel is an M-ideal (e.g. the twosided ideal of a C^*-algebra) and that the original space is completely reflexive we have that the quotient space is completely locally reflexive [ER94, Thm. 4.6].

Banach spaces are always locally reflexive (*Principle of local reflexivity* [Sch70]).

On the contrary, **not all** operator spaces are completely locally reflexive. For example the full C^*-algebra of the free group on two generators $C^*(F_2)$ and $B(\ell_2)$ are not completely locally reflexive [EH85, p. 124-125].

An operator space X is completely locally reflexive, if and only if one (and then every) of the following conditions is satisfied for all finite dimensional operator spaces L [EJR98, §1, 4.4, 5.8]:

1. $L \overset{\vee}{\otimes} X^{**} \overset{\mathrm{cb}}{=} (L \overset{\vee}{\otimes} X)^{**}$, where $\overset{\vee}{\otimes}$ denotes the injective operator space tensor product,

[53] The difference to the definition of local reflexivity is the fact that the φ_α are not only supposed to be contractive, but even completely contractive.

2. $CB(L^*, X^{**}) \overset{\text{cb}}{=} CB(L^*, X)^{**}$,
3. $L^* \overset{\wedge}{\otimes} X^* \overset{\text{cb}}{=} (L \overset{\vee}{\otimes} X)^*$, where $\overset{\wedge}{\otimes}$ denotes the projective operator space tensor product and $\overset{\vee}{\otimes}$ denotes the injective operator space tensor product,
4. $CN(X, L^*) \overset{\text{cb}}{=} CI(X, L^*)$, where $CN(\cdot,\cdot)$ denotes the completely nuclear and $CI(\cdot,\cdot)$ the completely integral mappings,
5. $\iota(\varphi) = \iota(\varphi^*)$ for all $\varphi \in CI(X, L^*)$.

In the conditions 1), 2) and 3) it suffices to check the usual isometry to prove the complete isometry.

Examples

The following classes of operator spaces are completely locally reflexive [EJR98, 6.1, 6.2], [EH85, Prop. 5.4]:

1. reflexive operator spaces (e.g. finite dimensional operator spaces L),
2. nuclear C^*- algebras (e.g. $K(\mathcal{H})$ or commutative C^*- algebras),[54]
3. preduals of von Neumann algebras, especially duals of C^*-algebras (e.g. $T(\mathcal{H}) = K(\mathcal{H})^* = B(\mathcal{H})_*$).

8 Completely Bounded Multilinear Mappings

Going beyond the linear case, one can introduce the concept of complete boundedness for multilinear maps. The motivation mainly lies in the study of higher dimensional Hochschild cohomology over C^*- and von Neumann algebras[55] [Christensen/Effros/Sinclair '87].

As in the linear case, the most important properties of the completely bounded multilinear maps make their appearance[56] in **representation**, **extension** and **decomposition theorems**.

Let $X_1, \dots, X_k$, Y be operator spaces and $\Phi : X_1 \times \cdots \times X_k \to Y$ a multilinear mapping. We define [Christensen/Effros/Sinclair '87, p. 281] a multilinear mapping

$$\begin{aligned} \Phi^{(n)} : M_n(X_1) \times \cdots \times M_n(X_k) &\to M_n(Y) \\ (x_1, \dots, x_k) &\mapsto \left[\sum_{j_1, j_2, \dots, j_{k-1}=1}^{n} \Phi(x_1^{l,j_1}, x_2^{j_1,j_2}, \dots, x_k^{j_{k-1},m}) \right], \end{aligned}$$

[54]a survey of the theory of nuclear C^*-algebras is e.g. to be found in [Mur90] and [Pat88].

[55]There is a close connection to the long-standing still open derivation problem for C^*-algebras (or, equivalently [Kirchberg '96, Cor. 1], the similarity problem). One should note that in the framework of operator spaces and completely bounded maps, considerable progress has been made in attacking these problems. For instance, Christensen [Christensen '82, Thm. 3.1] was able to show that the inner derivations from a C^*-algebra into $B(\mathcal{H})$ are precisely the completely bounded ones.

[56]possibly footnote!

where $n \in \mathbb{N}$, the **nth amplification** of Φ.

Φ is called **completely bounded** if $\|\Phi\|_{\mathrm{cb}} := \sup_n \|\Phi^{(n)}\| < \infty$. It is called **completely contractive** if $\|\Phi\|_{\mathrm{cb}} \le 1$.

Also compare the chapter Completely bounded bilinear maps.

Example: For bilinear forms on commutative C^*-algebras we have the following result on automatic complete boundedness [Christensen/Sinclair '87, Cor. 5.6]:

Let A be a commutative C^*-algebra. Then each continuous bilinear form $\Phi : A \times A \to \mathbb{C}$ is automatically completely bounded and

$$\|\Phi\| \le \|\Phi\|_{\mathrm{cb}} \le K_G \|\Phi\|,$$

where K_G denotes the Grothendieck constant. Furthermore, K_G is the least such constant.

One often studies completely bounded multilinear maps by considering the linearization on the Haagerup tensor product, where the following relation holds [Paulsen/Smith '87, Prop. 1.3; cf. also Sinclair/Smith '95, Prop. 1.5.1]: If $X_1, \dots, X_n$ are operator spaces and $\mathcal{H}$ is a Hilbert space, then a multilinear mapping $\Phi : X_1 \times \cdots \times X_n \to B(\mathcal{H})$ is completely bounded if and only if its linearization φ is a completely bounded mapping on $X_1 \otimes_h \cdots \otimes_h X_n$. In this case, $\|\Phi\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}$.

Also compare the chapter: Completely bounded bilinear mappings.

Representation theorem [Paulsen/Smith '87, Thm. 3.2, cf. also Thm. 2.9; Sinclair/Smith, Thm. 1.5.4]:
Let $A_1, \dots, A_k$ be C^*-algebras, $X_1 \subset A_1, \dots, X_k \subset A_k$ operator spaces and $\mathcal{H}$ a Hilbert space. Let further be $\Phi : X_1 \times \cdots \times X_k \to B(\mathcal{H})$ a completely contractive multilinear mapping. Then there exist Hilbert spaces $\mathcal{K}_i$ $(i = 1, \dots, k)$, *-representations $\pi_i : A_i \to B(\mathcal{K}_i)$ $(i = 1, \dots, k)$, contractions $T_i : \mathcal{K}_{i+1} \to \mathcal{K}_i$ $(i = 1, \dots, k-1)$ and two isometries $V_i : \mathcal{H} \to \mathcal{K}_i$ $(i = 1, k)$ such that

$$\Phi(x_1, \dots, x_k) = V_1^* \pi_1(x_1) T_1 \pi_2(x_2) T_2 \cdots T_{k-1} \pi_k(x_k) V_k.$$

Following [Ylinen '90, p. 296; cf. also Christensen/Effros/Sinclair '87] it is possible to eliminate the "bridging maps" T_i. One obtains the following simpler form for the **representation theorem**:

Let $A_1, \dots, A_k$ be C^*-algebras, $X_1 \subset A_1, \dots, X_k \subset A_k$ operator spaces and $\mathcal{H}$ a Hilbert space. Let further $\Phi : X_1 \times \cdots \times X_k \to B(\mathcal{H})$ be a completely contractive multilinear mapping. Then there exist a Hilbert space $\mathcal{K}$, *-representations $\pi_i : A_i \to B(\mathcal{K})$ and two operators $V_1, V_k \in B(\mathcal{H}, \mathcal{K})$ such that

$$\Phi(x_1, \dots, x_k) = V_1^* \pi_1(x_1) \pi_2(x_2) \cdots \pi_k(x_k) V_k.$$

From this result one can deduce the following:

Extension theorem [cf. Paulsen/Smith '87, Cor. 3.3 and Sinclair/Smith '95, Thm. 1.5.5]:
Let $X_i \subset Y_i$ $(i = 1, \dots, k)$ be operator spaces and $\mathcal{H}$ a Hilbert space. Let further $\Phi : X_1 \times \dots X_k \to B(\mathcal{H})$ be a completely contractive multilinear mapping. Then there

exists a multilinear mapping $\widetilde{\Phi} : Y_1 \times \cdots \times Y_k \to B(\mathcal{H})$ which extends Φ preserving the cb-norm: $\|\Phi\|_{\mathrm{cb}} = \|\widetilde{\Phi}\|_{\mathrm{cb}}$.

Let A and B be C^*-algebras. For a k-linear mapping $\Phi : A^k \to B$ we define [Christensen/Sinclair '87, pp. 154-155] another k-linear mapping $\Phi^* : A^k \to B$ by

$$\Phi^*(a_1, \ldots, a_k) := \Phi(a_k^*, \ldots, a_2^*, a_1^*)^*,$$

where $a_1, \ldots, a_k \in A$. A k-linear map $\Phi : A^k \to B$ is called **symmetric** if $\Phi = \Phi^*$. In this case, ${\Phi^{(n)}}^* = \Phi^{(n)}$ $(n \in \mathbb{N})$.

A k-linear map $\Phi : A^k \to B$ is called **completely positive** if

$$\Phi^{(n)}(A_1, \ldots A_k) \geq 0$$

for all $n \in \mathbb{N}$ and $(A_1, \ldots, A_k) = (A_k^*, \ldots, A_1^*) \in M_n(A)^k$, where $A_{\frac{k+1}{2}} \geq 0$ for odd k.

Caution is advised: In the multilinear case complete positivity does **not** necessarily imply complete boundedness! For an example (or more precisely a general method of constructing such), cf. Christensen/Sinclair '87, p. 155.

There is a multilinear version of the decomposition theorem for completely bounded symmetric multilinear mappings:

Decomposition theorem [Christensen/Sinclair '87, Cor. 4.3]:
Let A and B be C^*-algebras, where B is injective, and let further $\Phi : A^k \to B$ be a completely bounded symmetric k-linear mapping. Then there exist completely bounded, completely positive k-linear mappings $\Phi_+, \Phi_- : A^k \to B$ such that $\Phi = \Phi_+ - \Phi_-$ and $\|\Phi\|_{\mathrm{cb}} = \|\Phi_+ + \Phi_-\|_{\mathrm{cb}}$.

9 Automatic Complete Boundedness

Completely bounded linear and multilinear mappings share strong structural properties. Thanks to the complete boundedness they have a very specific form (cf. the corresponding representation theorems) whence they are much more accessible than arbitrary bounded linear (or multilinear) mappings.[57] Moreover, even in the multilinear case we have at our disposal an extension theorem for completely bounded mappings – which again is in striking contrast to the situation of arbitrary bounded (multi)linear mappings.

Because of the very nice structure theory of completely bounded linear and multilinear mappings, respectively, it is highly interesting to decide whether or not a given bounded (multi)linear mapping actually is completely bounded. The most elegant way to proceed is of course to check some simple conditions concerning the spaces involved and/or some (purely) algebraic properties of the mapping which automatically imply the complete boundedness of the latter.

In the following we shall collect various such criteria relying

(1) on the initial and/or target space

[57] For example, the representation theorems provide a very useful tool in calculating cohomology groups (cf. for example the monograph [SS95]).

(2) mainly on algebraic properties of the mapping (being, e.g., a *-homomorphism of C^*-algebras or a module homomorphism).

(1) "Criterion: spaces"

(1.1) The linear case

* **Smith's lemma**: If X is a matricially normed space and $\Phi : X \to M_n$, where $n \in \mathbb{N}$, is a bounded linear operator, then we have $\|\Phi\|_{\text{cb}} = \|\Phi^{(n)}\|$. In particular, Φ is completely bounded if and only if $\Phi^{(n)}$ is bounded [Smi83, Thm. 2.10].
* Let X be an operator space, A a commutative C^*-algebra and $\Phi : X \to A$ a bounded linear operator. Then Φ is completely bounded and $\|\Phi\|_{\text{cb}} = \|\Phi\|$ ([Loe75, Lemma 1], cf. also [Arv69, Prop. 1.2.2]).
  In particular, every bounded linear functional is automatically completely bounded with the same cb-norm.
* The following theorem shows that, roughly speaking, the above situations are the only ones where every bounded linear operator between (arbitrary) C^*-algebras is automatically completely bounded. More precisely:
  Let A and B be C^*-algebras. In order to have the complete boundedness of very bounded linear operator from A to B it is necessary and sufficient that either A is finite dimensional or B is a subalgebra of $M_n \otimes C(\Omega)$ for a compact Hausdorff space Ω [HT83, Cor. 4], [Smi83, Thm. 2.8], cf. also [Smi83, p. 163].

(1.2) The bilinear case

* Let A be a commutative C^*-algebra. Then every bounded bilinear form $\Phi : A \times A \to \mathbb{C}$ is automatically completely bounded and

$$\|\Phi\| \leq \|\Phi\|_{\text{cb}} \leq K_G \|\Phi\|,$$

  where K_G denotes the Grothendieck constant; furthermore, K_G is the least possible constant [CS87, Cor. 5.6]

(2) "Criterion: mappings"

(2.1) The linear case

* Every *-homomorphism between C^*-algebras is completely contractive.[58]
* Let A and B be C^*-subalgebras of $B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Let further $E \subset B(\mathcal{H})$ be an (A, B)-operator module and $\Phi : E \to B(\mathcal{H})$ be a bounded (A, B)-module homomorphism. Let A and B be **quasi-cyclic**[59] , i.e., for every finite set of vectors $\xi_i, \eta_j \in \mathcal{H}$, $1 \leq i, j \leq n$, there exist

[58] This is evident because $\pi^{(n)} : M_n(A) \to M_n(B)$ is a *-homomorphism for every $n \in \mathbb{N}$.

[59] For example, cyclic C^*-algebras are quasi-cyclic. A von Neumann algebra M is quasi-cyclic if every normal state on the commutant M' is a vector state [Smi91, Lemma 2.3].

$\xi, \eta \in \mathcal{H}$ such that $\xi_i \in \overline{A\xi}$, $\eta_j \in \overline{B\eta}$, $1 \le i, j \le n$. Then Φ is completely bounded and $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|$ ([SS95, Thm. 1.6.1]; cf. also [Smi91, Thm. 2.1] and [Smi91, Remark 2.2], [Sat82, Satz 4.16]). – For special cases of this result obtained earlier see [Haa80], [EK87, Thm. 2.5], [PPS89, Cor. 3.3] and [DP91, Thm. 2.4].

(2.2) The bilinear case

∗ Let $A \subset B(\mathcal{H})$ be a C^*-algebra and $E \subset B(\mathcal{H})$ an A-operator bimodule. A bilinear mapping $\Phi : E \times E \to B(\mathcal{H})$ is called **A-multimodular** if

$$\Phi(aeb, fc) = a\Phi(e, bf)c$$

holds for all $a, b, c \in A$ and $e, f \in E$.
We have a bilinear analogue of the above theorem:
Let $A \subset B(\mathcal{H})$ be a quasi-cyclic C^*-algebra and $E \subset B(\mathcal{H})$ be an A-operator bimodule. Let further $\Phi : E \times E \to B(\mathcal{H})$ be a bilinear, A-multimodular mapping such that the corresponding linearization $\widetilde{\Phi} : E \otimes E \to B(\mathcal{H})$ is bounded on $E \otimes_h E$.
Then Φ is completely bounded and $\|\Phi\|_{\mathrm{cb}} = \|\widetilde{\Phi}\|$ [SS95, Thm. 1.6.2].

10 Convexity

Since convex sets are important in the study of ordered or normed vector spaces, it is only natural to ask for a non-commutative version of convexity that is more suited for vector spaces of operators. Thus matrix convex sets were introduced, which play the same role in operator space theory as ordinary convex sets in classical functional analysis.

Some time earlier C^*-convex sets were defined for C^*-algebras. Both notions are different but similar, so a section about C^*-convex sets is also included in this survey.

Some publications about non-commutative convexity are [EW97b], [WW99], [FZ98], [Mor94], [Fuj94].

10.1 Matrix convexity

Let V be a complex vector space. A **set K of matrices over V** consists of subsets $K_n \subset M_n(V)$ for all $n \in \mathbb{N}$. Subsets $K_1 \subset V$ can be considered as sets of matrices over V by putting $K_n = \emptyset$ for $n \ge 2$. More generally, if for some $n \in N$ the corresponding set K_n is not given a priori, put this K_n as the empty set. For sets of matrices, we have the following notions of convexity:[60]

[60] With the notation $M(V) = \bigcup\{M_n(V) \mid n \in \mathbb{N}\}$, a set K of matrices over V is simply a subset of $M(V)$. The sets K_n can be regained as $K_n = K \cap M_n(V)$.
Then the definitions have the following form:
Let $K \subset M(V)$. K is called **matrix convex** if

$$K \oplus K \subset K \text{ and } \alpha^* K \alpha \subset K \text{ for all matrices } \alpha \text{ with } \alpha^*\alpha = \mathbb{1}.$$

A set K of matrices over V is called **matrix convex** or a **matrix convex set** [Wit84b] if for all $x \in K_n$ and $y \in K_m$

$$x \oplus y \in K_{n+m},$$

and for all $x \in K_n$ and $\alpha \in M_{n,m}$ with $\alpha^* \alpha = \mathbb{1}_m$

$$\alpha^* x \alpha \in K_m.$$

A set K of matrices over V is called **absolutely matrix convex** [EW97a] if for all $x \in K_n$ and $y \in K_m$

$$x \oplus y \in K_{n+m},$$

and for all $x \in K_n$, $\alpha \in M_{n,m}$ and $\beta \in M_{m,n}$ with $\|\alpha\|$, $\|\beta\| \le 1$

$$\alpha x \beta \in K_m.$$

A set K of matrices over V is called a **matrix cone** [Pow74] if for all $x \in K_n$ and $y \in K_m$

$$x \oplus y \in K_{n+m},$$

and for all $x \in K_n$ and $\alpha \in M_{n,m}$

$$\alpha^* x \alpha \in K_m.$$

A set K of matrices over V is matrix convex if and only if all matrix convex combinations of elements of K are again in K. K is absolutely matrix convex if and only if all absolutely matrix convex combinations of elements of K are again in K. Here, a **matrix convex combination** of $x_1, \ldots, x_n$ ($x_i \in K_{k_i}$) is a sum of the form $\sum_{i=1}^n \alpha_i^* x_i \alpha_i$ with matrices $\alpha_i \in M_{k_i,j}$ such that $\sum_{i=1}^n \alpha_i^* \alpha_i = \mathbb{1}_j$. An **absolutely matrix convex combination** of $x_1, \ldots, x_n$ is a sum of the form $\sum_{i=1}^n \alpha_i x_i \beta_i$ with matrices $\alpha_i \in M_{j,k_i}$ and $\beta_i \in M_{k_i,j}$ such that $\sum_{i=1}^n \alpha_i \alpha_i^* \le \mathbb{1}_j$ and $\sum_{i=1}^n \beta_i^* \beta_i \le \mathbb{1}_j$.

If V is a topological vector space, topological terminology is to be considered at all matrix levels: For instance, a set K of matrices over V is called **closed** if all K_n are closed.

K is called **absolutely matrix convex** if

$$K \oplus K \subset K \text{ and } \alpha K \beta \subset K \text{ for all matrices } \alpha, \beta \text{ with } \|\alpha\|, \|\beta\| \le 1.$$

K is called a **matrix cone** if

$$K \oplus K \subset K \text{ and } \alpha^* K \alpha \subset K \text{ for all matrices } \alpha.$$

Examples

matrix convexity

1. The unit ball of an operator space X, given by the family $\mathrm{Ball}(X)_n = \{x \in M_n(X) \mid \|x\| \leq 1\}$ for all $n \in \mathbb{N}$,[61] is absolutely matrix convex and closed.

2. The set of matrix states of a unital C^*-algebra A, given by the family $\mathrm{CS}(A)_n = \{\varphi : A \to M_n \mid \varphi$ completely positive and unital$\}$ for all $n \in \mathbb{N}$, is matrix convex and weak-$*$-compact.

3. If A is a C^*-algebra, the positive cones $M_n(A)^+$ for all $n \in \mathbb{N}$ form a closed matrix cone.

10.1.1 Separation theorems

An important tool in the theory are the following **separation theorems**.

Let $\langle V, W\rangle$ be a (non degenerate) duality of complex vector spaces. Thus V and W have weak topologies, and the matrix levels have the corresponding product topology. For $v = [v_{i,j}] \in M_n(V)$ and $w = [w_{k,l}] \in M_m(W)$, $\langle v, w\rangle$ is defined by the joint amplifications of the duality:

$$\langle v, w\rangle = [\langle v_{i,j}, w_{k,l}\rangle]_{(i,k),(j,l)}.$$

Note that the matrices, ordered by the cone of the positive semidefinite matrices, are not totally ordered; $\not\leq$ does not imply $\geq$.

Theorem:[62] Let $\langle V, W\rangle$ be a duality of complex vector spaces, K a closed set of matrices over V and $v_0 \in M_n(V) \setminus K_n$ for some n.

a) [WW99, Thm. 1.6] If K is matrix convex, then there are $w \in M_n(W)$ and $\alpha \in (M_n)_{\mathrm{sa}}$ such that for all $m \in \mathbb{N}$ and $v \in K_m$

$$\mathrm{Re}\langle v, w\rangle \leq \mathbb{1}_m \otimes \alpha, \text{ but } \mathrm{Re}\langle v_0, w\rangle \not\leq \mathbb{1}_n \otimes \alpha.$$

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[61] Also $\mathrm{Ball} = \{x \in M(X) \mid \|x\| \leq 1\}$.

[62] From this theorem one can easily get the following sharper version of the parts a), b) and d):

a) If K is matrix convex, then there are $w \in M_n(W)$, $\alpha \in (M_n)_{\mathrm{sa}}$ and $\varepsilon > 0$ such that for all $m \in \mathbb{N}$ and $v \in K_m$

$$\mathrm{Re}\langle v, w\rangle \leq \mathbb{1}_m \otimes (\alpha - \varepsilon \mathbb{1}_n), \text{ but } \mathrm{Re}\langle v_0, w\rangle \not\leq \mathbb{1}_n \otimes \alpha.$$

b) If K is matrix convex and $0 \in K_1$, then there are $w \in M_n(W)$ and $\varepsilon > 0$ such that for all $m \in \mathbb{N}$ and $v \in K_m$

$$\mathrm{Re}\langle v, w\rangle \leq (1 - \varepsilon)\mathbb{1}_{nm}, \text{ but } \mathrm{Re}\langle v_0, w\rangle \not\leq \mathbb{1}_{n^2}.$$

d) If K is absolutely matrix convex, then there are $w \in M_n(W)$ and $\varepsilon > 0$ such that for all $m \in \mathbb{N}$ and $v \in K_m$

$$\|\langle v, w\rangle\| \leq 1 - \varepsilon, \text{ but } \|\langle v_0, w\rangle\| > 1.$$

b) [EW97b, Thm. 5.4] If K is matrix convex and $0 \in K_1$, then there is $w \in M_n(W)$ such that for all $m \in \mathbb{N}$ and $v \in K_m$

$$\operatorname{Re}\langle v, w\rangle \le \mathbb{1}_{nm}, \text{ but } \operatorname{Re}\langle v_0, w\rangle \not\le \mathbb{1}_{n^2}.$$

c) [Bet97, p. 57] If K is a matrix cone, then there is $w \in M_n(W)$ sucht that for all $m \in \mathbb{N}$ and $v \in K_m$

$$\operatorname{Re}\langle v, w\rangle \le 0, \text{ but } \operatorname{Re}\langle v_0, w\rangle \not\le 0.$$

d) [EW97a, Thm. 4.1] If K is absolutely matrix convex, then tere is $w \in M_n(W)$ such that for all $m \in \mathbb{N}$ and $v \in K_m$

$$\|\langle v, w\rangle\| \le 1, \text{ but } \|\langle v_0, w\rangle\| > 1.$$

One can prove Ruan's theorem using the separation theorem for absolutely matrix convex sets, applied to the unit ball of a matricially normed space [].

If V is a complex involutive vector space, one can find selfadjoint separating functionals:

Theorem: [63] Let $\langle V, W\rangle$ be a duality of complex involutive vector spaces, K a closed set of selfadjoint matrices over V and $v_0 \in M_n(V) \setminus K_n$ for some n.

b) If K is matrix convex and $0 \in K_1$, then there is a $w \in M_n(W)_{\text{sa}}$ such that for all $m \in \mathbb{N}$ and $v \in K_m$

$$\langle v, w\rangle \le \mathbb{1}_{nm}, \text{ but } \operatorname{Re}\langle v_0, w\rangle \not\le \mathbb{1}_{n^2}.$$

10.1.2 Bipolar theorems

Let $\langle V, W\rangle$ be a duality of complex vector spaces and K a set of matrices over V.

The **matrix polar** of K is a set D of matrices over W, given by[64]

$$D_n = \{w \in M_n(W) \mid \operatorname{Re}\langle v, w\rangle \le \mathbb{1}_{nm} \text{ for all } m \in \mathbb{N},\ v \in K_m\}.$$

The **absolute matrix polar** of K is a set D of matrices over W, given by[65]

$$D_n = \{w \in M_n(W) \mid \|\langle v, w\rangle\| \le 1 \text{ for all } m \in \mathbb{N},\ v \in K_m\}.$$

Polars of sets of matrices over W are defined analogously.

We have the **bipolar theorems**: Let $\langle V, W\rangle$ be a duality of complex vector spaces and K a set of matrices over V.

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[63] This theorem can be obtained from the above separation theorem, part a) and b). Note that for selfadjoint v, the mapping $w \mapsto \langle v, w\rangle$ is selfadjoint.

[64] $D = \{w \in M(W) \mid Re\langle v, w\rangle \le \mathbb{1} \text{ for all } v \in K\}$.

[65] $D = \{w \in M(W) \mid \|\langle v, w\rangle\| \le 1 \text{ for all } v \in K\}$.

a) [EW97b] K equals its matrix bipolar if and only if K is closed and matrix convex and $0 \in K_1$.

b) [EW97a] K equals its absolute matrix bipolar if and only if K is closed and absolutely matrix convex.

The matrix bipolar of a set K of matrices over V is therefore the smallest closed and matrix convex set which contains K and 0.

The absolute matrix bipolar of a set K of matrices over V is therefore the smallest closed and absolutely matrix convex set which contains K.

So we get a characterization of the unit balls of $MIN(E)$ and $MAX(E)$ for a normed space E.

10.2 Matrix extreme points

A part of convexity theory studies the possibility of representing all points of a convex set as convex combinations of special points, the so called extreme points. The well-known results about this are the Krein-Milman theorem and its sharpenings, the Choquet representation theorems. The question arises whether there are analogous results for non-commutative convexity. The sections Matrix extreme points and C^*-extreme points give partial answers to this question.

Let V be a vector space. A matrix convex set of matrices over V is called matrix convex set in V for short. Let A be a set of matrices over V. The **matrix convex hull** of A is the smallest matrix convex set in V containing A. Its closure is the smallest closed matrix convex set containing A because the closure of matrix convex sets is matrix convex. Two elements $x, y \in M_n(V)$ are **unitarily equivalent**, if there is a unitary $u \in M_n$ such that $x = u^*yu$. Let $U(S)$ be the set of all elements, that are unitarily equivalent to elements of $S \subset M_n(V)$. $x \in M_n(V)$ is called **reducible**, if it is unitarily equivalent to some block matrix $\left(\begin{smallmatrix} y & 0 \\ 0 & z \end{smallmatrix}\right) \in M_n(V)$. A matrix convex combination $\sum_{i=1}^k \alpha_i^* x_i \alpha_i$ is called **proper**, if all α_i are square matrices different from 0.

Let K be a matrix convex set in V. Then $x \in K_n$ is a **structural element**[66] of K_n, if whenever $x = \sum_{i=1}^k \alpha_i^* x_i \alpha_i$ is a proper matrix convex combination of $x_i \in K_n$, then every x_i is unitarily equivalent to x. The set of all structural elements of K_n is denoted by $\mathrm{str}(K_n)$. The set of structural elements of K is the set of matrices over V consisting of $\mathrm{str}(K_n)$ for all $n \in \mathbb{N}$.

Example: Let L be an operator system. The generalized state space of L is the matrix convex set $\mathrm{CS}(L)$ in the dual L^*, which consists of the matrix states

$$\mathrm{CS}(L)_n = \{\psi \mid \psi : L \to M_n \text{ completely positive and unital}\}.$$

The generalized state space is weak*-compact. It follows from [CE77, Lemma 2.2] that the structural elements of $\mathrm{CS}(L)_n$ are exactly those completely positive and unital

[66] The term *structural element* is defined by Morenz [Mor94]. Another equivalent definition is given in [WW99]. There the structural elements are called *matrix extreme points*.

mappings which are pure. To every compact and matrix convex set K there is an operator system which has K as its generalized state space [WW99, Prop. 3.5].

Let V be a locally convex space and induce the product topology on $M_n(V)$. The **matrix convex Krein-Milman Theorem** is: Let K be a compact matrix convex set in V. Then K is equal to the closed matrix convex hull of the structural elements of K. If V has finite dimension, then K is the matrix convex hull of its structural elements.

The converse result is: Let K be a compact matrix convex set in V. Let S be a closed set of matrices, such that $S_n \subset K_n$ and $v^* S_l v \subset S_m$ for all partial isometries $v \in M_{lm}$ and for all $n, m, l \in \mathbb{N}$, $l \geq m$. If the closed matrix convex hull of S equals K, then all structural elements of K are in S.([WW99], [Fis96]).

It is possible to sharpen these results for more special matrix convex sets. A matrix convex set K is called **simple**, if there are $n \in \mathbb{N}$ and $A \subset M_n(V)$, such that K is equal to the matrix convex hull of A. K is a simple matrix convex set, if and only if there is $n \in \mathbb{N}$ such that $\operatorname{str}(K_m) = \emptyset$ for all $m > n$.

Suppose that K is a matrix convex set in V. Then $x \in K_m$ is a **matrix extreme point**, if $x \in \operatorname{str}(K_m)$ and

$$x \notin \cup_{m<l} \mathbb{1}_{lm}^* \operatorname{str}(K_l) \mathbb{1}_{lm}.$$

Let $\operatorname{mext}(K)$ be the set of matrices consisting of all matrix extreme points of K.

Suppose that K is a simple compact matrix convex set in V. Then K is equal to the closed matrix convex hull of $\operatorname{mext}(K)$. If V has finite dimension, then the closure is not needed, that means K is the matrix convex hull of $\operatorname{mext}(K)$. In this case the following result also holds: Let S be a set of matrices over V not containing reducible elements such that the matrix convex hull of S equals K, then $\operatorname{mext}(K)_m \subset U(S_m)$ for all $m \in \mathbb{N}$ ([Mor94], [Fis96]).

If K is compact and not simple, $\operatorname{mext}(K)$ may be empty. As an example take the generalized state space $\operatorname{CS}(A)$ of a C^*-algebra A. Its matrix extreme points are exactly the irreducible finite dimensional representations of A. These need not exist in general.

10.3 C^*-convexity

Let A be a unital C^*-algebra. A subset $K \subset A$ is a **C^*-convex** set, if

$$\sum_{i=1}^{n} a_i^* x_i a_i \in K$$

for all $x_i \in K$ and $a_i \in A$ such that $\sum_{i=1}^n a_i^* a_i = \mathbb{1}$. The sum $\sum_{i=1}^n a_i^* x_i a_i$ is called a **C^*-convex combination**.

A subset $K \subset A$ is a C^***-absolutely convex** set, if

$$\sum_{i=1}^{n} a_i^* x_i b_i \in K$$

for all $x_i \in K$ and $a_i,\ b_i \in A$ such that $\sum_{i=1}^n a_i^* a_i,\ \sum_{i=1}^n b_i^* b_i \leq \mathbb{1}$.

In particular C^*-convex sets are convex and C^*-absolutely convex sets are absolutely convex.

Example: Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. The n-th matrix range of T is the set

$$W(T)_n := \{\varphi(T) | \varphi : B(\mathcal{H}) \to M_n \text{ completely positive and unital}\}.$$

$W(T)_n$ is a compact and C^*-convex subset of M_n and to every compact and C^*-convex subset $K \subset M_n$ there exist a separable Hilbert space $\mathcal{H}$ and $S \in B(\mathcal{H})$ such that $K = W(S)_n$ ([LP81, Prop. 31]).

The sets

$$\mathrm{Ball}(B(\mathcal{H})) = \{x \in B(\mathcal{H}) | \|x\| \leq 1\} \quad \text{and} \quad P = \{x \in B(\mathcal{H}) | 0 \leq x \leq \mathbb{1}\}$$

are C^*-convex und wot-compact subsets of $B(\mathcal{H})$. $\mathrm{Ball}(B(\mathcal{H}))$ is also C^*-absolutely convex.

Loebl and Paulsen introduced C^*-convex sets in [LP81]. At the beginning of the nineties Farenick and Morenz studied C^*-convex subsets of M_n ([Far92],[FM93]). Eventually Morenz succeeded in proving an analogue of the Krein-Milman theorem for a compact C^*-convex subset of M_n ([Mor94]). At the end of the nineties Magajna generalized the notion of C^*-convex sets to the setting of operator modules and proved some separation theorems [Mag00, Th. 1.1] and also an analogue of the Krein-Milman theorem [Mag98, Th. 1.1].

10.3.1 Separation theorems

Suppose that $A, B \subset B(\mathcal{H})$ are unital C^*-algebras and $Y \subset B(\mathcal{H})$ is a (A, B)-bimodul. Then $K \subset Y$ is (A, B)**-absolutely convex**, if

$$\sum_{i=1}^{n} a_i^* x_i b_i \in K$$

for all $x_i \in K$ and $a_i \in A,\ b_i \in B$ such that $\sum_{i=1}^n a_i^* a_i,\ \sum_{i=1}^n b_i^* b_i \leq \mathbb{1}$. Let Y be a A-bimodul. Then K is A**-convex**, if

$$\sum_{i=1}^{n} a_i^* x_i a_i \in K$$

for all $x_i \in K$ and $a_i \in A$ such that $\sum_{i=1}^n a_i^* a_i = \mathbb{1}$. In the case $Y = A$ this definition is equivalent to the definition of C^*-convex sets.

There are following separation theorems: Let $A, B \subset B(\mathcal{H})$ be unital C^*-algebras and $Y \subset B(\mathcal{H})$ a A, B-bimodul. Let $K \subset Y$ be norm closed and $y_0 \in Y \setminus K$.

1) If $A = B$, $0 \in K$ and K is A-convex, then there is a Hilbert space $\mathcal{H}_\pi$, a cyclic representation $\pi : A \to B(\mathcal{H}_\pi)$ and a completely bounded A-bimodul-homomorphism, such that for all $y \in K$

$$\operatorname{Re}\phi(y) \leq \mathbb{1}, \text{ but } \operatorname{Re}\phi(y_0) \not\leq \mathbb{1}.$$

2)If K is (A, B)-absolutely convex, then there is a Hilbert space $\mathcal{H}_\pi$, representations $\pi : A \to B(\mathcal{H}_\pi)$ and $\sigma : B \to B(\mathcal{H}_\pi)$ and a completely bounded (A, B)-bimodul-homomorphism $\phi : Y \to B(\mathcal{H}_\pi)$, such that for all $y \in K$

$$\|\phi(y)\| \leq 1, \text{ but } \|\phi(y_0)\| > 1.$$

10.4 C^*-extreme points

Let A be an unital C^*-algebra and $Y \subset A$. The C^*-convex hull of Y is the smallest C^*-convex set that contains Y.

Suppose $K \subset A$ is C^*-convex. Then $x \in K$ is a **C^*-extreme point**, if whenever $x = \sum_{i=1}^n a_i^* x_i a_i$ is a C^*-convex combination of $x_i \in K$ with invertible $a_i \in A$, then there are unitaries $u_i \in A$ such that $x = u_i^* x_i u_i$ for $i = 1, \dots, n$.

Suppose now $A = M_n$ and let $K \subset M_n$ be compact and C^*-convex. Let $\tilde{K}$ be the matrix convex hull of K. Then $\tilde{K}$ is a simple compact and matrix convex set in $\mathbb{C}$, such that $\tilde{K}_n = K$ ([Fis96]). Thus it is possible to conceive a C^*-convex subset of M_n as a matrix convex set in $\mathbb{C}$. Now the matrix convex Krein-Milman theorem can be used. Moreover, it follows from the work of Farenick and Morenz that the structural elements of $\tilde{K}_n$ are exactly the not reducible C^*-extreme points of K. So following theorem holds: Let $K \subset M_n$ be compact and C^*-convex, then K is equal to the C^*-convex hull of its C^*-extreme points.

In order to get a somewhat more general result, the definition of the extreme points can be changed. Suppose that R is a hyperfinite factor and that $K \subset R$ is C^*-convex. Then $x \in K$ is a **R-extreme point**, if whenever $x = \sum_{i=1}^n a_i x_i a_i$ is a C^*-convex combination of $x_i \in K$ such that all $a_i \in A$ are positive and invertible, then it follows that $x = x_i$ and $a_i x = x a_i$ forr $i = 1, \dots, n$.[67]

With this definition following theorem hold: Let $K \subset R$ be C^*-convex and weak* compact. Then K is equal to the weak* closure of the C^*-convex hull of its R-extreme points.

[67] If $R = M_n$ the R-extreme points are exactly the C^*-extreme points. In general every R-extreme point is also C^*-extreme, but not vice versa.

11 Mapping Spaces

Let E, F be Banach spaces. We consider a linear subspace $A(E,F)$ of the space $B(E,F)$ of the continuous operators between E and F which contains all finite rank maps and is a Banach space with respect to a given norm. Furthermore, it is usually required that $A(E,F)$ be defined for all pairs of Banach spaces E and F. Such a space is called – according to A. Grothendieck – a **mapping space** .

Analogously, we call an operator space $A(X,Y)$ which is a linear subspace of $CB(X,Y)$ a ***CB*-mapping space**. Note that generally the algebraic identification of $M_n(A(X,Y))$ with $A(X,M_n(Y))$ fails to be isometric and that the norms on $A(X,M_n(Y))$ do not generate an operator space structure for $A(X,Y)$.

There is a close relationship between mapping spaces and tensor products: The space $F(X,Y)$ of all finite rank maps between X and Y and the algebraic tensor product of X^* with Y are isomorphic:

$$X^* \otimes_{\text{alg}} Y \cong F(X,Y).$$

This identification enables us to transfer norms from one space to the other one. To this end, we consider the extension of the mapping $X^* \otimes Y \to F(X,Y)$ to the completion with respect to an operator space tensor norm $X^* \widetilde{\otimes} Y$:

$$\Phi : X^* \widetilde{\otimes} Y \to CB(X,Y).$$

Φ is in general neither injective nor surjective. As a CB-mapping space one obtains

$$\text{Im}(\Phi) \subset CB(X,Y)$$

with the operator space norm of

$$(X^* \tilde{\otimes} Y)/\text{Ker}(\Phi).$$

We consider now assignments that assign a mapping space $A(\cdot,\cdot)$ with operator space norm $\alpha(\cdot)$ to every pair of operator spaces. In the Banach space theory A. Pietsch intensified the notion of mapping spaces to that of the operator ideals [Pie78]. Analogously, we consider operator ideals which are mapping spaces with the ***CB*-ideal property** [ER94], i.e , the composition

$$\begin{array}{rcl} CB(X_1,X_2) \times A(X_2,Y_2) \times CB(Y_2,Y_1) & \to & A(X_1,Y_1) \\ (\Psi_1,\Phi,\Psi_2) & \mapsto & \Psi_2 \circ \Phi \circ \Psi_1 \end{array}$$

is for all operator spaces X_1, X_2, Y_1, Y_2 defined and jointly completely contractive.

A *CB*-ideal is called local [EJR98], if its norm satisfies:

$$\alpha(\varphi) = \sup\{\alpha(\varphi|_L) : L \subset X,\ \dim L < \infty\}.$$

11.1 Completely nuclear mappings

Let X and Y be operator spaces. The **completely nuclear** mappings from X to Y [ER94, §2], [EJR98, §3] are defined by the projective operator space tensor norm. We consider the extension of the canonical identity $X^* \otimes Y = F(X, Y)$:

$$\Phi : X^* \overset{\wedge}{\otimes} Y \to X^* \overset{\vee}{\otimes} Y \subset CB(X, Y).$$

$\overset{\vee}{\otimes}$ is the injective tensor product and $\overset{\wedge}{\otimes}$ the projective tensor product.

A mapping in the range of Φ is called **completely nuclear**. One denotes by

$$CN(X, Y) := (X^* \overset{\wedge}{\otimes} Y)/\mathrm{Ker}(\Phi)$$

the space of the completely nuclear mappings and endows it with the quotient operator space structure. The operator space norm is denoted by $\nu(\cdot)$. $M_n(CN(X, Y))$ and $CN(X, M_n(Y))$ are in general not isometric.

Nuclear [68] mappings are completely nuclear. [ER94, 3.10]

In general, the projective tensor norm does not respect complete isometries. Hence,even for subspaces $Y_0 \subset Y$ the canonical embedding $CN(X, Y_0) \to CN(X, Y)$ is generally only completely contractive and not isometric. Since the projective tensor norm respects quotient mappings, every nuclear map $\varphi : X_0 \to Y$ on a subspace $X_0 \subset X$ with $\nu(\varphi) < 1$ has an extension $\tilde{\varphi}$ to the whole of X satisfying $\nu(\tilde{\varphi}) < 1$.

The completely nuclear mappings enjoy the *CB*-ideal property. Furthermore, the adjoint φ^* is completely nuclear if φ is, and the inequality: $\nu(\varphi^*) \leq \nu(\varphi)$ [EJR98, Lemma 3.2] holds.

A mapping φ is completely nuclear, if and only if there is a factorization of the form

$$\begin{array}{ccc} B(\ell_2) & \overset{M(a,b)}{\to} & T(\ell_2) \\ \uparrow r & & \downarrow s \\ X & \overset{\varphi}{\to} & Y \end{array}$$

Here a, b are Hilbert-Schmidt operators defining the mapping $M(a, b) : x \mapsto axb$. For the completely nuclear norm we have: $\nu(\varphi) = 1$ precisely if for all $\epsilon > 0$ there exists a factorization with $\|r\|_{cb}\|a\|_2\|b\|_2\|s\|_{cb} \leq 1 + \epsilon$ [69] [ER94, Thm. 2.1].

[68] The completely nuclear mappings owe their definition to the one of the nuclear mappings of the Banach space theory. There, one considers a corresponding mapping $\Phi_B : E^* \otimes_\gamma F \to B(E, F)$ for two Banach spaces E and F.

[69] In the Banach space theory one has an analogous statement: A mapping φ is nuclear, if and only if there's a diagram

$$\begin{array}{ccc} \ell_\infty & \overset{d}{\to} & \ell_1 \\ r \uparrow & & \downarrow s \\ E & \overset{\varphi}{\to} & F, \end{array}$$

where d is a diagonal operator, i.e. , there is a $(d_i) \in \ell_1$, such that $d((a_i)) = (d_i \cdot a_i)$ for all $(a_i) \in \ell_\infty$. The nuclear norm is then computed as: $\nu_B(\varphi) = \inf \|r\|\|d\|_{\ell_1}\|s\|$, where the infimum runs over all possible factorizations.

The completely nuclear mappings are not local.

11.2 Completely integral mappings

Completely integral mappings are defined by the aid of completely nuclear mappings. A mapping $\varphi : X \to Y$ is said to be **completely integral**, if there are a constant $c > 0$ and a net of finite rank maps $\varphi_\alpha \in CN(X,Y)$ with $\nu(\varphi_\alpha) \le c$ converging to φ in the point norm topology [70].

The set of all these mappings forms the space $CI(X,Y)$ of the completely integral mappings.

The infimum of all those constants c satisfying the above condition is actually attained and denoted by $\iota(\varphi)$. $\iota(\cdot)$ is a norm turning $CI(X,Y)$ into a Banach space. The unit sphere of $CI(X,Y)$ is merely the point norm closure of the unit sphere of $CN(X,Y)$.

One obtains the canonical operator space structure by defining the unit sphere of $M_n(CI(X,Y))$ as the point norm closure of unit sphere of $M_n(CN(X,Y))$.

By definition we have $\iota(\varphi) \le \nu(\varphi)$; for finite dimensional X we have moreover [EJR98, Lemma 4.1]

$$CI(X,Y) \overset{\text{cb}}{=} CN(X,Y).$$

Integral [71] mappings are completely integral [ER94, 3.10].

The canonical embedding

$$CI(X,Y) \hookrightarrow (X \overset{\vee}{\otimes} Y^*)^*$$

is a complete isometry [EJR98, Cor. 4.3]. One has moreover that [EJR98, Cor. 4.6] φ is completely integral, if and only if there is a factorization of the form:

$$\begin{array}{ccccc} B(\mathcal{H}) & & \overset{M(\omega)}{\to} & & B(\mathcal{K})^* \\ \uparrow r & & & & \downarrow s \\ X & & \overset{\varphi}{\to} Y \hookrightarrow & & Y^{**} \end{array}$$

with weak*-continuous s . The mapping $M(\omega) : B(\mathcal{H}) \to B(\mathcal{K})^*$ is for two elements $a \in B(\mathcal{H})$, $b \in B(\mathcal{K})$ given by $(M(\omega)(a))(b) = \omega(a \otimes b)$. We have for the norm $\iota(\varphi) = 1$, if there is a factorization with $\|r\|_{cb}\|\omega\|\|s\|_{cb} = 1$ (note that generally $\|M(\omega)\|_{cb} \ne \|\omega\|$).

The completely integral mappings enjoy also the *CB*-ideal property. Contrasting the situation of completely nuclear mappings they are local. In general one only has $\iota(\varphi) \le \iota(\varphi^*)$ [EJR98].

[70] $\varphi_\alpha \to \varphi$ in the point norm topology, if $\|\varphi_\alpha(x) - \varphi(x)\| \to 0$ for all $x \in X$.

[71] The unit ball of the integral mappings of the Banach space theory is just the point norm closure of the unit ball of the nuclear mappings. One should note that the formulas $\iota_B(\varphi) = \iota_B(\varphi^*)$ and $I_B(E,F^*) = (E \otimes_\lambda F)^*$ have no counterparts for completely integral mappings.

12 Appendix

12.1 Tensor products

12.1.1 Tensor products of operator matrices

As usual we define the algebraic tensor product of operator matrices $x = [x_{ij}] \in M_p(X)$, $y = [y_{kl}] \in M_q(Y)$ by setting

$$x \otimes y := [x_{ij} \otimes [y_{kl}]_{k,l}]_{i,j} \in M_p(X \otimes M_q(Y)).$$

Here we have used the definition $M_p(X) = M_p \otimes X$ and the associative law

$$M_p(X) \otimes M_q(Y) = M_p \otimes (X \otimes M_q(Y)) = M_p(X \otimes M_q(Y)). \tag{5}$$

In view of the next identification one should note that the *shuffle*-map is an algebraic isomorphism:

$$X \otimes (M_q \otimes Y) \to M_q \otimes (X \otimes Y), \tag{6}$$

$$x \otimes (\beta \otimes y) \mapsto \beta \otimes (x \otimes y),$$

for $\beta \in M_q$, $x \in X$, $y \in Y$. The *shuffle*-isomorphism at hand we obtain the identification:

$$x \otimes y = [x_{ij} \otimes [y_{kl}]_{k,l}]_{i,j} = [[x_{ij} \otimes y_{kl}]_{k,l}]_{i,j} \in M_p(M_q(X \otimes Y)).$$

Finally we use the usual[72] identification $M_p(M_q) = M_{pq}$

$$[[x_{ij} \otimes y_{kl}]_{k,l}]_{i,j} = [x_{ij} \otimes y_{kl}]_{(i,k),(j,l)}$$

to obtain

$$M_p(X) \otimes M_q(Y) = M_{pq}(X \otimes Y). \tag{7}$$

We call this algebraic isomorphism the ***shuffle*-isomorphism**.

One should note that for operator space tensor products the algebraic identifications (5) and (6) are only complete contractions:

$$\mathbb{M}_p(X) \otimes_\alpha Y \quad \to \quad \mathbb{M}_p(X \otimes_\alpha Y), \tag{8}$$

$$X \otimes_\alpha \mathbb{M}_q(Y) \quad \to \quad \mathbb{M}_q(X \otimes_\alpha Y). \tag{9}$$

In general these are not isometries even for $p = 1$ resp. $q = 1$.

For an operator space tensor product the ***shuffle*-map**

$$\mathbb{M}_p(X) \otimes_\alpha \mathbb{M}_q(Y) \to \mathbb{M}_{pq}(X \otimes_\alpha Y) \tag{10}$$

[72] In the matrix so obtained (i,k) are the row indices and (j,l) are the column indices, where $i,j = 1,\dots,p$ and $k,l = 1,\dots,q$. The indices (i,k) resp. (j,l) are ordered lexicographically.

in general is only completely contractive.

In the case of the injective operator space tensor product this is of course a complete isometry.

More generally, one considers the *shuffle*–map for rectangular matrices[73]:

$$\mathbb{M}_{m,n}(X) \otimes_\alpha \mathbb{M}_{p,q}(Y) \to \mathbb{M}_{mp,nq}(X \otimes_\alpha Y).$$

Another example is provided by the Blecher-Paulsen equation.

12.1.2 Joint amplification of a duality

The **matrix duality**which is fundamental in the duality theory of operator spaces, is a special case of the joint amplification of a bilinear mapping.

The joint amplification of a duality $\langle X, X^* \rangle$ of vector spaces is defined by

$$\langle x, \varphi \rangle^{p \times q} = \langle [x_{ij}], [\varphi_{\kappa\lambda}] \rangle^{p \times q} := [\langle x_{ij}, \varphi_{\kappa\lambda} \rangle] \in M_p(M_q) = M_{pq}$$

for $x = [x_{ij}] \in M_p(X)$, $\varphi = [\varphi_{\kappa\lambda}] \in M_q(X^*)$.

Interpreting φ as a mapping $\varphi : X \to M_q$ we have

$$\varphi^{(p)}(x) = \langle x, \varphi \rangle^{p \times q}.$$

Associated to the **duality of tensor products**[74] $\langle X \otimes Y, X^* \otimes Y^* \rangle$ is the joint amplification

$$\langle M_p(X \otimes Y), M_q(X^* \otimes Y^*) \rangle.$$

Especially, the equation

$$\begin{aligned} \langle x \otimes y, \varphi \otimes \psi \rangle &= \langle [x_{ij}] \otimes [y_{kl}], [\varphi_{\kappa\lambda}] \otimes [\psi_{\mu\nu}] \rangle \\ &:= [\langle x_{ij}, \varphi_{\kappa\lambda} \rangle \langle y_{kl}, \psi_{\mu\nu} \rangle]_{(ik\kappa\mu),(jl\lambda\nu)} \in M_{p_1}(M_{p_2}(M_{q_1}(M_{q_2}))) = M_{p_1 p_2 q_1 q_2} \end{aligned}$$

obtains, where $x = [x_{ij}] \in M_{p_1}(X)$, $y = [y_{kl}] \in M_{p_2}(Y)$, $\varphi = [\varphi_{\kappa\lambda}] \in M_{q_1}(X^*)$, $\psi = [\psi_{\mu\nu}] \in M_{q_2}(Y^*)$.

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[73] The *shuffle*-map

$$(U \otimes X) \otimes (V \otimes Y) \to (U \otimes V) \otimes (X \otimes Y),$$

$$(u \otimes x) \otimes (v \otimes y) \mapsto (u \otimes v) \otimes (x \otimes y),$$

U,V,X,Y operator spaces, has been studied for various combinations of operator space tensor products [EKR93, Chap. 4].

[74] The duality $\langle X \otimes Y, X^* \otimes Y^* \rangle$ is defined by $\langle x \otimes y, \varphi \otimes \psi \rangle := \langle x, \varphi \rangle \langle y, \psi \rangle$ for $x \in X$, $y \in Y$, $\varphi \in X^*$, $\psi \in Y^*$.

12.1.3 Tensor matrix multiplication

The definition of the completely bounded bilinear maps as well as the Haagerup tensor product relies on the **tensor matrix multiplication** [Eff87]

$$x \odot y = [x_{ij}] \odot [y_{jk}] := \left[\sum_{j=1}^{l} x_{ij} \otimes y_{jk}\right] \in M_n(X \otimes Y)$$

of operator matrices $x = [x_{ij}] \in M_{n,l}(X)$, $y = [y_{jk}] \in M_{l,n}(Y)$.

The amplification of the bilinear mapping $\otimes : X \times Y \to X \otimes Y$ is given by

$$\otimes^{(n,l)} = \odot : M_{n,l}(X) \times M_{l,n}(Y) \to M_n(X \otimes Y).$$

For scalar matrices $\alpha, \gamma \in M_n$, $\beta \in M_l$ we have

$$(\alpha x \beta) \odot (y\gamma) = \alpha(x \odot (\beta y))\gamma.$$

We use the short hand notation $\alpha x \beta \odot y \gamma$.

For linear maps

$$\begin{aligned} \Phi = [\Phi_{ij}] : x &\to M_{n,l}(V), \\ \Psi = [\Phi_{jk}] : x &\to M_{l,n}(W) \end{aligned}$$

we denote by $\Phi \odot \Psi$ the mapping

$$\Phi \odot \Psi = \left[\sum_{j=1}^{l} \Phi_{ij} \otimes \Psi_{jk}\right] : X \otimes Y \to M_n(V \otimes W),$$

$$\Phi \odot \Psi : x \otimes y \mapsto \left[\sum_{j=1}^{l} \Phi_{ij}(x) \otimes \Psi_{jk}(y)\right].$$

We then have

$$(\Phi \odot \Psi)^{(p)}(x \odot y) = (\Phi^{(p,q)}(x)) \odot (\Psi^{(q,p)}(y))$$

for $x \in M_{p,q}(X)$, $y \in M_{q,p}(Y)$.

Let $\otimes_\alpha$ be an operator space tensor product. We define the tensor matrix multiplication $\odot_\alpha$ of operator matrices $S = [S_{i,j}] \in M_{n,l}(CB(X_1, X_2))$, $T = [T_{k,l}] \in M_{l,n}(CB(Y_1, Y_2))$ of completely bounded maps by setting

$$S \odot_\alpha T = \left[\sum_{j=1}^{l} S_{ij} \otimes_\alpha T_{jk}\right] \in M_n(CB(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2)).$$

12.2 Interpolation

Intersection and sum

Let X and Y be operator spaces such that $M_1(X)$ and $M_1(Y)$ are embedded in a Hausdorff topological vector space. $M_n(X \cap Y)$ is given a norm via $M_n(X \cap Y) := M_n(X) \cap M_n(Y)$. So we have:

$$\|[x_{ij}]\|_{M_n(X\cap Y)} = \max\left\{\|[x_{ij}]\|_{M_n(X)}, \|[x_{ij}]\|_{M_n(Y)}\right\}.$$

The operator space $X \cap Y$ is called the **intersection** of X and Y.
For operator spaces[75] X and Y, by embedding $X \oplus Y$ in $(X^* \oplus_\infty Y^*)^*$ we obtain an operator space structure $X \oplus_1 Y$. We write $\diamondsuit := \{(x, -x)\} \subset X \oplus_1 Y$. The quotient operator space $(X \oplus_1 Y)/\diamondsuit$ is called the **sum** of X and Y and is denoted by $X + Y$. We have

$$\|[x_{ij}]\|_{M_n(X+Y)} = \inf_{[x_{ij}]=[x_{ij}]_X+[x_{ij}]_Y} \|[(x_{ij_X}, x_{ij_Y})]\|_{M_n(X\oplus_1 Y)}.$$

Interpolation

Let E_0, E_1 be Banach spaces continuously embedded in a Hausdorff topological vector space. The pair (E_0, E_1) is called a compatiple couple in the sense of interpolation theory [BL76]. Then we can define the interpolation space $E_\theta := (E_0, E_1)_\theta$ for $0 < \theta < 1$.

Pisier introduced the analogous construction for operator spaces [Pis96, §2]: Let X_i $(i = 0, 1)$ be operator spaces continuously embedded in a Hausdorff topological vector space V. Then we have specific norms on $M_n(X_i)$ and continuous linear inclusions $M_n(X_i) \hookrightarrow M_n(V)$ for all $n \in \mathbb{N}$.[76] The **interpolated operator space** X_θ is defined via $M_n(X_\theta) := (M_n(X_0), M_n(X_1))_\theta$.
Let X be an operator space, $\mathcal{H}$ a Hilbert space and $V : \mathcal{H} \to X$ a bounded linear and injective mapping with dense range such that the mapping[77] $VV^* : \overline{X^*} \to X$ also is bounded, linear and injective with dense range. Then we have completely isometrically [Pis96, Cor. 2.4]:

$$(\overline{X^*}, X)_{\frac{1}{2}} \overset{\text{cb}}{=} OH_{\mathcal{H}}.$$

Examples

1. $(\mathcal{R}_{\mathcal{H}}, \mathcal{C}_{\mathcal{H}})_{\frac{1}{2}} \overset{\text{cb}}{=} OH_{\mathcal{H}} \overset{\text{cb}}{=} (MIN_{\mathcal{H}}, MAX_{\mathcal{H}})_{\frac{1}{2}}$

[75] Let E, F be Banach spaces. Then we have their 1-direct sum $E \oplus_1 F$ with the norm

$$\|(x_E, x_F)\| = \|x_E\| + \|x_F\|$$

and their sum $E + F$ with the quotient norm

$$\|x\|_{E+F} = \inf_{x=x_E+x_F} \left(\|x_E\|_E + \|x_F\|_F\right).$$

[76] We identify $M_n(V)$ with V^{n^2}.
[77] As usual we identify $\mathcal{H}$ with its dual.

2. $(\mathcal{C}_\mathcal{H} \otimes_h \mathcal{R}_\mathcal{H}, \mathcal{R}_\mathcal{H} \otimes_h \mathcal{C}_\mathcal{H})_{\frac{1}{2}} \overset{\text{cb}}{=} OH_\mathcal{H} \otimes_h OH_\mathcal{H} \overset{\text{cb}}{=} OH_{\mathcal{H}\otimes\mathcal{H}}$

In this manner one also obtains operator space structures on the Schatten ideals $S_p = (S_\infty, S_1)_{\frac{1}{p}}$ for $1 \leq p \leq \infty$.

13 Symbols

Sets

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers

Banach spaces

E, F	Banach spaces
$\mathcal{H}$, $\mathcal{K}$	Hilbert spaces
$\bigoplus_2$	Hilbert space direct sum
$\mathcal{H}^n$	$\bigoplus_{i=1}^n \mathcal{H}$
ℓ_2^n	the n-dimensional Hilbert space

Algebras

$B(\mathcal{H})$	the algebra of bounded linear operators on $\mathcal{H}$
A, B	C^*-algebras
A^{op}	the *opposite algebra* of an algebra A
$\mathbb{1}_A$	the identity in A
M, N	von Neumann algebras

Ideals

S_p	the Schatten-p-classes
$K(\cdot,\cdot)$	the ideal of compact operators
$\mathbf{K}(\cdot,\cdot)$	the closed linear span of the elementary operators; ideal in $B^{\text{ad}}(\cdot,\cdot)$
$N(\cdot,\cdot)$	the ideal of nuclear operators
$HS(\cdot,\cdot)$	the ideal of Hilbert-Schmidt operators (S_2)
$\Pi_2(\cdot,\cdot)$	the ideal of absolutely 2-summing operators
$\pi_2(\cdot)$	the absolutely 2-summing norm

Mapping spaces

$\Gamma_2(\cdot,\cdot)$	operators factorizing through the column Hilbert space
$\gamma_2(\cdot)$	the norm on Γ_2
$CN(\cdot,\cdot)$	completely nuclear operators
$\nu(\cdot)$	the norm on CN
$CI(\cdot,\cdot)$	completely integral operators
$\iota(\cdot)$	the norm on CI

Operator spaces

X, Y	operator spaces
$B(\mathcal{H})$	aqlgebra of bounded linear operators on $\mathcal{H}$
$M_n(X)$	$M_n \otimes X$ matrices with entries from X (algebraically)
$M_1(X)$	first level of the operator space X
$CB(X,Y)$	the operator space of completely bounded mappings
$CB(X,Y)_A$	the operator space of completely bounded right A-module homomorphisms
$CB(X \times Y; Z)$	the operator space of completely bounded bilinear mappings
$JCB(X \times Y; Z)$	the operator space of jointly completely bounded bilinear mappings
$\\|\cdot\\|_{\mathrm{jcb}}$	norm of a jointly completely bounded bilinear mapping
X_0, Y_0	operator subspace of the corresponding operator spaces
X^*	dual of the operator space X

Special operator spaces

$MIN(E)$	the minimal operator space on E
$MAX(E)$	the maximal operator space on E
$MIN_{\mathcal{H}}$	the minimal operator space on $\mathcal{H}$
$MAX_{\mathcal{H}}$	the maximal operator space on $\mathcal{H}$
$\mathcal{R}_{\mathcal{H}}$	the row Hilbert space
$\mathcal{R}_n$	$\mathcal{R}_{\ell_2^n}$
$\mathcal{C}_{\mathcal{H}}$	the column Hilbert space
$\mathcal{C}_n$	$\mathcal{C}_{\ell_2^n}$
$OH_{\mathcal{H}}$	the operator Hilbert space

Norms

$\\|\cdot\\|_{\mathrm{cb}}$	completely bounded norm
$\\|\cdot\\|_{\mathrm{row}}$	row norm
$\\|\cdot\\|_{\mathrm{col}}$	column norm
$\\|\cdot\\|_n$	norm on $M_n(X)$
$\\|\cdot\\|_{m,n}$	norm on $M_{m,n}(X)$

Matrices

$M_{n,m}(X)$	$n \times m$-matrices over X
$M_n(X)$	$M_{n,n}(X)$
$M_{n,m}$	$M_{n,m}(\mathbb{C})$
M_n	$M_{n,n}$
$\mathbb{M}_{n,m}(X)$	the operator space of $n \times m$-matrices over X
$\mathbb{M}_n(X)$	$\mathbb{M}_{n,n}(X)$
$\mathbb{M}_{n,m}$	$\mathbb{M}_{n,m}(\mathbb{C})$
$\mathbb{M}_n$	$\mathbb{M}_{n,n}$
$C_n(X)$	$\mathbb{M}_{n,1}(X)$, the columns of an operator space
$R_n(X)$	$\mathbb{M}_{1,n}(X)$, the rows of an operator space

Tensor products

$\odot$	tensor matrix product
$\otimes$	algebraic tensor product
$\otimes_A$	algebraic module tensor product
$\widetilde{\otimes}$	completion of the algebraic tensor product
$\otimes_\alpha$	operator space tensor product
$\otimes_{\alpha^*}$	dual operator space tensor product
$\otimes_h$	the Haagerup tensor product
$\otimes_{hA}$	the module Haagerup tensor product
$\overset{\vee}{\otimes}$	the injective tensor product
$\overset{\wedge}{\otimes}$	the projective tensor product
$\otimes_\lambda$	the injective Banach space tensor product
$\otimes_\gamma$	the projective Banach space tensor product
$\otimes_{\text{ext}}$	the external tensor product
$\otimes_\Theta$	the internal tensor product
$S \otimes_\alpha T$	α-tensor product of the completely bounded operators S, T

Tensor norms

$\|\cdot\|_\alpha$	α-operator space tensor norm
$\|\cdot\|_{\alpha,n}$	α-operator space tensor norm on the nth level
$\|\cdot\|_{\alpha^*}$	dual operator space tensor norm of $\|\cdot\|_\alpha$
$\|\cdot\|_\vee$	injective operator space tensor norm
$\|\cdot\|_\wedge$	projective operator space tensor norm
$\|\cdot\|_h$	Haagerup operator space tensor norm
$\|\cdot\|_\lambda$	injective Banach space tensor norm
$\|\cdot\|_\gamma$	projective Banach space tensor norm

Operator modules

A_1, A_2	C^*-algebras
M, N	von Neumann algebras
$\mathrm{Aut}(M)$	the set of *-automorphisms of M
$CB_{(A_1,A_2)}(X,Y)$	the operator space of completely bounded left A_1- right A_2-module homomorphisms between X and Y
$CB^\sigma_{(A_1,A_2)}(X,Y)$	the operator space of normal completely bounded left A_1- right A_2-module homomorphisms between X and Y
$CB^s_{(A_1,A_2)}(X,Y)$	the operator space of singular completely bounded left A_1- right A_2-module homomorphisms between X and Y

Hilbert-C^*-modules

$\overline{X}$	conjugate Hilbert-C^*-module
$B(\cdot,\cdot)_A$	the space of bounded right A-module homomorphisms
$\langle\cdot,\cdot\rangle_A$	inner product of a right Hilbert-A-module
$_A\langle\cdot,\cdot\rangle$	inner product of a left Hilbert-A-module
$\langle\cdot,\cdot\rangle$	inner product of a Hilbert space
$B^{\mathrm{ad}}(X,Y)$	the operator space of adjointable A-module homomorphisms between Hilbert-C^*-modules X and Y

Mappings

$\mathbb{1}_{B(\mathcal{H})}$	the identity of $B(\mathcal{H})$
$\mathbb{1}_n$	the identity of M_n
π	representations
$\Phi^{(n)}$	nth amplification of a linear mapping Φ
$\Phi^{(n,l)}$	(n,l)th amplification of a bilinear mapping Φ
$\Phi^{(n)}$	$\Phi^{(n,n)}$, nth amplification of a bilinear mapping Φ
$\Phi^{(n)}$	nth amplification of a multilinear mapping Φ
$\Phi^{(p\times q)}$	joint amplification of a bilinear mapping Φ
$\tilde{\Phi}$	linearization of a bilinear mapping Φ
Φ^σ	the normal (=w^*-w^*-continuous) part of a mapping Φ between dual spaces
Φ^s	the singular part of a mapping Φ between dual spaces
Θ	faithful non-degenerate $*$-representation of a C^*-algebra

Isomorphisms

$\overset{\mathrm{cb}}{=}$	completely isometrically isomorphic
$\overset{\mathrm{cb}}{\simeq}$	completely isomorphic
$\simeq$	isomorphic

Miscellanea

Ball	unit ball
$\bigoplus_{i=1}^n X_i$	nth direct sum
$\bigoplus_{n\in\mathbb{N}} X_n$	countable direct sum
T^*	the adjoint of the operator T

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