THIRD-GRADERS' BLOCK-BUILDING: HOW DO THEY EXPRESS THEIR KNOWLEDGE OF CUBOIDS AND CUBES?

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The study we report on here intends to detect third-graders' conceptual knowledge on cuboids and cubes, respectively. Avoiding methods which are restricted to commenting verbally or drawing to investigate young children's knowledge on geometrical solids, we used wooden blocks in construction tasks: German and Malaysian children aged 8 to 9 were asked to take wooden cubes, cuboids, prisms or blocks from Froebel's Gifts and to construct cuboids (cubes) by assembling the blocks according to their knowledge and visualization. First observations are interpreted according to the Van Hiele framework. In addition, we have a closer look on the variety of constructions some children produced and raise concluding hypotheses concerning the development of children's conceptual knowledge on geometrical solids.

INTRODUCTION

Geometry education in primary school plays a fundamental role for the development of basic knowledge on geometrical shapes and solids. Thus, classroom activities often focus on naming and sorting shapes. Besides, the primary curriculum has also been extended to activities with hands-on-materials and tasks which have to be solved mentally (Franke & Reinhold, 2016). This includes "working on the composing/ decomposing, classifying, comparing and mentally manipulating both two- and three-dimensional figures" (Sinclair & Bruce, 2015, p. 319). Obviously, both sides of the coin – namely visualizing and mentally manipulating and multi-sensory or haptic experiences – facilitate young children's ability of recognizing shapes and foster their acquisition of geometrical knowledge (e. g. Kalenine et al., 2011). As younger children often face difficulties in articulating this knowledge, we consider block building activities to be a meaningful way for them to express their geometrical concepts on solids. Yet, we do not investigate how constructions with tangible blocks foster the development of conceptual knowledge on geometrical solids, in this study.

THEORETICAL FRAMEWORK

Conceptualizing Conceptual Knowledge on Geometrical Solids

The customary conception of a *concept* comprises the "(…) ideal representation of a class of objects, based on their common features" (Fischbein, 1993, p. 139). In this sense, geometrical concepts refer to common features of a class of geometrical shapes or solids which can be visualized or perceived (visually and haptic) when encountering concrete representatives. For example, specific figural properties like the shape of a solid's surfaces or the angles which determine the way the surfaces are interrelated may indicate that a representative is part of a certain class of solids. Based on this notion, students' conceptual knowledge on geometrical solids reaches beyond the capability of correctly naming concrete representatives or giving a verbal definition, later on at secondary level. It rather comprehends the perception, visualization and identification of distinctive properties which refers to individual mental images students have while thinking of a specific solid (cf. Tall & Vinner, 1981). In addition, Vollrath (1984, p. 9-10) suggests that geometrical concept knowledge can be operationalized by illustrating examples of a certain category of shapes or solids, by assigning the term to a superordinate term, or by solving problems which correspond to the used term and its associated properties.

Development of Conceptual Knowledge on Geometrical Solids

The development of geometrical concept knowledge from primary to secondary has been described by the well-known Van Hiele Model which defines five levels of development which are based on previous level(s) and include specific characteristics: School starters and younger children most often classify shapes according to their holistic appearance which is limited to recognition of resemblance. At this level of *VISUALIZATION* "There is no why, one just sees it." (Van Hiele, 1986, p. 83) Thus, identification of prototypes at this level is fairly easy and enables children to identify other shapes or to visually distinguish different types of four-side figures (e. g. rectangles, parallelograms). Yet, shape recognition is limited to recognition of resemblance and does not pay attention to reasoning on properties or (sub-ordinate) relations between different shapes. In addition, Clements et al. (1999) and others discuss a pre-recognitive level which characterizes young children's abilities before reaching the level of *VISUALIZATION*. Based on this and at the ensuing level of *ANALYSIS*, children are capable of taking a shape's properties into account when they decide upon categorization. Activities of (de)composing, discussing and reflecting upon those activities facilitate children's noticing of properties, but still, they do not realize relationships between properties and are unable to give a concise definition (with necessary and sufficient conditions). Thus, they are usually not able to tell that a cube is a very special cuboid. Only when children are able to cope with questions concerning relationships of shapes and when they start arguing about the impact of various properties on relations among shapes in their definitions, children have reached the level of *ABSTRACTION* (Van Hiele, 1999, p. 311).

Expressing Geometrical Knowledge in Drawings and Constructions

In numerous previous studies, scholars have analyzed children's drawing processes and products to get access to children's understanding and their developmental stages of conceptual knowledge on geometrical shapes. For example, knowledge on the variety of triangles and quadrilaterals in terms of identifying, sorting and comparing representatives was detected by Burger & Shaughnessy (1986). Maier & Benz (2014) stated an immense variety in understanding the concept of triangles according to their analysis of German and English primary children's drawings, too. A significant relationship between children´s drawings and their geometric understanding was also stated by Thom & McGarvey (2015), and Hasegawa (1997) tried to identify stages on the development of an *n*-gon-concept by using drawing activities and rotations. These and other studies regard student´s drawings as a representation of student´s geometric concepts (cf. Hasegawa, 1997, p. 177). In line with this research, children's drawing processes and products are widely accepted as individual expressions of spatial abilities (Milbrath & Trautner, 2008) or spatial structuring of two-dimensional shapes (Mulligan et al., 2004; Mulligan & Mitchelmore, 2009). Based on the work of Lewis (1963) who was among the first to investigate how children draw a cube, Mitchelmore (1978) examined how children aged 7 to 15 draw cubes, cuboids, cylinders and four-sided pyramids. Yet, these and following studies have to cope with children's limited drawing skills concerning three-dimensional shapes in primary age. Hence, we derive only very specific information on children's geometrical knowledge on solids when we ask them to draw a solid.

A promising alternative can be found in concrete constructions with blocks: When playing with blocks, even young children deal with geometrical congruence or they distinguish solids according to their properties which is an important aspect of geometrical concept knowledge (see above). Besides, they reflect on spatial relations, orientations or the structure of a three-dimensional array. In Reinhold et al. (2013), we reported on (young) children's difficulties in the (re)construction of cube arrays for purposes of enumeration, but we also found evidence in many ensuing studies[1] that children's fine motor function and their general haptic competence to assemble single blocks or components to three-dimensional arrays is usually entirely developed at the age of 9.

RESEARCH QUESTIONS, DATA COLLECTION AND ANALYSIS

Based on this theoretical framework, we assume that analyses of differences in individual construction processes and products (which may, additionally, be commented verbally) provide deeper insight into children's visualization of solids. This is expected to contribute to a deeper understanding of children's concept knowledge on geometrical solids, while we were interested in exploring to what extent third-graders can articulate their conceptual knowledge on geometrical solids via constructing activities with wooden blocks:

- What kind (and sizes) of cubes and cuboids do third-graders construct and which variations occur?
- Are these constructions in line with their verbal explanations?
- How can we interrelate these results with Van Hiele framework and is there a necessity and supportive data to enrich the framework?

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[1] Data was gained in various unpublished Master Theses research studies which reported on part-studies of the project (Y)CUBES at the Universities of Braunschweig and Leipzig, Germany (cf. Reinhold et al, 2013).

Data collection focused on one-on-one-interviews with ten children aged eight to nine in a primary school in one of the larger East-German cities and with twelve nine year-olds in a primary school in a Northern Malaysian city in 2015 (“Grundschule” in Germany and “Malay-medium National School” in Malaysia)[2]. In the beginning, children were asked to explain their ideas and knowledge concerning cubes and cuboids in a short dialogue with the interviewer. Afterwards, a variety of tasks (e. g. “Please, build a cuboid using these blocks.”) invited them to express their knowledge on cubes and cuboids via construction activities with wooden cubes, cuboids, prisms and a collection of different blocks (Froebel´s Gift 6). During their constructions, they were encouraged to describe their proceeding. A manual for all interviews referred to previous research related to the development of geometrical thought (e. g. Crowley, 1987). All interviews were transcribed verbatim and coded with software support by Atlas.ti. A coding guideline was developed mainly according to Grounded Theory Methods (Corbin & Strauss, 2015), trying to detect new facets of articulating conceptual knowledge on geometrical solids and to generate new hypotheses concerning the development of third-graders’ geometrical concepts.

EXCERPTS FROM THE RESULTS

Qualitative analyses of the data reveal a wide variety among either the German or the Malaysian children’s construction activities, and thereby indicate a wide variety in third-graders geometrical concept knowledge on the selected solids.

The range of *PRODUCTS FOR CUBOIDS* (using cubic blocks) included regular cubes (e. g. 2 x 2 x 2 or 3 x 3 x 3), convex constructions with various identical layers (e. g. 3 x 4 x 2), and flat constructions made of only one layer of attached cubes (put as a “lying layer” or as “walls”, e. g. made of 2 x 5 x 1 or 3 x 1 x 1 cubes). Additionally, we observed children who (correctly) identified rows of entirely connected cubes (e. g. 3 x 1 x 1) as cuboids (see first row in figure 1).

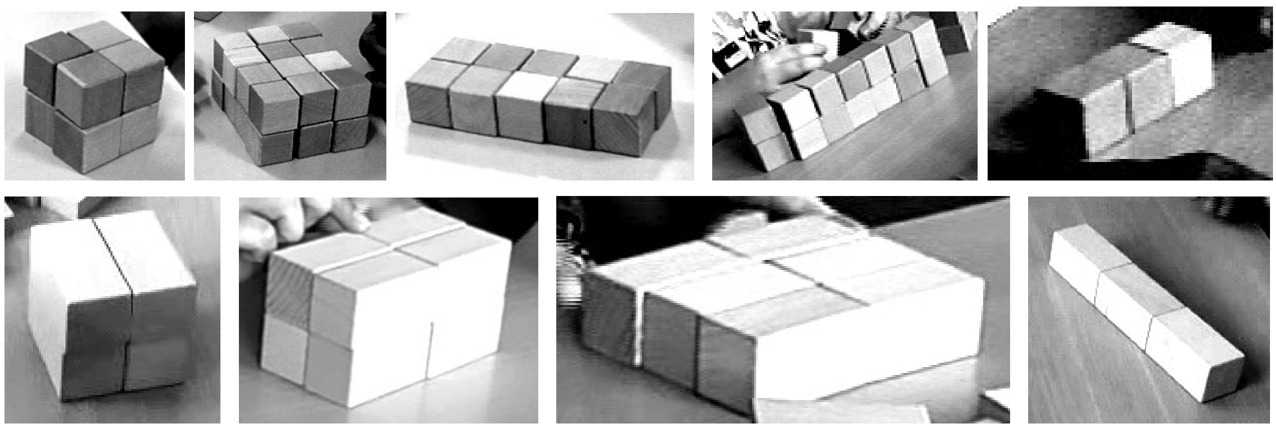

Figure 1: Variety of cuboids constructed by third-graders

[2] Data collection in Malaysia was supported by the DAAD (Higher Education Dialogue with the Muslime World; Faculty of Education, Leipzig University, Germany and Universiti Sains Malaysia, Penang; “Pupil’s Diversity and Success in Education in Germany and Malaysia”).

Most interestingly, solutions which led to prototypical representatives (convex with various layers or flat lying, e. g. a 2 x 3 x 4 cuboid) were prevailing, whereas constructions resembling “thin and long” objects (with several cubes which are aligned as a row in horizontal position) were rare (see table 1 for a brief overview on types of (correct) representatives for cuboids ten German and twelve Malaysian children constructed with some children finding various solutions). Very similar types of products were constructed when children used cuboid blocks for the construction of bigger cuboids (see second row in figure 1).

type of product	total among German children (using cubes)	total among Malaysian children (using cubes)
Cube	0	1
convex with various layers	7	4
flat lying	12	1
flat wall	1	0
row	5	2

Table 1: Total number of correct representatives of cuboids in constructions

Taking a closer look on the *PRODUCTS FOR CUBES* children constructed during the interviews, we made the general observation that the property of quadratic surfaces is obviously a fairly dominant split of knowledge children express in their constructions. Yet, most children focus on a square base area during their constructions (see figure 2, two examples on the right side). For example, we found that three (out of ten) German children constructed only the quadratic base of the solid and named this building a “cube”. Similarly, three (out of twelve) Malaysian children presented the same kind of construction.

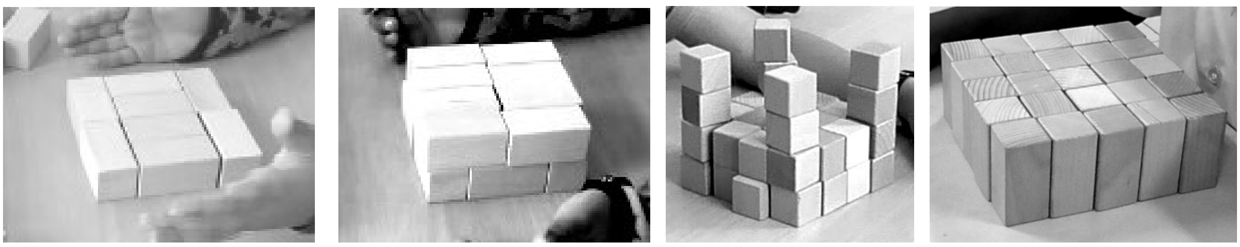

Figure 2: Two “cubes” constructed during the same sequence and further constructions named as “cubes” (with common feature of a quadratic base).

The German third-grader Anna struggles with the demands she has to cope with when constructing a cube, too: Within a longer sequence of the interview she initially constructs a flat lying cuboid with all blocks arranged in a quadratic array. Next, she constructs a second quadratic layer on the lower quadratic layer – naming both constructions a "cube" (see figure 2, first and second picture). Additionally, her focus lies on quadratic arrays as a starting point when building cuboids, as well. She does not identify the thin and long cuboid (from Froebel´s Gift) as a cuboid ("No, this one is not a cuboid, because it is too long."), but identifies another cuboid (from Froebel´s Gift) with the feature of two quadratic surfaces correctly. These comments and constructions are in line with Anna's verbal explanation in the beginning of the interview "A cube is quadratic." and "A cuboid has equal long sides, except for this side (*showing the lateral quadratic surfaces of a block lying on the table.*)." In summary, we can state that Anna is on her way to the level of *ANALYSIS* as she tries to use descriptive mathematical knowledge when giving comments on her construction (e. g. using mathematical terms like "side" or "edge").

On one hand, these observations obviously reveal problems in developing a sound geometrical concept of "cuboid" and the sub-ordinate concept of "cube". On the other hand, most German children tried to name properties and offered answers like "because it has equally long edges" when they were asked to explain why they considered their own building to be a cube. Some Malaysian children were capable of arguing in a similar way and offered arguments like "It looks the same from all sides." or "All surfaces are the same and it's three-dimensional."

Another interesting aspect was to observe cognitive conflicts some German children faced when using the material: For example, they said "With cuboid-bricks I can´t build a cube.", "With this strange bricks (*referring to prisms)* I can´t build a cube or cuboid." or "With triangles I can´t build a cube." This reveals that the participating third-graders often *DO* identify at least a limited set of common features of cuboids (and of the sub-ordinate class of cubes) in the sense of Fischbein (1993). Yet, they obviously often have difficulties in considering all relevant features at the same time.

Compared to German children, children's block constructions in Malaysia revealed a wider distribution on different developmental stages of geometrical concept knowledge (e. g. several children stating "I just know this is a cube." at the level of *VISUALIZATION*, but only a few children listing properties of the constructed object in detail at the level of *ANALYSIS*). These differences could be due to language peculiarities: In German, the term "Wuerfel" is used in children's every-day-life. It serves both for dice and cubes and is particularly different from "Quader" (cuboid), whereas there is a significant similarity of the words "cubes" and "cuboids" (which is also obvious in Bahasa Malay some children speak at home: "Bentuk Kiub" or "Bentuk Kubus" for "cube" and "Dadu" for "dice").

CONCLUSIONS AND OUTLOOK

Aiming at more detailed information on the question how third-graders articulate their geometrical knowledge via constructions with wooden blocks, we found an impressive variety of different types of products and of individual approaches which provided the opportunity to interrelate the constructive activities with the Van Hiele framework. According to our analyses of third-graders' conceptual knowledge on cubes and cuboids, none of the participating German and Malaysian children was in the phase of transition from *ANALYSIS* to *ABSTRACTION* – a result which is basically in line with similar studies (e. g. Szinger, 2008, p. 173). All children faced difficulties in realizing relationships between the geometrical solids cube and cuboid. The more surprising results were the difficulties some children had in constructing *ANY* correct representative of adjacent blocks for either cubes or cuboids or both.

Additionally, the results from our work with children of different cultural backgrounds may serve as an empirically grounded enrichment of the Van Hiele framework – keeping in mind that all data only derived from a fairly small sample (N = 22). The results also raise new hypotheses concerning the development of children's conceptual knowledge on geometrical solids: As the variety we detected among third-graders is likely to enlarge in ensuing years of children's development, the individual variety and flexibility in constructing cuboids and cubes and the ability to give comments might extent and change during a longer phase of children's individual development (especially from grade three until grade five). In this sense, the results of our initial study in this field provides the starting point for a longitudinal study we have set up recently. This is encouraged by a particular interest in children's development on geometrical concept knowledge on cuboids and cubes which has not been tracked intensely, so far.

References

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the Van Hiele Levels of Development in Geometry. *Journal for Research in Mathematics Education*, 17(1), 31-48.

Crowley, M. (1987). The Van Hiele Model of the Development of Geometric Thought. In M. Montgomery Lindquist (Ed.), *Learning and Teaching Geometry, K-12* (Yearbook of the National Council of Teachers of Mathematics, pp. 1-16). Reston, VA: NCTM.

Clements, D. H., Swaminathan, S., Zeitler Hannibal, M. A., & Sarama, J. (1999). Young Children´s Concept of Shape. *Journal for Research in Mathematics Education*, *30*(2), 192-212.

Corbin, J., & Strauss, A. (2015). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory*. Thousand Oaks: Sage.

Fischbein, E. (1993). The Theory of Figural Concepts. *Educational Studies in Mathematics*, *24*(2), 139-162.

Franke, M., & Reinhold, S. (2016). *Didaktik der Geometrie in der Grundschule*. Wiesbaden: Springer.

Hasegawa, J. (1997). Concept Formation of Triangles and Quadrilaterals in the Second Grade. *Educational Studies in Mathematics*, *32*(2), 157-179.

Kalenine, S., Pinet, L., & Genta, E. (2011). The Visual and Visio-haptic Exploration of Geometrical Shapes Increases their Recognition in Preschoolers. *Int. Journal of Behavioral Development*, *35*(1), 18-26.

Lewis, H. P. (1963). Spatial Representation in Drawing as a Correlate of Development and a Basis for Picture Preference. *Journal of Genetic Psychology*, 102, 95-107.

Maier, A. S., & Benz, C. (2014). Children’s Conceptual Knowledge on Triangles Manifested in their Drawings. In P. Liljedahl, S. Oesterle, C. Nicol, & D. Alann (Eds.*), Proceedings of the Joint Meeting of PME 38 and PME-NA36* (Vol. 4, pp. 153-160). Vancouver, Canada: PME.

Milbrath, C., & Trautner, H. M. (Eds.) (2008). *Children’s Understanding and Production of Pictures, Drawings and Art.* Göttingen: Hogrefe.

Mitchelmore, M. C. (1978). Developmental Stages in Children’s Representation of Regular Solid Figures. *The Journal of Genetic Psychology*, *133*(2), 229-239.

Mulligan, J., Prescott, A., & Mitchelmore, M. (2004). Children’s Development of Structure in Early Mathematics. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the IGPME* (Vol. 2, pp. 465-472). Bergen, Norway: PME.

Mulligan, J., & Mitchelmore, M. (2009). Awareness of Pattern and Structure in Early Mathematical Development. *Mathematics Education Research Journal, 21*(2), 33-49.

Reinhold, S., Beutler, B., & Merschmeyer-Brüwer, C. (2013). Preschoolers Count and Construct: Spatial Structuring and its Relation to Building Strategies in Enumeration-Construction Tasks. In A. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the IGPME* (Vol. 4, pp. 81-88). Kiel: PME.

Sinclair, N., & Bruce, C. D. (2015). New Opportunities in Geometry Education at the Primary School. *ZDM Mathematics Education*, *47*(3), 319-329.

Szinger, I. S. (2008). The Evolvement of Geometrical Concepts in Lower Primary Mathematics. *Annales Mathematicae et Informaticae*, *35*, 173-188.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Thom, J. S., & McGarvey, L. M. (2015). The Act and Artifact of Drawing(s): Observing Geometric Thinking with, in, and through children´s drawings. *ZDM Mathematics Education*, *47*(3), 465-481.

Van Hiele, P. M. (1986). *Structure and Insight: A Theory of Mathematics Education*. Orlando: Academic Press.

Van Hiele, P. M. (1999). Developing Geometric Thinking through Activities That Begin with Play. *Teaching Children Mathematics*, *5*, 310-316.

Vollrath, H.-J. (1984). *Methodik des Begriffslehrens im Mathematikunterricht.* Stuttgart: Klett.