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A note on the finiteness of certain cuspidal divisor class groups

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0. Introduction

The groups in question are the cuspidal divisor class groups of Drinfeld modular curves. Like elliptic modular curves, Drinfeld modular curves $\overline{M}$ arise from compactifying certain smooth affine algebraic curves M (defined through a moduli problem) by attaching a finite number of cusps [5]. The cuspidal divisor class group $\mathcal{C}$ of $\overline{M}$ is the subgroup of the Jacobian $J = J(\overline{M})$ generated by the cusps. Its finiteness (the analogue of the Manin-Drinfeld theorem) has been proven in two important special cases in [4] and in [5]. The strategy of proof consisted in both cases in the construction of a sufficiently large number of meromorphic functions with effectively calculable divisors supported by the cusps. It was also stated in [5] that a proof in the general case could be worked out by combining the respective methods of [4] and [5]. Such a proof (straightforward, but laborious) would have the advantage to produce an explicit (albeit unreasonably large) bound for the order of $\mathcal{C}$.

Recent work of C. Schoen and J. Top [9] and of the author [6] as well as other reasons now indicate the necessity of disposing of a complete proof, valid in the general case, at least of the weaker qualitative statement, i.e., of the finiteness of $\mathcal{C}$ without an effective bound. It is the aim of the present note to fill that gap, see Theorem 1.2.

Surprisingly, the basic idea originally used by Manin [8] and Drinfeld [2] in the case of elliptic modular curves also applies to our situation. Main ingredients are the descriptions, given in [7] and [6], of J and $\mathcal{C}$ by means of theta functions, and the analogue of Ramanujan's conjecture, which provides control over the eigenvalues of Hecke operators. All we have to do is to arrange this material conveniently. Once the finiteness of $\mathcal{C}$ is established, the methods of [6] allow to investigate its group theoretical structure and to find bounds for its order. A brief account of this is given in the final section, Proposition 4.1.

1. The set-up.

We collect here the necessary notations and definitions. These, like the stated results, are largely taken from [5], [6], or [7], to which we also refer for motivation and further references.

(1.1) We let K be a function field in one variable over a finite field $\mathbb{F}_q$ with q elements algebraically closed in K. We fix once for all a place "∞" of K and let A be the Dedekind subring of elements of K regular away from ∞. The completion of K at ∞ and its completed algebraic closure with respect to the normalized absolute value $| \, . \, | = | \, . \, |_\infty$ are denoted by K_∞ and $C = \hat{\overline{K}}_\infty$, respectively. The *Drinfeld upper half-plane* is $\Omega := C - K_\infty$, the set of C-valued points of an analytic space labelled by the same symbol Ω.

The group $\mathrm{GL}(2, K_\infty)$ acts as a matrix group from the right on K_∞^2 and from the left on Ω through $\binom{a\,b}{c\,d}z = \frac{az+b}{cz+d}$. A *congruence subgroup* Γ of $\mathrm{GL}(2,K)$ is some intermediate group $\Gamma(Y,\mathfrak{n}) \subset \Gamma \subset \Gamma(Y)$, where $Y \subset K^2$ is a projective A-submodule of rank two, $\Gamma(Y) = \{\gamma \in \mathrm{GL}(2,K) \mid Y\gamma = Y\}$, and $\Gamma(Y,\mathfrak{n}) \subset \Gamma(Y)$ is the kernel of the "reduction mod $\mathfrak{n}$" mapping for some ideal $\mathfrak{n} \neq 0$ of A. For each congruence subgroup Γ, we let M_Γ be the (smooth connected affine) algebraic curve over C with set of C-points $M_\Gamma(C) = \Gamma \setminus \Omega$ and $\overline{M}_\Gamma$ its canonical smooth compactification. The *Drinfeld modular curves* $\overline{M}_\Gamma$ are actually defined over finite separable extensions of the field K. We will however only work over C, and will, in view of the GAGA theorems, not distinguish between $\overline{M}_\Gamma$, the associated analytic space, and the point set $\overline{M}_\Gamma(C)$. Elements of $\overline{M}_\Gamma(C) - M_\Gamma(C)$ are called *cusps*, they form a set cusp(Γ) which is in canonical one-to-one correspondence with the set $\Gamma \setminus \mathbb{P}^1(K)$ of orbits of Γ on $\mathbb{P}^1(K)$. By $J = J_\Gamma$, we denote the Jacobian of $\overline{M}_\Gamma$, by $\mathcal{C} = \mathcal{C}_\Gamma$ the group of divisors of degree zero on $\overline{M}_\Gamma$ supported by the cusps, modulo principal divisors. The purpose of the present note is to prove the following assertion.

1.2 Theorem. *For each congruence subgroup* Γ *of* $\mathrm{GL}(2,K)$*, the cuspidal divisor class group* $\mathcal{C}_\Gamma$ *is finite.*

Fixing a base point x on $\overline{M}_\Gamma$ (for example, the cusp represented by $\infty \in \mathbb{P}^1(K)$), we regard $\mathcal{C}_\Gamma$ as a subgroup of J_Γ. In order to describe the Abel-Jacobi mapping with respect to x from $\overline{M}_\Gamma$ to J_Γ, we introduce some more material.

(1.3) The *Bruhat-Tits tree* $\mathcal{T}$ of $\mathrm{PGL}(2,K_\infty)$ is a homogeneous tree of valency $q_\infty + 1$ on which $\mathrm{PGL}(2,K_\infty)$ acts (vertex- and edge-) transitively. Here $q_\infty = q^{\deg\,\infty}$ is the cardinality of the residue class field at infinity, and the above says that each vertex of $\mathcal{T}$ is incident with precisely $q_\infty + 1$ non-oriented edges. The sets $X(\mathcal{T})$ of vertices and $Y(\mathcal{T})$ of oriented edges of $\mathcal{T}$ may be described in terms of the group $\mathrm{GL}(2,K_\infty)$ ([12] II 1.3, [7] 1.3). Both Ω and $\mathcal{T}$ are analogues of the complex upper half-plane; in fact, $\mathcal{T}$ is some sort of combinatorial picture of Ω. This is expressed through the existence of a $\mathrm{GL}(2,K_\infty)$-equivariant map λ from Ω to the set $\mathcal{T}(\mathbb{Q})$ of real points of $\mathcal{T}$ with rational barycentric coordinates ([7] 1.5). Moreover, the quotient graph $\Gamma \setminus \mathcal{T}$ is "almost finite"; it is the union of a finite graph with $g(\Gamma) :=$ genus of $\overline{M}_\Gamma$ independent cycles and a finite number of infinite half-lines • − − − • − − − − • − − − − • − $\cdots$, one for each cusp of $\overline{M}_\Gamma$ ([12] II Théorème 9, [5] V.2).

(1.4) For each coefficient ring B $(= \mathbb{Z}, \mathbb{Q}, \mathbb{C})$, we let $\underline{H}(\mathcal{T}, B)$ be the B-module of harmonic B-valued cochains on $\mathcal{T}$ as defined in [7] 1.7. Hence a harmonic cochain is a function $\varphi : Y(\mathcal{T}) \longrightarrow B$ that satisfies $\varphi(e) + \varphi(\overline{e}) = 0$ ($\overline{e}$ = edge

e with inverse orientation) and $\sum_{o(e)=v} \varphi(e) = 0$ for each $v \in X(\mathcal{T})$, where the summation is over the edges e with origin v. A congruence subgroup Γ acts on $\underline{H}(\mathcal{T}, B)$, and we put $\underline{H}(\mathcal{T}, B)^{\Gamma}$ and $\underline{H}_!(\mathcal{T}, B)^{\Gamma}$ for the submodules of invariant cochains, of invariant cochains with compact support modulo Γ, respectively. As is described in [7] sect. 3, elements of $\underline{H}(\mathcal{T}, B)^{\Gamma}$ and $\underline{H}_!(\mathcal{T}, B)^{\Gamma}$ have an intuitive description as weighted flows on the quotient graph $\Gamma \backslash \mathcal{T}$, as weighted flows vanishing on the cuspidal half-lines of $\Gamma \setminus \mathcal{T}$, respectively. If B is torsion-free (as will always be the case),

$$\begin{array}{rcl} \underline{H}_!(\mathcal{T}, B)^{\Gamma} &=& \underline{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma} \otimes B \\ \underline{H}(\mathcal{T}, B)^{\Gamma} &=& \underline{H}(\mathcal{T}, \mathbb{Z})^{\Gamma} \otimes B, \end{array}$$

which are free B-modules of ranks $g(\Gamma)$ and $g(\Gamma) + c(\Gamma) - 1$, respectively. Here $c(\Gamma) := \#(\mathrm{cusp}(\Gamma))$ is the number of cusps. For a prime number $l \neq p := \mathrm{char}(\mathbb{F}_q)$, $\underline{H}_!(\mathcal{T}, \mathbb{Q}_l)^{\Gamma}$ is "one half of $H^1_{\mathrm{et}}(\overline{M}_{\Gamma}, \mathbb{Q}_l)$", the l-adic cohomology of $\overline{M}_{\Gamma}$. (See [7] 4.13.1 for a precise statement. All the references to follow are with respect to [7]). If $B = \mathbb{C}$, these modules have an automorphic interpretation (*loc. cit.* 4.7.6). In particular, they are provided with Hecke operators (see (2.6)) and a Petersson product

$$(\ ,\ ) : \underline{H}(\mathcal{T}, \mathbb{C})^{\Gamma} \times \underline{H}_!(\mathcal{T}, \mathbb{C})^{\Gamma} \longrightarrow \mathbb{C}.$$

The latter is hermitian and positive definite on $\underline{H}_! \times \underline{H}_!$, and integral-valued on $\underline{H}(\mathcal{T}, \mathbb{Z})^{\Gamma} \times \underline{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma}$.

Let $\overline{\Gamma}$ be the maximal torsion-free abelian quotient of Γ. By *loc. cit.* 3.3.3, there is a canonical injection $j : \overline{\Gamma} \hookrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma}$ with finite cokernel. In fact, j is bijective in all known cases, and presumably always.

(1.5) In *loc. cit.* sect. 5, we associated a meromorphic theta function $\theta(\omega, \eta, z)$ on Ω with each pair ω, η of elements of Ω. This was generalized in [6], where the parameters ω, η are allowed to lie in $\overline{\Omega} = \Omega \cup \mathbb{P}^1(K)$. The $\theta(\omega, \eta, \cdot)$ satisfy the functional equation

$$\theta(\omega, \eta, \gamma z) = c(\omega, \eta, \gamma)\ \theta(\omega, \eta, z) \qquad (\gamma \in \Gamma)$$

with a homomorphism $c(\omega, \eta, \cdot) : \Gamma \longrightarrow C^*$ that factors over $\overline{\Gamma}$. The divisor of $\theta(\omega, \eta, \cdot)$ is Γ-invariant and, as a divisor on $\overline{M}_{\Gamma}$, equals $[\omega] - [\eta]$, where $[\omega]$ is the class of $\omega \in \overline{\Omega}$ in $\overline{M}_{\Gamma}(C) = \Gamma \setminus \overline{\Omega}$. Furthermore, for each $\alpha \in \Gamma$, the function $u_{\alpha}(z) = \theta(\omega, \alpha\omega, z)$ is holomorphic invertible, independent of $\omega \in \overline{\Omega}$, depends only on the class $\overline{\alpha}$ of α in $\overline{\Gamma}$, and satisfies $u_{\alpha\beta} = u_{\alpha} \cdot u_{\beta}$. If $c_{\alpha}(\ .\ ) = c(\omega, \alpha\omega, \cdot)$ denotes its multiplier, the bilinear map from $\overline{\Gamma} \times \overline{\Gamma}$ to C^* induced from $(\alpha, \beta) \longmapsto c_{\alpha}(\beta)$ is symmetric and satisfies

$$\log_{q_\infty} |c_{\alpha}(\beta)| = (j(\overline{\alpha}), j(\overline{\beta})), \tag{1.5.1}$$

where the right hand side is the Petersson product and "$|\ .\ |$" the absolute value on C extending the normalized absolute value on K_{∞}. In particular,

$(\alpha, \beta) \longmapsto \log_{q_\infty} |c_\alpha(\beta)|$ is positive definite as a bilinear form on $\overline{\Gamma}$. Hence $\alpha \longmapsto c_\alpha$ induces an injection

$$\overline{c} : \overline{\Gamma} \hookrightarrow \mathrm{Hom}(\overline{\Gamma}, C^*)$$

into the C-valued points of the torus $T_\Gamma := \mathrm{Hom}(\overline{\Gamma}, \mathbb{G}_m)$ of dimension $g(\Gamma)$. The following is one of the main results of [7] and [6].

1.6 Theorem. *The Jacobian J_Γ of $\overline{M}_\Gamma$ is the quotient of T_Γ by the multiplicative lattice $\overline{c}(\overline{\Gamma})$. That is, we have an exact sequence*

$$(1.6.1) \qquad 1 \longrightarrow \overline{\Gamma} \longrightarrow \mathrm{Hom}(\overline{\Gamma}, C^*) \longrightarrow J_\Gamma(C) \longrightarrow 0.$$

The Abel-Jacobi mapping with base point $[\omega]$ maps $[\eta]$ to the class of $c(\omega, \eta, \cdot)$ in J_Γ $(\omega, \eta \in \overline{\Omega})$.

Proof. [7] 7.4.1 and [6] sect. 2.

(1.7) We still need another link between functions on Ω and on $\mathcal{T}$. To each invertible holomorphic function f on Ω we let $r(f) \in \underline{H}(\mathcal{T}, \mathbb{Z})$ be the associated harmonic cochain defined in [7] 1.7.3. It measures the growth of f, and is a substitute for the logarithmic derivative of f. It is characterized by the short exact sequence

$$(1.7.1) \qquad \begin{array}{ccccc} 1 \longrightarrow C^* \longrightarrow \mathcal{O}_\Omega(\Omega)^* & \longrightarrow & \underline{H}(\mathcal{T}, \mathbb{Z}) \longrightarrow 0 \\ f & \longmapsto & r(f) \end{array}$$

of $\mathrm{GL}(2, K_\infty)$-modules, where the middle term is the multiplicative group of invertible holomorphic functions on Ω. For $\alpha \in \Gamma$, $r(u_\alpha)$ is Γ-invariant, and in fact,

$$(1.7.2) \qquad r(u_\alpha) = j(\overline{\alpha}) \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$$

(*loc. cit.*, Theorem 5.6.1).

2. Reductions and preparations.

(2.1) Let $Y \subset K^2$ be an A-submodule of rank two. There exists an ideal $0 \neq \mathfrak{n}$ of A sucht that both $\mathfrak{n}Y$ and $\mathfrak{n}A^2$ are contained in $Y \cap A^2$. Therefore $\Gamma(Y, \mathfrak{n})$ and $\Gamma(A^2, \mathfrak{n}) =: \Gamma(\mathfrak{n})$ are subgroups of $\Gamma(Y) \cap \mathrm{GL}(2, A)$. Consequently, each congruence subgroup Γ of $\mathrm{GL}(2, K)$ has the property that its intersection with $\Gamma(1) := \mathrm{GL}(2, A)$ contains some principal congruence subgroup $\Gamma(\mathfrak{n})$ of $\Gamma(1)$. Since the assertion of Theorem 1.2 is obviously stable under passing from Γ to overgroups Γ', we are reduced to showing (1.2) for groups of shape $\Gamma(\mathfrak{n})$. We therefore make the *assumption*, in force from now on:

$$(2.2) \qquad \begin{array}{l} \Gamma = \Gamma(\mathfrak{n}) = \{\gamma \in \mathrm{GL}(2, A) \mid \gamma \equiv 1 \ (\mathrm{mod}\ \mathfrak{n})\}, \\ \text{where } \mathfrak{n} \text{ is a non-trivial ideal of } A. \end{array}$$

Then Γ has no prime-to-p torsion, which has the important consequence ([7] Proposition 3.4.5):

(2.3) The map $j : \overline{\Gamma} \xrightarrow{\cong} \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is bijective.

(2.4) A *cuspidal theta function* for Γ is an invertible holomorphic function f on Ω subject to the functional equation

$$f(\gamma z) = c_f(\gamma)\ f(z)$$

for $\gamma \in \Gamma$ with some $c_f(\gamma) \in C^*$. Such a function has a well-defined zero order $\mathrm{ord}_{[s]} f$ at each cusp $[s]$ of Γ ([6] 3.3.2; here and in the sequel, we write $[\omega]$ for the class (mod Γ) of $\omega \in \overline{\Omega}$). Hence we can associate with each cuspidal theta function f its divisor $\mathrm{div}\ f \in D_\infty := \mathbb{Z}[\mathrm{cusp}(\Gamma)]$. Our assumption 2.2 implies (*loc. cit.* 3.5, 3.9):

(2.4.1) The group of cuspidal theta functions is generated by C^* and the functions $\theta(s,t,z)$, where $s,t \in \mathbb{P}^1(K)$.

A fortiori (*loc. cit.* 2.14):

(2.4.2) The multiplier $c_f : \Gamma \longrightarrow C^*$ of a cuspidal theta function f takes its values in K_∞^*.

Let $D_\infty^0 \hookrightarrow D_\infty$ be the subgroup of divisors of degree zero. Then

$$\text{(2.4.3)} \qquad \overline{\mathrm{div}} : \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma / \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \xrightarrow{\cong} D_\infty^0,$$

which is a special case of *loc. cit.*, Theorem 3.8.

We briefly describe the isomorphism $\overline{\mathrm{div}}$. Each $\varphi \in \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma$ may be written as $r(f)$ with some cuspidal theta function f; then $\overline{\mathrm{div}}(\text{class of } \varphi) := \mathrm{div}(f)$ is a well-defined element of D_∞^0. Conversely, we may associate with the divisor $[s]-[t]$ the class mod $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ of $r(\theta(s,t,\cdot))$, which yields the inverse mapping of $\overline{\mathrm{div}}$. Hence, in terms of the description provided by (1.6), the cuspidal group $\mathcal{C}_\Gamma \hookrightarrow J_\Gamma$ consists precisely of the classes mod $\overline{c}(\overline{\Gamma})$ of multipliers c_f of cuspidal theta functions f. We now get a criterion for cuspidal divisors to be principal.

Let $D \in \mathbb{Z}[\mathbb{P}^1(K)]$ be a divisor of degree zero on $\mathbb{P}^1(K)$. Writing D as a linear combination $\sum m_{s,t}(s-t)$ of divisors $(s-t)$ with $s,t \in \mathbb{P}^1(K)$, the function

$$\theta(D,z) := \prod \theta(s,t,z)^{m_{s,t}}$$

depends only on D but not on the presentation chosen.
We let $c_D := \prod c(s,t,\cdot)^{m_{s,t}} \in \mathrm{Hom}(\overline{\Gamma}, K_\infty^*)$ be its multiplier. If $[D]$ denotes the induced cuspidal divisor on $\overline{M}_\Gamma$, we have:

$$\text{(2.4.4)} \qquad [D] \text{ is principal } \Leftrightarrow \text{ There exists } \alpha \in \Gamma \text{ such that } c_D = c_\alpha,$$

as follows from (1.6) and the discussion above.

(2.5) Next, we need to discuss Hecke operators $T_{\mathfrak{p}}$. These are naturally defined for all the maximal ideals $\mathfrak{p}$ of A coprime to $\mathfrak{n}$ and act in a compatible way on certain spaces of automorphic forms and, as correspondences, on the compactified moduli scheme $\overline{M}(\mathfrak{n})$ of Drinfeld A-modules of rank two with a level-$\mathfrak{n}$ structure (see [7] section 4). Now $\overline{M}(\mathfrak{n}) \times_A C$ splits into irreducible components parametrized by $\mathrm{Pic}_{\mathfrak{n}}(A)$, the generalized ideal class group of A with conductor $\mathfrak{n}$ ([5] II 1.8 + 1.4). Our curve $\overline{M}_\Gamma = \overline{M}_{\Gamma(\mathfrak{n})}$ is naturally identified with the component corresponding to $1 \in \mathrm{Pic}_{\mathfrak{n}}(A)$. In general, a Hecke operator $T_{\mathfrak{p}}$ doesn't respect the above decomposition; it does so however if $\mathfrak{p} = (\pi)$ is principal and generated by some $\pi \in A$ that satisfies $\pi \equiv 1 \pmod{\mathfrak{n}}$. This follows from the adelic description of $T_{\mathfrak{p}}$ given in [7] 4.9. Such Hecke operators $T_{\mathfrak{p}}$ are called *admissible* for brief; they are the only ones we presently need. The action of admissible Hecke operators on the component $\overline{M}_\Gamma$ may be translated from the adelic language into formulae that only involve $\Omega, \mathcal{T}$, and the groups $\Gamma \hookrightarrow \mathrm{GL}(2, K)$. This is the background of the description given in the next subsection. Some compatibility properties of $T_{\mathfrak{p}}$ have been proved in [7] in the special case of Hecke congruence subgroups Γ of $\mathrm{GL}(2, A)$; the respective proofs in the case of our present full congruence subgroups $\Gamma = \Gamma(\mathfrak{n})$ are identical.

(2.6) Thus let the prime $\mathfrak{p} = (\pi)$ be principal, generated by π subject to $\pi \equiv 1 \pmod{\mathfrak{n}}$, and let τ be the matrix $\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{GL}(2, K)$. Put $\Delta = \Gamma \cap \tau\Gamma\tau^{-1}$, and choose a system $\{\alpha\}$ of representatives for $\Delta \setminus \Gamma$. The $\mathfrak{p}$-th Hecke operator is given

(a) as a correspondence on the curve $\overline{M}_\Gamma(C) = \Gamma \setminus \overline{\Omega}$ through $\omega \longmapsto \sum_{\{\alpha\}} \tau^{-1}\alpha\omega$;

(b) as a correspondence on the graph $\Gamma \setminus \mathcal{T}$ through $x \longmapsto \sum_{\{\alpha\}} \tau^{-1}\alpha x$. Here x may be a vertex, an oriented edge, or a real point of $\mathcal{T}$;

(c) as an endomorphism on $\underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma$ and its submodule $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ through

$$(T_{\mathfrak{p}}\varphi)(e) = \sum_{\{\alpha\}} \varphi(\tau^{-1}\alpha e).$$

Of course, the coefficient ring $\mathbb{Z}$ may be replaced by any subring B of $\mathbb{C}$;

(d) as an endomorphism of $\overline{\Gamma}$ in purely group-theoretical terms as some sort of *Verlagerung* ([7] 9.3);

(e) as an endomorphism of the torus $T_\Gamma = \mathrm{Hom}(\overline{\Gamma}, \mathbb{G}_m)$ through its action on the first argument $\overline{\Gamma}$.

In (a), (b), (c) we have chosen representatives for classes modulo Γ of $\omega \in \Omega$, $x \in \mathcal{T}$, $e \in Y(\mathcal{T})$. Of course, the respective description of $T_{\mathfrak{p}}$ depends neither on these choices nor on the choice of $\{\alpha\}$. The different actions satisfy all kinds of compatibilities, viz., (ab) $T_{\mathfrak{p}}$ commutes with the map

$$\lambda_\Gamma : \ M_\Gamma(C) = \Gamma \setminus \Omega \longrightarrow \Gamma \setminus \mathcal{T}(\mathbb{Q})$$

which is obtained from the map λ referred to in (1.3) by dividing out Γ. This is obvious from the $\mathrm{GL}(2, K_\infty)$-equivariance of λ;

(cd) $T_{\mathfrak{p}}$ commutes with the isomorphism $j : \overline{\Gamma} \longrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ of (2.3) ([7] 9.3.2);

(ade) $T_{\mathfrak{p}}$ induces via (a) an endomorphism of the Jacobian J_Γ of $\overline{M}_\Gamma$, and the exact sequence (1.6.1)

$$1 \longrightarrow \overline{\Gamma} \longrightarrow \mathrm{Hom}(\overline{\Gamma}, C^*) \longrightarrow J_\Gamma(C) \longrightarrow 0$$

is compatible with the respective Hecke operators (*loc. cit.* 9.4).

Furthermore, by its very construction, the isomorphism $\overline{\mathrm{div}}$ of (2.4.3) commutes with the action of $T_{\mathfrak{p}}$.

2.7 Remark. In all cases, $T_{\mathfrak{p}}$ is represented as a sum over $\{\tau^{-1}\alpha \mid \alpha \in \Delta \setminus \Gamma\}$, or equivalently, over $R(\mathfrak{n}, \pi) := \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi \end{smallmatrix}\right)\alpha \mid \alpha \in \Delta \setminus \Gamma\}$, since the scalar matrix $\left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix}\right)$ acts trivially. Now $R(\mathfrak{n}, \pi)$ is a system of representatives for the set $S(\mathfrak{n}, \pi) := \{\gamma \in \mathrm{Mat}(2, A) \mid \gamma \equiv 1 \pmod{\mathfrak{n}}, \ \det \gamma = \pi\}$ modulo left equivalence under $\Gamma = \Gamma(\mathfrak{n})$. (This is a consequence e.g. of the elementary divisor theorem, or of the strong approximation theorem for SL(2).) We may therefore represent $T_{\mathfrak{p}}$ also as the correspondence $x \longmapsto \{\gamma x \mid \gamma \in R\}$, where R can be any system of representatives for $\Gamma \setminus S(\mathfrak{n}, \pi)$. The descriptions given in (2.6) are chosen to be compatible with those in [7] sect. 9, which allows to check the stated properties.

3. Proof of the main result.

Let now $\mathfrak{n}$ be any non-trivial ideal of A. It suffices to prove Theorem 1.2 for $\overline{M}_\Gamma = \overline{M}_{\Gamma(\mathfrak{n})}$, i.e., to prove that the cuspidal group $\mathcal{C} = \mathcal{C}_\Gamma$ is finite. Choose a prime ideal $\mathfrak{p}$ of A generated by $\pi \in A$ that satisfies $\pi \equiv 1 \pmod{\mathfrak{u}}$. Such a $\mathfrak{p}$ exists by the K-version of Dirichlet's theorem. We let $T = T_{\mathfrak{p}}$ be the corresponding admissible Hecke operator and $m := q^{\deg \mathfrak{p}} + 1$.

3.1 Proposition. *The subgroup $\mathcal{C}$ of $J = J_\Gamma$ is stable under T, and T restricted to $\mathcal{C}$ is multiplication by m.*

Proof. (i) With notations as in (2.6), T is given by summation over some set R of representatives of $\Gamma \setminus S(\mathfrak{n}, \pi)$. It is straightforward that $\#(R) = m$. In view of the descriptions of $\mathcal{C}$ and T given in (2.4) and (2.6), it suffices to see that R can be chosen such that each $\gamma \in R$ stabilizes each class $[s]$ in $\Gamma \setminus \mathbb{P}^1(K) = \mathrm{cusp}(\Gamma)$.

(ii) Consider first the analogous question for $\Gamma(1) = \mathrm{GL}(2, A)$. In that case ([1]

VII 4.10), $s \longmapsto$ class of P_s induces a bijection

$$\mathrm{cusp}(\Gamma(1)) = \Gamma(1) \setminus \mathbb{P}^1(K) \xrightarrow{\cong} \mathrm{Pic}(A).$$

Here the invertible A-module P_s is the intersection with A^2 of the line $Ks \subset K^2$ corresponding to $s \in \mathbb{P}^1(K)$. Let $\gamma \in S(\mathfrak{n}, \pi)$ and $s \in \mathbb{P}^1(K)$. Since det $\gamma = \pi$, we have an exact sequence

$$0 \longrightarrow \gamma(A^2) \longrightarrow A^2 \longrightarrow A/\mathfrak{p} \longrightarrow 0.$$

Hence $P_{\gamma s}$ fits into an exact sequence

$$0 \longrightarrow K\gamma s \cap \gamma(A^2) \longrightarrow P_{\gamma s} \longrightarrow X \longrightarrow 0$$

with X isomorphic to one of $A/\mathfrak{p}$ or 0. Now since multiplication by γ yields

$$P_s = Ks \cap A^2 \xrightarrow{\cong} K\gamma s \cap \gamma(A^2)$$

and $\mathfrak{p}$ is principal, the classes in $\mathrm{Pic}(A)$ of P_s and $P_{\gamma s}$ agree in either case. Thus γ stabilizes at least $s \pmod{\Gamma(1)}$.

(iii) Let G be the subgroup of $\mathrm{GL}(2, K)$ generated by $\Gamma(1)$ and the fixed element γ of R, and put $G(\mathfrak{n}) := \{\alpha \in G \mid \alpha \equiv 1 \pmod{\mathfrak{n}}\}$. Then $\Gamma = \Gamma(\mathfrak{n}) = \Gamma(1) \cap G(\mathfrak{n})$. By (ii),

$$\Gamma(1) \setminus \mathbb{P}^1(K) \xrightarrow{\cong} G \setminus \mathbb{P}^1(K).$$

It remains to verify that also

$$\mathrm{cusp}(\Gamma(\mathfrak{n})) = \Gamma(\mathfrak{n}) \setminus \mathbb{P}^1(K) \xrightarrow{\cong} G(\mathfrak{n}) \setminus \mathbb{P}^1(K).$$

Let $s \in \mathbb{P}^1(K)$ have stabilizer B in G. The orbit

$$\Gamma(1)s = Gs \xrightarrow{\cong} G/B = \Gamma(1)/\Gamma(1) \cap B$$

decomposes modulo $G(\mathfrak{n})$ into $G(\mathfrak{n})\setminus G/B$ and modulo $\Gamma(\mathfrak{n})$ into $\Gamma(\mathfrak{n})\setminus\Gamma(1)/\Gamma(1) \cap B = \Gamma(1) \cap G(\mathfrak{n}) \setminus \Gamma(1)/\Gamma(1) \cap B$, which agree since $G = \Gamma(1)B$. Hence γ even stabilizes $s \pmod{\Gamma}$, as was to be shown. □

3.2 Remark. Note that, as soon as $T = T_{\mathfrak{p}}$ is constructed as a correspondence on $\Gamma \setminus \mathcal{T}$ preserving the simplicial structure, each vertex v is mapped under T to a collection of m vertices v' whose distance from v is uniformly bounded by deg $\mathfrak{p}$. Given the structure of $\Gamma \setminus \mathcal{T}$ as a union of a finite graph and a number of half-lines $h_{[s]}$ indexed by $[s] \in \mathrm{cusp}(\Gamma)$, T cannot but stabilize each of the $[s]$, i.e., associate m times $[s]$ to $[s]$. Thus (3.1) corresponds to a quite intuitive geometrical fact.

Next, consider the endomorphism T on $\underline{H}_!(\mathcal{T}, \mathbb{Q})^{\Gamma}$. All its eigenvalues ϵ are known to satisfy the Ramanujan bound

$$|\epsilon| \leq 2q^{\deg\ \mathfrak{p}/2}. \qquad (3.3)$$

This fact is essentially due to Drinfeld [3]; it follows from combining

(a) the observation that $\underline{H}_!(\mathcal{T}, \mathbb{Q}_l)^\Gamma$ is a T-stable subspace of $H^1_{\text{et}}(\overline{M}(\mathfrak{n}) \times_A C, \mathbb{Q}_l)$. Here $l \neq p$ is a prime number, and the admissibility condition (2.5) on $T = T_\mathfrak{p}$ is crucial;

(b) the Eichler-Shimura relation [3] 5.5 that links T to the action of the Frobenius endomorphism $F_\mathfrak{p}$ on the first l-adic cohomology of $\overline{M}(\mathfrak{n}) \times_A \overline{\mathbb{F}}_\mathfrak{p}$ ($\overline{\mathbb{F}}_\mathfrak{p}$ = algebraic closure of $\mathbb{F}_\mathfrak{p} = A/\mathfrak{p}$);

(c) "Weil's conjecture", proved by Deligne, which states that

$$|\varphi| = q^{\deg\ \mathfrak{p}/2}$$

for eigenvalues φ of F_p.

Perhaps the discussion given in [7] sect. 4 is helpful, where some more details are given.

Since $2q^{\deg\ \mathfrak{p}/2} < m = q^{\deg\ \mathfrak{p}} + 1$, the operator $T - m \cdot \text{id}$ is invertible on $\underline{H}_!(\mathcal{T}, \mathbb{Q})^\Gamma$ and has a finite cokernel if restricted to $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$. Taking into account (1.6), (2.3) and the compatibilities (2.6), the corresponding endomorphism $T - m \cdot \text{id}$ on J is an isogeny. Hence there exists a quasi-inverse, i.e., an endomorphism Q of J that satisfies

$$Q \circ (T - m \cdot \text{id}) = n \cdot \text{id}$$

for some natural number n. By (3.1), $n \cdot \text{id}$ restricted to $\mathcal{C}$ is trivial, thus $n\mathcal{C} = 0$, and Theorem 1.2 is proved.

4. Consequences and remarks.

For brevity, we write $\underline{H}_! \subset \underline{H}$ for the groups $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \subset \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma$, respectively, where $\Gamma = \Gamma(\mathfrak{n})$ is as in the preceding sections. We further let $\underline{H}_!^\perp$ be the orthogonal complement of $\underline{H}_!$ in $\underline{H}$ with respect to the pairing (,) of (1.4). Then $\underline{H}_!^\perp$ is a quasi-complement of $\underline{H}_!$ in $\underline{H}$, i.e., the index of $\underline{H}_! \oplus \underline{H}_!^\perp$ in $\underline{H}$ is finite. Recall that $q_\infty = q^{\deg\ \infty}$ denotes the size of the residue field of K at ∞ and $c(\Gamma) = \#\ (\text{cusp}(\Gamma))$.

4.1 Proposition. *For $\Gamma = \Gamma(\mathfrak{n})$ there exists a short exact sequence*

$$0 \longrightarrow U \longrightarrow \mathcal{C}_\Gamma \longrightarrow \underline{H}/(\underline{H}_! \oplus \underline{H}_!^\perp) \longrightarrow 0,$$

where U is a factor group of $(\mathbb{Z}/(q_\infty - 1))^{c(\Gamma)-1}$.

Proof. Let P be the subgroup of $\underline{H}$ that under the map $\overline{\text{div}}$ of (2.4.3) corresponds to the principal divisors in D^0_∞. Then P contains $\underline{H}_!$. We show that

$$(q_\infty - 1)\underline{H}_!^\perp \oplus \underline{H}_! \subset P \subset \underline{H}_!^\perp \oplus \underline{H}_!,$$

which in view of $\underline{H}_!^\perp \cong \mathbb{Z}^{c(\Gamma)-1}$ will imply the assertion.

Suppose that $\varphi = r(f) \in \underline{H}$ represents a principal divisor under $\overline{\mathrm{div}}$. By (2.4.4), the multiplier $c_f : \Gamma \longrightarrow C^*$ of f equals c_α for some $\alpha \in \Gamma$, and $g := f \cdot u_\alpha^{-1}$ is Γ-invariant. In terms of the pairing (,) this reads

$$0 = (g, j(\overline{\Gamma})) = (g, \underline{H}_!).$$

That is, $\varphi = r(g) + r(u_\alpha)$ lies in $\underline{H}_!^\perp \oplus \underline{H}_! \subset \underline{H}$.

Conversely, let $\varphi = r(f) \in \underline{H}_!^\perp$, where f has multiplier $c_f : \Gamma \longrightarrow C^*$. From (the extension to $\underline{H} \times \underline{H}_!$ of) formula 1.5.1, $|c_f(\gamma)| = 1$ for $\gamma \in \Gamma$. By the finiteness of $\mathcal{C}$ and (2.4.4), there exists $n \in \mathbb{N}$ and $\alpha \in \Gamma$ such that $(c_f)^n = c_\alpha$. Hence $|c_\alpha(\gamma)| = 1$ for $\gamma \in \Gamma$, and, in view of (1.5.1) and the non-degeneracy of (,) on $\underline{H}_!$, $c_\alpha = 1$. Thus finally f^n is Γ-invariant and c_f takes its values in the n-th roots of unity, but also in K_∞^* (see 2.4.2). Together, $(c_f)^{q_\infty - 1} = 1$ and therefore $(q_\infty - 1)\varphi \in P$. □

4.2 Corollary. *The p-components of* $\mathcal{C}_\Gamma$ *and of* $\underline{H}/(\underline{H}_! \oplus \underline{H}_!^\perp)$ *agree.*

4.3 Remarks. Some more material about $\mathcal{C}_\Gamma$ and its relationship to $\underline{H}/(\underline{H}_! \oplus \underline{H}_!^\perp)$ and to the component group of the Néron model of J_Γ (even for more general congruence subgroups) can be found in [6] sections 4 and 5. Corollary 4.2 remains valid for arbitrary congruence groups Γ. The case of Hecke congruence subgroups over polynomial rings $A = \mathbb{F}_q[T]$ is studied in detail. Of course, it would be desirable to know more about the group U that figurs in 4.1, e.g., does it vanish? In that case, $\mathcal{C}_\Gamma$ had a description entirely in terms of the graph $\Gamma \setminus \mathcal{T}$.

Theorem 1.2 is about the curve $\overline{M}_\Gamma$, where Γ is a congruence subgroup of some $\mathrm{GL}(Y)$. The hypothesis was used twice; first for the reduction to the case of $\Gamma = \Gamma(\mathfrak{n}) \subset \mathrm{GL}(2, A)$ and secondly, to have the frame in which we can apply the crucial property (3.3). As is proved in [12] II Théorème 12, there is an abundance of non-congruence subgroups Γ of finite index in $\mathrm{GL}(2, A)$. Nothing seems to be known about the corresponding curves $\overline{M}_\Gamma = \Gamma \setminus \overline{\Omega}$. By analogy with the case of non-congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$ ([10], [11]), it appears unlikely that their cuspidal groups $\mathcal{C}_\Gamma$ are finite.

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