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A lemma on the higher integrability of functions with applications to the regularity theory of two-dimensional generalized Newtonian fluids

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Abstract

In this paper we prove a lemma on the higher integrability of functions and discuss its applications to the regularity theory of two-dimensional generalized Newtonian fluids.

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1 Introduction

As a starting point, let us look at the following non-linear generalization of the classical Stokes problem studied in [5]: given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and a system of volume forces $g : \Omega \to \mathbb{R}^n$ together with a boundary function u_0: $\partial\Omega \to \mathbb{R}^n$, find a velocity field v: $\Omega \to \mathbb{R}^n$ and a pressure function π: $\Omega \to \mathbb{R}$ such that

$$\left.\begin{array}{lll} -\mathrm{div}\,\{T(\varepsilon(v))\} + \nabla\pi = g & \text{in} & \Omega, \\ \mathrm{div}\, v = 0 & \text{on} & \Omega, \\ v = u_0 & \text{on} & \partial\Omega. \end{array}\right\} \tag{1.1}$$

Here T is the gradient of a potential f: $\mathbb{S}^{n\times n} \to \mathbb{R}$ defined on the space of all symmetric matrices of order n, which is assumed to be of class C^2, and $\varepsilon(v) = (Dv + (Dv)^T)/2$ denotes the symmetric gradient of v. In [5] the first two authors consider the anisotropic (i.e. non-uniformly elliptic) case, that is, the potential f satisfies

$$\lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \leq D^2 f(\varepsilon)(\sigma,\sigma) \leq \Lambda(1 + |\varepsilon|^2)^{\frac{q-2}{2}}|\sigma|^2 \tag{1.2}$$

for all ε, $\sigma \in \mathbb{S}^{n\times n}$ with positive constants λ, Λ and with exponents $1 < p \leq q < \infty$. As outlined in [5], (1.1) is equivalent to a variational problem. Thus, from the point of view of regularity theory, it is reasonable just to study the local minimizers u of the energy

$$J[u] = \int_\Omega f(\varepsilon(u))\, dx$$

within the class

$$\mathbb{K} = \{u \in W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^n) : \mathrm{div}\, u = 0\}$$

and to neglect the unproblematic volume forces g. A detailed discussion is given in Section 2 of [5]; here we just recall the main results of this paper.

(i). If (1.2) holds with $q < p(1 + 2/n)$, then the local J-minimizers are partially of class $C^{1,\alpha}$ in the interior of Ω.

(ii). Let $n = 2$ together with $q = 2$. Then singular points can be excluded, i.e. the first derivatives of local minimizers are Hölder continuous in the interior of Ω.

As it stands, the 2d-case suffers from one essential restriction on the data: if (1.2) holds with $q < 2$, we way just replace q with $\tilde{q} := 2$ on the right hand side of (1.2) and obtain the result (ii). Of course, such a trivial replacement is not possible in the case $q > 2$. It seems that the argument in [5] does not work for this case (compare to the short discussion given in [5] at the end of Section 6). The reason is that one of the main tools – the lemma of Frehse and Seregin (see [13]) – does not apply to this case. In the paper at hand, we prove a new lemma, formulated as Lemma 1.2 below, and with the help of this lemma we obtain

Theorem 1.1. *Let $n = 2$. Suppose that f satisfies (1.2) with $q < \min(2p, p + 2)$. Then any local J-minimizer is of class $C^{1,\alpha}(\Omega;\mathbb{R}^2)$ for any $0 < \alpha < 1$.*

Let us remark that the condition $q < p + 2$ is well known in the study of problems in the calculus of variations with non-standard growth. The condition was first introduced in [7]. In [4] it turned out to be the appropriate assumption to establish regularity results for bounded solutions. A detailed discussion can be found in Chapter 5 of [4]. Moreover, let us remark that if we assume $q \le 2$ (as in [5]), then we have $\min(2p, p+2) = 2p$.
In Section 4, we will further apply Lemma 1.2 to the regularity theory of the stationary flow of generalized Newtonian fluids and give an extension (in the spirit of Theorem 1.1) of the 2d-results from [3] where the convective term $[\nabla v]v$ is included in the first line of (1.1). Moreover, we reinvestigate two-dimensional electrorheological fluids discussed in [10] and [6] and remove the restriction $p(x) \le 2$ imposed in [6].
Now we formulate our lemma on the higher integrability of functions.

Lemma 1.2. *Let $d > 1, \beta > 0$ be two constants. With a slight abuse of notation, let f, g, h now denote any non-negative functions in $\Omega \subset \mathbb{R}^n$ satisfying*

$$f \in L^d_{\mathrm{loc}}(\Omega), \quad \exp(\beta g^d) \in L^1_{\mathrm{loc}}(\Omega), \quad h \in L^d_{\mathrm{loc}}(\Omega).$$

Suppose that there is a constant $C > 0$ such that

$$\left(\fint_B f^d\, dx\right)^{\frac{1}{d}} \le C \fint_{2B} fg\, dx + C\left(\fint_{2B} h^d\, dx\right)^{\frac{1}{d}} \tag{1.3}$$

holds for all balls $B = B_r(x)$ with $2B = B_{2r}(x) \Subset \Omega$. Then there is a real number $c_0 = c_0(n,d,C) > 0$ such that if $h^d \log^{c_0\beta}(e+h) \in L^1_{\mathrm{loc}}(\Omega)$, then the same is true for f. Moreover, for all balls B as above we have

$$\begin{aligned}\fint_B f^d \log^{c_0\beta}\left(e + \frac{f}{||f||_{d,2B}}\right) dx \le & c\left(\fint_{2B} \exp(\beta g^d)\, dx\right)\left(\fint_{2B} f^d\, dx\right) \\ & + c \fint_{2B} h^d \log^{c_0\beta}\left(e + \frac{h}{||f||_{d,2B}}\right) dx,\end{aligned} \tag{1.4}$$

where $c = c(n,d,\beta,C) > 0$ and $||f||_{d,2B} = (\fint_{2B} f^d\, dx)^{1/d}$.

Let us give several comments on Lemma 1.2. First, if we assume that the function g in Lemma 1.2 is bounded, then (1.3) is the well-known (weak) reverse Hölder's inequality with a nonhomogeneous term involving h. The celebrated Gehring Lemma [14] then shows that the function f actually enjoys higher integrability, $f \in L^{d+\epsilon}_{\mathrm{loc}}(\Omega)$ for some $\epsilon > 0$. We refer to the monographs [15] and [19] for detailed discussions of Gehring's lemma and its applications in analysis. Second, our assumption on the function g is exponential integrability. In this case, we cannot expect the same sort of higher integrability as we obtain from Gehring's lemma. We merely can expect a very slight degree of improved regularity. Precisely, Lemma 1.2 shows that the scale of improved degree is logarithmic. Moreover, the lemma gives a precise analysis how the degree of improved regularity depends on β, i.e. on the constant occurring in the exponential integrability condition on g. The following example shows that Lemma 1.2 just can be improved by finding the precise value of c_0. Let $d = (n+1)/n$ and let

$$\begin{aligned} f^d(x) &= \frac{\beta}{|x|^n}\left(\log\frac{1}{|x|}\log\log\frac{1}{|x|}\right)^{-\beta-1}\left(1 + \log\log\frac{1}{|x|}\right), \\ g^d(x) &= \frac{n}{\beta}\left(\log\frac{1}{|x|}\right)\frac{\log\log\frac{1}{|x|}}{1+\log\log\frac{1}{|x|}}, \\ h(x) &= 0 \end{aligned}$$

be defined on $B_{e^{-e}}(0)$. Then these functions satisfy all integrability assumptions in Lemma 1.2 and (1.3) holds with a constant $C = C(n) > 0$. It is easy to check that $f^d \log^{c_0\beta}(e+f) \in L^1$ with $c_0 < 1$, but not with $c_0 > 1$.

The idea for the proof of Lemma 1.2 originates in [11], where similar arguments are used for the study of the regularity properties of mappings of finite distortion. Lemma 1.2 and Corollary 1.3 given below should be considered as higher integrability results in the framework of Orlicz spaces. We remark that the nice paper [18] largely extends Gehring's lemma to Orlicz spaces but unfortunately our hypotheses do not fit into the setting of [18]. We therefore give a separate proof of Lemma 1.2 which is based on a modification of Gehring's original work.

Let us finally compare Lemma 1.2 with the Frehse-Seregin lemma. First, our assumption on g is weaker. Second, in the Frehse-Seregin lemma the first integral of the right hand side of (1.3) has to be replaced by an integral w.r.t. the annulus $2B \setminus B$, such that a delicate "hole-filling" argument for the proof is possible. Third, the Frehse-Seregin lemma shows the Dirichlet growth of functions, whereas Lemma 1.2 gives the higher integrability of functions, which implies the Dirichlet growth. With a bit of work, we have from Lemma 1.2

Corollary 1.3. *Suppose that f, g, h are the same as in Lemma 1.2 and that (1.3) is true for all balls $2B \Subset B_1(0)$. Suppose also that $h^d \log^{c_0\beta}(e+h) \in L^1_{\mathrm{loc}}(B_1(0))$, where c_0 is as in Lemma 1.2. Then*

$$\int_E f^d \, dx \le c \log^{-c_0\beta}\left(e + \frac{1}{|E|}\right) \tag{1.6}$$

for all measurable set $E \subset B_{1/2}(0)$, where the constant $c > 0$ depends only on n, d, C, β, f, h but not on the set E.

We remark here that on one hand, our conclusion (1.6) is stronger than that in the Frehse-Seregin lemma, since it holds for any measurable sets other than the balls. On the other hand, our conclusion (1.6) is weaker. In the Frehse-Seregin lemma, (1.6) is true with any exponent on the right hand side other than $c_0\beta$. The reason is that we have weaker assumptions on the function g.

The paper is organized as follows: the proof of Theorem 1.1 is given in Section 2. Here we first prove a Caccioppoli-type inequality for local minimizers in a standard way. We refer to [5], also for the approximation argument. Second, we apply Hölder's inequality and Sobolev-Poincaré's inequality to get a reverse inequality. In order to make this reverse inequality satisfy the assumptions of Lemma 1.2, we require the condition $q < p + 2$. Actually, in this step, we have to digress from [5], Section 6 (again compare to the discussion at the end of Section 6 of [5]). Finally, thanks to Lemma 1.2, we get the higher integrability of a suitable function which is then enough to finish the proof of Theorem 1.1.

The proof of Lemma 1.2 is given in Section 3 following the ideas outlined above.

In Section 4, we discuss the improvements of the regularity of stationary 2d-flows, where we concentrate on the study of "anisotropic" generalized Newtonian fluids with non-vanishing convective term as well as electrorheological fluids.

Our notation is standard: $B = B_r = B_r(x)$ is a ball in $\mathbb{R}^n$ and $2B = B_{2r}(x)$. $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{R}^n$. $\fint_E w \, dx = \frac{1}{|E|}\int_E w \, dx$ is the average integral of w over the set E. The Hardy-Littlewood maximal function Mg of a function $g \in L^1(\mathbb{R}^n)$ is defined as

$$Mg(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| \, dy.$$

For the definitions of the standard Lebesgue and Sobolev spaces, like $L^d_{(\mathrm{loc})}$, $W^k_{p(,\mathrm{loc})}$, $\overset{\circ}{W}{}^k_p$, we refer to [2].

2 Proof of Theorem 1.1

Suppose that $u \in \mathbb{K}$ is a local J-minimizer. In [5] the properties of u have been investigated via a suitable local regularization procedure. Since we need certain inequalities – stated in [5] only for the solutions of the regularized problems – we recall the terminology used in [5]. Fix a disc $B_{2R} = B_{2R}(\bar{x}) \Subset \Omega$. For a sequence $\{u_m\}$ of mollifications of u we define

$$\begin{aligned} \delta_m &= \left(1 + m + ||\varepsilon(u_m)||^{2q}_{L^q(B_{2R})}\right)^{-1}, \\ f_m(\varepsilon) &= f(\varepsilon) + \delta_m(1+|\varepsilon|^2)^{q/2}, \quad \varepsilon \in \mathbb{S}^{2\times 2}, \end{aligned} \tag{2.1}$$

and denote by v_m the unique solution of the minimization problem

$$\int_{B_{2R}} f_m(\varepsilon(w))\, dx \to \min$$

in the class $u_m + \overset{\circ}{W}{}^1_p(B_{2R};\mathbb{R}^2)$ subject to $\operatorname{div} w = 0$. According to Lemma 4.4 and Corollary 4.2 of [5], we know that

$$\sup_m ||v_m||_{W^1_t(B_r)} < \infty \tag{2.2}$$

for any radius $r < 2R$ and any finite exponent t, and that $\{v_m\}$ converges weakly in each class $W^1_t(B_r;\mathbb{R}^2)$ to the function u (by combining (2.2) with Lemma 4.1 ii) of [5]). We remark here that the proof of (2.2) just requires the assumption $q < 2p$. Let $B_{2r}(x_0) \Subset B_{2R}$ and choose a cut-off function $\eta \in C^\infty_0(B_{2r}(x_0))$ such that $\eta = 1$ in $B_r(x_0), \eta \geq 0$, and $|\nabla \eta| \leq 4/r$. We further let

$$H_m = \left(D^2 f_m(\varepsilon(v_m))(\partial_k \varepsilon(v_m), \partial_k \varepsilon(v_m))\right)^{1/2}$$

and recall (see (6.2) of [5]) that

$$\int_{B_\rho(\bar{x})} H_m^2\, dx \leq c(\rho) < \infty \tag{2.3}$$

for all $\rho < 2R$.

Next we recall (6.1) of [5] and use the estimates given after (4.11) of [5] to get from (6.1) of [5] the starting inequality

$$\int_{B_r(x_0)} H_m^2\, dx \leq c r^{-1} \int_{B_{2r}(x_0)} H_m h_m |\nabla v_m - Q|\, dx, \tag{2.4}$$

where

$$h_m = (1 + |\varepsilon(v_m)|^2)^{\frac{q-2}{4}}$$

and Q is an arbitrary 2×2 matrix, not necessarily symmetric. To estimate the integral on the right hand side of (2.4), we apply Hölder's and Sobolev-Poincaré's inequality to get ($\gamma = 4/3$)

$$\begin{aligned} \fint_{B_r(x_0)} H_m^2\, dx &\leq c r^{-1} \fint_{B_{2r}(x_0)} H_m h_m |\nabla v_m - Q|\, dx \\ &\leq c \left(\fint_{B_{2r}(x_0)} (H_m h_m)^\gamma\, dx\right)^{1/\gamma} \frac{1}{r} \left(\fint_{B_{2r}(x_0)} |\nabla v_m - Q|^4\, dx\right)^{1/4} \\ &\leq c \left(\fint_{B_{2r}(x_0)} (H_m h_m)^\gamma\, dx\right)^{1/\gamma} \left(\fint_{B_{2r}(x_0)} |\nabla^2 v_m|^\gamma\, dx\right)^{1/\gamma}, \end{aligned} \tag{2.5}$$

provided we choose $Q = \fint_{B_{2r}(x_0)} \nabla v_m \, dx$. Observe that we have the identity

$$\partial_j \partial_k v^i = \partial_j \varepsilon_{ik}(v) + \partial_k \varepsilon_{ij}(v) - \partial_i \varepsilon_{jk}(v),$$

hence

$$|\nabla^2 v_m| \le c |\nabla \varepsilon(v_m)|$$

and from (1.2) it follows that

$$|\nabla \varepsilon(v_m)| \le c H_m (1 + |\varepsilon(v_m)|^2)^{\frac{2-p}{4}}.$$

Thus (2.5) implies

$$\fint_{B_r(x_0)} H_m^2 \, dx \le c \left(\fint_{B_{2r}(x_0)} (H_m h_m)^\gamma \, dx \right)^{1/\gamma} \left(\fint_{B_{2r}(x_0)} H_m^\gamma (1 + |\varepsilon(v_m)|^2)^{\frac{2-p}{4}\gamma} \, dx \right)^{1/\gamma},$$

hence

$$\left(\fint_{B_r(x_0)} H_m^2 \, dx \right)^{1/2} \le c \left(\fint_{B_{2r}(x_0)} (H_m \tilde{h}_m)^\gamma \, dx \right)^{1/\gamma}, \tag{2.6}$$

where

$$\tilde{h}_m := \max\left\{ h_m, (1 + |\varepsilon(v_m)|^2)^{\frac{2-p}{4}} \right\}.$$

Inequality (2.6) is valid for any disc $B_{2r}(x_0) \Subset B_{2R}$ with constant c independent of m and $B_{2r}(x_0)$. With these preliminaries it is possible to apply Lemma 1.2 with the choices

$$d = 2/\gamma = 3/2, \quad f = H_m^\gamma, \quad g = \tilde{h}_m^\gamma, \quad h = 0,$$

provided that $\exp(\beta g^d) = \exp(\beta \tilde{h}_m^2) \in L^1_{\mathrm{loc}}(B_{2R})$ for some $\beta > 0$. Actually, we claim that

$$\int_{B_\rho} \exp(\beta \tilde{h}_m^2) \, dx \le c(\rho, \beta) < \infty \tag{2.7}$$

for any $\beta > 0$ and any $\rho < 2R$, where $B_\rho = B_\rho(\bar{x})$. Indeed, by (4.20) of [5], we know that the sequence

$$\phi_m = (1 + |\varepsilon(v_m)|^2)^{p/4}$$

is uniformly bounded in $W^1_{2,\mathrm{loc}}(B_{2R})$. By Trudinger's inequality (see e.g. Theorem 7.15 of [16]), this implies that

$$\int_{B_\rho} \exp(\beta_0 \phi_m^2) \, dx \le c(R) < \infty$$

where $\beta_0 > 0$ depends only on the uniform bound of the $W^1_2(B_\rho)$ norms of ϕ_m, and also that

$$\int_{B_\rho} \exp(\beta \phi_m^{2-\kappa}) \, dx \le c(\rho, \beta, \kappa) < \infty \tag{2.8}$$

for any $1 > \kappa > 0$, $\beta > 0$. On the other hand, we have

$$\tilde{h}_m^2 \le \phi_m^{2-\kappa} \tag{2.9}$$

for some small $\kappa = \kappa(p, q) > 0$. Indeed, from the assumption $q < p + 2$ it follows that

$$h_m^2 = \left(1 + |\varepsilon(v_m)|^2\right)^{\frac{q-2}{2}} \le \left(1 + |\varepsilon(v_m)|^2\right)^{\frac{p}{4}(2-\kappa)} = \phi_m^{2-\kappa}.$$

Clearly, since $p > 1$,

$$\left(1+|\varepsilon(v_m)|^2\right)^{\frac{2-p}{2}} \le \left(1+|\varepsilon(v_m)|^2\right)^{\frac{p}{4}(2-\kappa)}.$$

Thus (2.9) follows and (2.7) is a consequence of (2.8) and (2.9).
Lemma 1.2 implies that $H_m^2 \log^{c_0\beta}(e+H_m) \in L^1_{\mathrm{loc}}(B_{2R})$ for any $\beta > 0$, where c_0 is the constant in Lemma 1.2. Moreover by (1.4), we have

$$\int_{B_\rho} H_m^2 \log^{c_0\beta}(e+H_m)\,dx \le c(\beta,\rho) < \infty \tag{2.10}$$

for all $\rho < 2R$. Here we used (2.3) and (2.7).
As in [5], we let $\sigma_m = Df_m(\varepsilon(v_m))$ and observe (compare (1.2))

$$|\nabla\sigma_m| \le cH_m(1+|\varepsilon(v_m)|^2)^{\frac{q-2}{4}} = cH_m h_m.$$

We will show that (2.10) and (2.7) imply the inequality

$$\int_{B_R} |\nabla\sigma_m|^2 \log^\alpha(e+|\nabla\sigma_m|)\,dx \le c(R,\alpha) < \infty \tag{2.11}$$

for any $\alpha > 0$. To prove (2.11), we need the following elementary inequality. Let $a, b \ge 0$.Then for any $\alpha > 0$, there is a constant $c(\alpha) > 0$ such that

$$(ab)^2\log^\alpha(e+ab) \le 2^\alpha a^2 \log^{\alpha+2}(e+a) + c(\alpha)\exp(6b), \tag{2.12}$$

which can be easily proven by considering two cases: $b \le \log(e+a)$ and $b > \log(e+a)$. Thus

$$\begin{aligned}
\int_{B_R} |\nabla\sigma_m|^2 \log^\alpha(e+|\nabla\sigma_m|)\,dx &\le \int_{B_R} (cH_m h_m)^2 \log^\alpha(e+cH_m h_m)\,dx \\
&\le c(\alpha)\int_{B_R} H_m^2 \log^{\alpha+2}(e+H_m)\,dx \\
&\quad + c(\alpha)\int_{B_R} \exp(6ch_m)\,dx \\
&\le c(R,\alpha),
\end{aligned}$$

where the first inequality follows from the fact that the function $y^2\log^\alpha(e+y)$ is an increasing function in $[0,\infty)$, the second inequality follows from (2.12) and the final one is a consequence of (2.10) and (2.7). This proves (2.11). If $\alpha > 1$, (2.11) shows that the tensors σ_m are continuous uniformly w.r.t. m, see e.g. [20] (in particular Example 5.3). Note that it is also possible to argue with the help of Corollary 1.3 and a lemma due to Frehse (see [12], p. 287).
Now we may argue as in [5], Section 6, compare also to [3], Corollary 5.1, to deduce first the Hölder continuity of $\varepsilon(u)$, which then implies the Hölder continuity of ∇u. The proof of Theorem 1.1 is finished.

3 Proof of Lemma 1.2

We fix a ball $B_0 = B_{r_0}(x_0) \Subset \Omega$. We will prove that (1.4) is true for $2B = B_0$. Notice that (1.3) and (1.4) are invariant under replacing f, h, g by cf, ch, g, respectively, for any constant $c > 0$. Thus, we may assume that

$$\int_{B_0} f^d\,dx = 1, \tag{3.1}$$

otherwise we argue with f replaced by $f/(\int_{B_0} f^d dx)^{1/d}$ and with h replaced by $h/(\int_{B_0} f^d dx)^{1/d}$. Let us introduce the auxiliary functions defined in $\mathbb{R}^n$ by

$$\begin{aligned} \tilde{f}(x) &= d(x)^{n/d} f, \\ w(x) &= \chi_{B_0}(x), \\ \tilde{h}(x) &= d(x)^{n/d} h(x), \end{aligned} \tag{3.2}$$

where $d(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus B_0)$ and χ_E is the characteristic function of the set E. We claim that

$$\begin{aligned} \left(\fint_B \tilde{f}^d \, dx \right)^{\frac{1}{d}} \leq & c(n,d,C) \fint_{2B} \tilde{f} g \, dx + c(n,d,C) \left(\fint_{2B} \tilde{h}^d \, dx \right)^{\frac{1}{d}} \\ & + c(n,d) \left(\fint_{2B} w \, dx \right)^{\frac{1}{d}} \end{aligned} \tag{3.3}$$

for all balls $B \subset \mathbb{R}^n$. Indeed, we may assume that B intersects B_0 since otherwise (3.3) is trivial. Our derivation of (3.3) splits naturally into two cases.
Case 1. We assume that $3B \subset B_0$. By an elementary geometric consideration we find that

$$\max_{x \in B} d(x) \leq 4 \min_{x \in 2B} d(x).$$

Applying (1.3) yields

$$\begin{aligned} \left(\fint_B \tilde{f}^d \, dx \right)^{\frac{1}{d}} & \leq \max_B d(x)^{n/d} \left(\fint_B f^d \, dx \right)^{\frac{1}{d}} \\ & \leq 4^{n/d} C \min_{2B} d(x)^{n/d} \left[\fint_{2B} f g \, dx + \left(\fint_{2B} h^d \, dx \right)^{\frac{1}{d}} \right] \\ & \leq 4^{n/d} C \left[\fint_{2B} \tilde{f} g \, dx + \left(\fint_{2B} \tilde{h}^d \, dx \right)^{\frac{1}{d}} \right]. \end{aligned}$$

Case 2. We assume that $3B$ is not contained in B_0 and recall that B intersects B_0. We have that

$$\max_{x \in B} d(x) \leq \max_{x \in 2B} d(x) \leq c(n) |2B \cap B_0|^{\frac{1}{n}}.$$

Hence we conclude that

$$\begin{aligned} \left(\fint_B \tilde{f}^d \, dx \right)^{\frac{1}{d}} & \leq \max_B d(x)^{n/d} \left(\frac{1}{|B|} \int_{B \cap B_0} f^d \, dx \right)^{\frac{1}{d}} \\ & \leq c(n,d) \left(\frac{|2B \cap B_0|}{|B|} \int_{B_0} f^d \, dx \right)^{\frac{1}{d}} \\ & \leq c(n,d) \left(\frac{1}{|2B|} \int_{2B} w \, dx \right)^{\frac{1}{d}}, \end{aligned}$$

where we used (3.1). Combining these two cases proves inequality (3.3).
Since (3.3) is true for all balls $B \subset \mathbb{R}^n$, we have the following point-wise inequality for the maximal functions. For all $y \in \mathbb{R}^n$,

$$M(\tilde{f}^d)(y)^{\frac{1}{d}} \leq c M(\tilde{f} g)(y) + c M(\tilde{h}^d)(y)^{\frac{1}{d}} + c M(w)(y)^{\frac{1}{d}},$$

from which it follows that for $\lambda > 0$

$$
\begin{aligned}
|\{x \in \mathbb{R}^n : M(\tilde{f}^d)(x) > \lambda^d\}| \le & |\{x \in \mathbb{R}^n : cM(\tilde{f}g)(x) > \lambda\}| \\
& + |\{x \in \mathbb{R}^n : cM(\tilde{h}^d)(x) > \lambda^d\}| \\
& + |\{x \in \mathbb{R}^n : cM(w)(x) > \lambda^d\}|,
\end{aligned} \tag{3.6}
$$

where $c = c(n, d, C) > 0$. We recall that $w(x) = \chi_{B_0}(x)$. So $M(w)(x) \le 1$ in $\mathbb{R}^n$, and then the set $\{x \in \mathbb{R}^n : cM(w)(x) > \lambda^d\}$ is empty for $\lambda > \lambda_1 = \lambda_1(n, d, C)$. Hence

$$
\begin{aligned}
|\{x \in \mathbb{R}^n : M(\tilde{f}^d)(x) > \lambda^d\}| \le & |\{x \in \mathbb{R}^n : cM(\tilde{f}g)(x) > \lambda\}| \\
& + |\{x \in \mathbb{R}^n : cM(\tilde{h}^d)(x) > \lambda^d\}|
\end{aligned}
$$

for all $\lambda > \lambda_1$. Now applying Proposition 2.1 in [11] yields

$$
\int_{\tilde{f}>\lambda} \tilde{f}^d \, dx \le c(n)\lambda^{d-1} \int_{c\tilde{f}g>\lambda} \tilde{f}g \, dx + c(n) \int_{c\tilde{h}>\lambda} \tilde{h}^d \, dx \tag{3.8}
$$

for all $\lambda > \lambda_1$. We may assume that the constant c in (3.8) is bigger than one.
Let $\alpha > 0$ be a constant, which will be chosen later and set

$$
\Psi(\lambda) = \frac{d-1}{\alpha} \log^{\alpha} \lambda + \log^{\alpha-1} \lambda.
$$

Notice that

$$
\Phi(\lambda) := \frac{d}{d\lambda} \Psi(\lambda) = \frac{d-1}{\lambda} \log^{\alpha-1} \lambda + \frac{\alpha-1}{\lambda} \log^{\alpha-2} \lambda > 0
$$

for all $\lambda > \lambda_2 = \exp(1/(d-1))$, and that

$$
\lambda^{d-1}\Phi(\lambda) = \frac{d}{d\lambda} \left(\lambda^{d-1} \log^{\alpha-1} \lambda \right).
$$

We multiply both sides of (3.8) by $\Phi(\lambda)$, and integrate with respect to λ over (λ_0, j) for $\lambda_0 = \max(\lambda_1, \lambda_2)$ and j large, and finally change the order of the integration to obtain that

$$
\begin{aligned}
\int_{\tilde{f}>\lambda_0} \tilde{f}^d \int_{\lambda_0}^{\min(\tilde{f},j)} \Phi(\lambda) \, d\lambda dx \le & c(n) \int_{c\tilde{f}g>\lambda_0} \tilde{f}g \int_{\lambda_0}^{\min(c\tilde{f}g,j)} \lambda^{d-1}\Phi(\lambda) \, d\lambda dx \\
& + c(n) \int_{c\tilde{h}>\lambda_0} \tilde{h}^d \int_{\lambda_0}^{\min(c\tilde{h},j)} \Phi(\lambda) \, d\lambda dx,
\end{aligned}
$$

that is,

$$
\begin{aligned}
\int_{\tilde{f}>\lambda_0} \left(\Psi(\min(\tilde{f}, j)) - \Psi(\lambda_0) \right) \tilde{f}^d \, dx \le & c(n) \int_{c\tilde{f}g>\lambda_0} G(x) \, dx \\
& + c(n) \int_{c\tilde{h}>\lambda_0} \Psi(\min(c\tilde{h}, j))\tilde{h}^d \, dx,
\end{aligned} \tag{3.10}
$$

where

$$
G(x) = \tilde{f}g \min(c\tilde{f}g, j)^{d-1} \log^{\alpha-1} \min(c\tilde{f}g, j).
$$

Here the constant c is bigger than 1. Taking into account the normalization (3.1), we have

$$
\int_{\tilde{f}>\lambda_0} \Psi(\lambda_0)\tilde{f}^d \, dx \le c(n, d, C, \alpha)|B_0|. \tag{3.11}
$$

Thus, it follows from (3.10) and (3.11) that

$$
\begin{aligned}
\frac{d-1}{\alpha}\int_{\tilde f>\lambda_0}\tilde f^d\log^\alpha\min(\tilde f,j)\,dx \le{}& c(n)\int_{c\tilde f g>\lambda_0}G(x)\,dx\\
&+c(n,d,C,\alpha)\int_{B_0}\tilde h^d\log^\alpha(e+\tilde h)\,dx+c(n,d,C,\alpha)|B_0|\\
\le{}& c(n)\int_{c\tilde f g>\lambda_0,\tilde f>\lambda_0}G(x)\,dx+c(n)\int_{c\tilde f g>\lambda_0,\tilde f\le\lambda_0}G(x)\,dx\\
&+c(n,d,C,\alpha)\int_{B_0}\tilde h^d\log^\alpha(e+\tilde h)\,dx+c(n,d,C,\alpha)|B_0|\\
\le{}& c(n)\int_{c\tilde f g>\lambda_0,\tilde f>\lambda_0}G(x)\,dx+c(n,d,C,\alpha)\int_{B_0}\tilde h^d\log^\alpha(e+\tilde h)\,dx\\
&+c(n,d,C,\alpha,\beta)\int_{B_0}\exp(\beta g^d)\,dx,
\end{aligned}
\tag{3.12}
$$

where in the last step, we used the facts that $G(x)$ is supported in B_0 and that $G(x)\le c(n,d,C,\alpha,\beta)\exp(\beta g^d)$ in the set where $\tilde f\le\lambda_0$.
In the remaining part of the proof, we will choose a suitable constant $\alpha>0$ such that the first integral in the right hand side of (3.12) can be absorbed in the left. We claim that on the set $\{c\tilde f g>\lambda_0,\tilde f>\lambda_0\}$

$$
G(x)\le\frac{c(n,d,C)}{\beta}\tilde f^d\log^\alpha\min(\tilde f,j)+c(n,d,C,\alpha,\beta)\exp(\beta g^d). \tag{3.13}
$$

for any $j>\exp(\alpha/(d-1))$. We first finish the proof of the lemma and then prove (3.13). Letting $\alpha=\beta(d-1)/2c(n)c(n,d,C)$, (3.12) becomes

$$
\begin{aligned}
\int_{\tilde f>\lambda_0}\tilde f^d\log^\alpha\min(\tilde f,j)\,dx\le{}& c(n,d,C,\beta)\int_{B_0}\exp(\beta g^d)\,dx\\
&+c(n,d,C,\beta)\int_{B_0}\tilde h^d\log^\alpha(e+\tilde h)\,dx.
\end{aligned}
$$

Now letting $j\to\infty$, by the monotone convergence theorem, we end up with

$$
\int_{\tilde f>\lambda_0}\tilde f^d\log^\alpha\tilde f\,dx\le c(n,d,C,\beta)\left(\int_{B_0}\exp(\beta g^d)\,dx+\int_{B_0}\tilde h^d\log^\alpha(e+\tilde h)\,dx\right),
$$

that is,

$$
\begin{aligned}
\int_{B_0}d(x)^n f^d\log^\alpha(e+d(x)^{n/d}f)\,dx={}&\int_{B_0}\tilde f^d\log^\alpha(e+\tilde f)\,dx\\
\le{}& c(n,d,C,\beta)\left(\int_{B_0}\exp(\beta g^d)\,dx+\int_{B_0}\tilde h^d\log^\alpha(e+\tilde h)\,dx\right).
\end{aligned}
\tag{3.15}
$$

Noticing that in σB_0 with $0<\sigma<1$ we have $d(x)^n\ge(1-\sigma)^n r_0^n\ge c(n,\sigma)|B_0|$, and taking account of the normalization (3.1), we arrive at ($c_0:=\alpha/\beta=(d-1)/2c(n)c(n,d,C)$)

$$
\begin{aligned}
⨍_{\sigma B_0}f^d\log^{c_0\beta}\left(e+\frac{f}{||f||_{d,B_0}}\right)dx\le{}& c\left(⨍_{B_0}\exp(\beta g^d)\,dx\right)\left(⨍_{B_0}f^d\,dx\right)\\
&+c⨍_{B_0}h^d\log^{c_0\beta}\left(e+\frac{h}{||f||_{d,B_0}}\right)dx,
\end{aligned}
$$

which proves (1.4).

To finish the proof of the lemma, it remains to prove (3.13). To this end, first, we notice that we may assume that $g(x) \geq 1$ in Ω; otherwise, we may replace $g(x)$ by $g(x)+1$. Thus, since $c \geq 1$,

$$\min(c\tilde{f}g, j) \leq cg\min(\tilde{f}, j).$$

We also notice that $\lambda^{d-1}\log^{\alpha-1}\lambda$ is an increasing function of λ when $\lambda > \lambda_2 = \exp(1/(d-1))$. Thus on the set $E_1 = \{x \in \mathbb{R}^n : c\tilde{f}g > \lambda_0\}$ we have that

$$\begin{aligned} G(x) &= \tilde{f}g\min(c\tilde{f}g, j)^{d-1}\log^{\alpha-1}\min(c\tilde{f}g, j) \\ &\leq c\tilde{f}\min(\tilde{f}, j)^{d-1}g^d\log^{\alpha-1}(cg\min(\tilde{f}, j)). \end{aligned} \tag{3.17}$$

We continue to estimate the right hand side of (3.17) by the elementary inequality

$$ab \leq a\log a + \exp(2b)$$

for $a \geq 1, b \geq 0$. On the set $E_2 = \{x \in \mathbb{R}^n : \tilde{f} > \lambda_0\}$, we have (by letting $a = \min(\tilde{f}, j)^{d-1}, b = \beta g^d/2c(d)$)

$$\min(\tilde{f}, j)^{d-1}g^d \leq \frac{2c(d)}{\beta}\left(\min(\tilde{f}, j)^{d-1}\log\min(\tilde{f}, j)^{d-1} + \exp(\frac{\beta}{c(d)}g^d)\right), \tag{3.18}$$

where $c(d) > 0$ is a large constant, which will be chosen later. Combining (3.17) and (3.18), we have on the set $E_1 \cap E_2$ that

$$\begin{aligned} G(x) &\leq \frac{c}{\beta}\tilde{f}\left(\min(\tilde{f}, j)^{d-1}\log\min(\tilde{f}, j) + \exp(\frac{\beta}{c(d)}g^d)\right)\log^{\alpha-1}(cg\min(\tilde{f}, j)) \\ &\leq \frac{c}{\beta}\tilde{f}\left(\min(\tilde{f}, j)^{d-1} + \exp(\frac{\beta}{c(d)}g^d)\right)\log^{\alpha}(cg\min(\tilde{f}, j)). \end{aligned} \tag{3.19}$$

Next, we have that

$$\log^\alpha(cg\min(\tilde{f}, j)) \leq 2\log^\alpha\min(\tilde{f}, j) + c(\alpha)\log^\alpha(cg), \tag{3.20}$$

which follows from the elementary inequality

$$(x+y)^\alpha \leq 2x^\alpha + c(\alpha)y^\alpha \tag{3.21}$$

for $x > 0, y > 0, \alpha > 0$. Thus by (3.19) and (3.20), we arrive at

$$\begin{aligned} G(x) &\leq \frac{c}{\beta}\tilde{f}(\min(\tilde{f}, j)^{d-1} + \exp(\frac{\beta}{c(d)}g^d))(2\log^\alpha\min(\tilde{f}, j) + c(\alpha)\log^\alpha(cg)) \\ &\leq \frac{c}{\beta}(\tilde{f}^d\log^\alpha\min(\tilde{f}, j) + c(n, d, C, \alpha, \beta)\exp(\beta g^d)), \end{aligned}$$

which proves (3.13). The last inequality holds because of the estimates

$$\tilde{f}\log^\alpha\min(\tilde{f}, j)\exp(\frac{\beta}{c(d)}g^d) \leq \tilde{f}^d\log^\alpha\min(\tilde{f}, j) + c(d, \alpha)\exp(\beta g^d), \tag{3.22}$$

and

$$c(\alpha)\tilde{f}\min(\tilde{f}, j)^{d-1}\log^\alpha(cg) \leq \tilde{f}^d\log^\alpha\min(\tilde{f}, j) + c(n, d, \alpha, \beta)\exp(\beta g^d), \tag{3.23}$$

and

$$c(\alpha)\tilde{f}\exp(\frac{\beta}{c(d)}g^d)\log^\alpha(cg) \leq \tilde{f}^d\log^\alpha\min(\tilde{f}, j) + c(n, d, \alpha, \beta)\exp(\beta g^d), \tag{3.24}$$

valid for the lower order terms. We prove (3.22): if $\exp(\beta g^d/c(d)) \le \tilde{f}^{d-1}$, then the left hand side of (3.22) is bounded from above by the first item in the right hand side. Otherwise, it is bounded from above by the second item in the right hand side, by chosing the constant $c(d)$ large enough, for example, $c(d) = (d+1)/(d-1)$. (3.24) is easily seen to hold on the set under consideration and we finish the proof of Lemma 1.2 by establishing (3.23).
Let us first consider the case $c(\alpha)\log^\alpha(cg) \le \log^\alpha \tilde{f}$. Then the left hand side of (3.23) is bounded from above by $\tilde{f}\min(\tilde{f}, j)^{d-1}\log^\alpha \tilde{f}$, which is bounded from above by $\tilde{f}^d \log^\alpha \min(\tilde{f}, j)$. In fact, this is obviously true if $\tilde{f} \le j$. If $\tilde{f} > j$, this follows from the fact that the function $\lambda^{d-1}\log^{-\alpha}\lambda$ is increasing function when $\lambda > \exp(\alpha/(d-1))$. So the argument works if $j > \exp(\alpha/(d-1))$.
In the case $c(\alpha)\log^\alpha(cg) > \log^\alpha \tilde{f}$ we easily see that the left hand side of (3.23) is bounded from above by the second item in the right hand side. The proof is finished. □

4 Further improvements of the regularity of stationary 2d-flows

4.1 Steady states of anisotropic generalized Newtonian fluids with non-vanishing convective term

We replace the system (1.1) by the boundary value problem

$$\left.\begin{array}{lll} -\mathrm{div}\,\{T(\varepsilon(v))\} + [\nabla v]v + \nabla\pi = g & \text{in} & \Omega, \\ \mathrm{div}\, v = 0 & \text{on} & \partial\Omega, \\ v = 0 & \text{on} & \partial\Omega, \end{array}\right\} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, g is of class $L^\infty(\Omega; \mathbf{R}^2)$ and $T = Df$ for a potential f satisfying condition (1.2). The following result is shown in [3]: suppose that (1.2) holds with

$$p > 6/5, \quad p \le q < 2p. \tag{4.2}$$

Then there exists a function $u \in \overset{o}{W}{}^1_p(\Omega; \mathbf{R}^2)$ such that

$$u \in W^2_{2-\kappa,\mathrm{loc}}(\Omega; \mathbb{R}^2) \text{ for any } 0 < \kappa < 1; \tag{4.3}$$

$$u \text{ satisfies (4.1) a.e.} \tag{4.4}$$

The first claim (4.3) can be found in Remark 1.2 and Corollary 3.1 of [3], (4.4) follows from Theorem 3.1 of [3] together with (4.3). Note that Lemma 4.1 of [3] holds in the two-dimensional case as well, i.e. we have in addition to (4.3) and (4.4)

$$\nabla(1 + |\varepsilon(u)|^2)^{p/4} \in L^2_{loc}(\Omega; \mathbb{R}^2). \tag{4.5}$$

The $C^{1,\alpha}$-regularity of u is also established in [3], however the restriction $q = 2$ is needed there. Now we will apply Lemma 1.2 and prove the following result.

Theorem 4.1. *Suppose that (4.2) holds together with $q < p + 2$. Then the solution u belongs to the space $C^{1,\alpha}(\Omega; \mathbb{R}^2)$ for any $0 < \alpha < 1$.*

The idea of the proof of Theorem 4.1 is similar to that of the proof of Theorem 1.1. Essentially, the difference (and difficulty) comes from the convective term $[\nabla v]v$ in the system (4.1). Since the solution u fixed above itself is locally bounded, we have a nice estimate for this quantity. Nevertheless, the reverse inequality we get here is different from the one in Section 2. More precisely, now this inequality is of type (1.3) with a function h not identically zero. But we can still apply Lemma 1.2 and prove the higher integrability.

Proof of Theorem 4.1. Fix a disc $B_{2R}(\bar{x}) \Subset \Omega$. Let $B_{2r} = B_{2r}(x_0) \Subset B_{2R}(\bar{x})$ and consider a standard cut-off function $\eta \in C_0^\infty(B_{2r})$. The following inequality is shown at the beginning of Section 6 of [3] (being valid also in the case $q > 2$)

$$\begin{aligned}\fint_{B_{2r}} \eta^2 \partial_k \sigma : \partial_k \varepsilon(u)\, dx \leq & -2\fint_{B_{2r}} \eta \partial_k \sigma : [\nabla \eta \odot \partial_k(u - Qx)]\, dx \\ & + \fint_{B_{2r}} \partial_k[u \otimes u] : \varepsilon(\eta^2 \partial_k[u - Qx])\, dx \\ & - \fint_{B_{2r}} g \cdot \partial_k[\eta^2 \partial_k(u - Qx)]\, dx \\ & - 2\fint_{B_{2r}} \eta \partial_k \pi \mathbf{1} : [\nabla \eta \odot \partial_k(u - Qx)]\, dx \\ = & I_1 + I_2 + I_3 + I_4,\end{aligned} \tag{4.6}$$

where $\sigma = Df(\varepsilon(u))$, π denotes the pressure function related to the problem and Q is an arbitrary 2×2 matrix. As in Section 2, we let

$$\begin{aligned} H &= \left(D^2 f(\varepsilon(u))(\partial_k \varepsilon(u), \partial_k \varepsilon(u))\right)^{\frac{1}{2}}; \\ h &= (1 + |\varepsilon(u)|^2)^{\frac{q-2}{4}}; \\ \tilde{h} &= \max\left\{h, (1 + |\varepsilon(u)|^2)^{\frac{2-p}{4}}\right\}. \end{aligned} \tag{4.7}$$

Note that $H \in L^2_{\mathrm{loc}}(\Omega)$. This can be deduced from Lemma 3.3 of [3] by passing to the limit $\delta \to 0$. For the left hand side of inequality (4.6) we have the equality

$$\fint_{B_{2r}} \eta^2 H^2\, dx = \fint_{B_{2r}} \eta^2 \partial_k \sigma : \partial_k \varepsilon(u)\, dx. \tag{4.8}$$

We will estimate each term on the right hand side of (4.6) from above in order to prove

$$\fint_{B_{2r}} \eta^2 H^2\, dx \leq c \left(\fint_{B_{2r}} (H\tilde{h})^\gamma\, dx\right)^{2/\gamma} + c\fint_{B_{2r}} G\, dx, \tag{4.9}$$

where

$$G = (1 + |\nabla u|^2)^t, \quad t = \max\left\{\frac{4-p}{2}, 1\right\}.$$

Starting with the first integral on the right hand side of the inequality (4.6) we argue in the same way as in Section 2 when passing from (2.4) to (2.6). We note that

$$|\nabla \sigma| \leq cHh \tag{4.10}$$

and that

$$|\nabla^2 u| \leq c|\nabla \varepsilon(u)| \leq cH\tilde{h}. \tag{4.11}$$

Thus, as in Section 2, we have ($\gamma = 4/3$)

$$
\begin{aligned}
|I_1| \leq & c \fint_{B_{2r}} \eta |\nabla \sigma| |\nabla \eta| |\nabla u - Qx| \, dx \\
\leq & c r^{-1} \fint_{B_{2r}} H h |\nabla u - Q| \, dx \\
\leq & c \left(\fint_{B_{2r}} (Hh)^\gamma \, dx \right)^{1/\gamma} \frac{1}{r} \left(\fint_{B_{2r}} |\nabla u - Q|^4 \, dx \right)^{1/4} \\
\leq & c \left(\fint_{B_{2r}} (Hh)^\gamma \, dx \right)^{1/\gamma} \left(\fint_{B_{2r}} |\nabla^2 u|^\gamma \, dx \right)^{1/\gamma} \\
\leq & c \left(\fint_{B_{2r}} (H\tilde{h})^\gamma \, dx \right)^{2/\gamma}
\end{aligned}
\tag{4.12}
$$

by choosing $Q = \fint_{B_{2r}} \nabla u \, dx$. To estimate the second integral on the right hand side of (4.6), we use the local boundedness of u and obtain (compare (6.3) and (6.4) of [3])

$$
\begin{aligned}
|I_2| \leq & \kappa \fint_{B_{2r}} \eta^2 H^2 \, dx + c(\kappa) \fint_{B_{2r}} (1 + |\nabla u|^2)^{\frac{4-p}{2}} \, dx \\
& + c \fint_{B_{2r}} \eta |\nabla \eta| |\nabla u| |\nabla u - Q| \, dx,
\end{aligned}
\tag{4.13}
$$

where $\kappa > 0$ is arbitrary. Thus, by letting $\kappa = 1/4$, we can absorb the first integral on the right hand side of (4.13) in the left hand side of (4.6). We will keep the second integral since it is bounded from above by $c \fint_{B_{2r}} G \, dx$, and estimate the third one in the following way:

$$
\begin{aligned}
\fint_{B_{2r}} \eta |\nabla \eta| |\nabla u| |\nabla u - Q| \, dx & \leq \frac{c}{r} \left(\fint_{B_{2r}} |\nabla u - Q|^4 \, dx \right)^{1/4} \left(\fint_{B_{2r}} |\nabla u|^\gamma \, dx \right)^{1/\gamma} \\
& \leq c \left(\fint_{B_{2r}} |\nabla^2 u|^\gamma \, dx \right)^{1/\gamma} \left(\fint_{B_{2r}} |\nabla u|^\gamma \, dx \right)^{1/\gamma} \\
& \leq c \left(\fint_{B_{2r}} (H\tilde{h})^\gamma \, dx \right)^{1/\gamma} \left(\fint_{B_{2r}} |\nabla u|^2 \, dx \right)^{1/2} \\
& \leq c \left(\fint_{B_{2r}} (H\tilde{h})^\gamma \, dx \right)^{2/\gamma} + c \fint_{B_{2r}} |\nabla u|^2 \, dx,
\end{aligned}
\tag{4.14}
$$

where we used Hölder's inequality in the first inequality, Sobolev-Poincaré's inequality in the second one (recall the choice of Q), (4.11) and Hölder's inequality in the third one and Schwarz's inequality in the final one. In particular, the l.h.s. of (4.14) is bounded by terms occuring on the r.h.s. of (4.9).

Next, let us consider I_3, the volume force term. Since g is bounded, we have

$$
\begin{aligned}
|I_3| \leq & c \fint_{B_{2r}} |\eta| |\nabla \eta| |\nabla u - Q| \, dx + c \fint_{B_{2r}} \eta^2 |\nabla^2 u| \, dx \\
\leq & \frac{c}{r} \left(\fint_{B_{2r}} |\nabla u - Q|^4 \, dx \right)^{1/4} + c \left(\fint_{B_{2r}} |\nabla^2 u|^\gamma \, dx \right)^{1/\gamma} \\
\leq & c \left(\fint_{B_{2r}} |\nabla^2 u|^\gamma \, dx \right)^{1/\gamma} \leq c \left(\fint_{B_{2r}} (H\tilde{h})^\gamma \, dx \right)^{1/\gamma} \\
\leq & c \left(\fint_{B_{2r}} (H\tilde{h})^\gamma \, dx \right)^{2/\gamma} + c,
\end{aligned}
\tag{4.15}
$$

where as before, we used Hölder's inequality, Sobolev-Poincaré's inequality and (4.11). Thus I_3 is also controlled by the right hand side of (4.9). Finally we discuss I_4, the pressure term. Actually, we can "ignore" this integral, as explained in Section 6 of [3]. We have

$$\begin{aligned}|I_4| \leq & c \fint_{B_{2r}} \eta |\nabla \sigma||\nabla \eta||\nabla u - Qx| \, dx \\ & + c \fint_{B_{2r}} \eta |\nabla (u \otimes u)||\nabla \eta||\nabla u - Q| \, dx \\ & + c \fint_{B_{2r}} \eta |g||\nabla \eta||\nabla u - Q| \, dx \\ \leq & c \left(\fint_{B_{2r}} (H\tilde{h})^\gamma \, dx \right)^{2/\gamma} + c \fint_{B_{2r}} (1 + |\nabla u|^2) \, dx \end{aligned} \tag{4.16}$$

by the estimates (4.12), (4.14) and (4.15). This finishes the proof of (4.9).
Now we rewrite (4.9) as

$$\left(\fint_{B_r} H^2 \, dx \right)^{\gamma/2} \leq c \fint_{B_{2r}} (H\tilde{h})^\gamma \, dx + c \left(\fint_{B_{2r}} G \, dx \right)^{\gamma/2} \tag{4.17}$$

and apply Lemma 1.2 with the choices

$$d = 2/\gamma = 3/2, \quad f = H^\gamma, \quad g = \tilde{h}^\gamma, \quad h = G^{\gamma/2}.$$

The assumptions of Lemma 1.2 are all satisfied. Indeed, we have

$$\int_{B_{2R}} \exp(\beta g^d) \, dx = \int_{B_{2R}} \exp(\beta \tilde{h}^2) \, dx \leq c(R, \beta) < \infty$$

for any $\beta > 0$ which follows from the same argument as in Section 2, since now we also know that $(1 + |\varepsilon(u)|^2)^{p/4} \in W_2^1(B_{2R}(\bar{x}))$ (recall (4.5)). Thus $g = \tilde{h}^\gamma$ is an admissible choice in Lemma 1.2, the assumptions on f and h clearly hold as well.
Lemma 1.2 implies that $H^2 \log^{c_0 \beta}(e + H_m) \in L^1_{\text{loc}}(B_{2R}(\bar{x}))$ for any $\beta > 0$. We can therefore proceed in the same way as outlined in Section 2 after (2.10) and complete the proof of Theorem 4.1. We omit the details.

4.2 Stationary electrorheological fluids

In this subsection, we discuss the the following system for the velocity field of so-called electrorheological fluids in the stationary case

$$\left. \begin{array}{lll} -\text{div}\,\{S(\varepsilon(v), E)\} + [\nabla v]v + \nabla \pi = g & \text{in} & \Omega, \\ \text{div}\, v = 0 & \text{on} & \Omega, \\ v = 0 & \text{on} & \partial\Omega, \end{array} \right\} \tag{4.18}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, g is a vector field of class $L^\infty(\Omega; \mathbb{R}^2)$, E denotes the smooth electrical field and π stands for the a priori unknown pressure function. A detailed discussion can be found in [24] (see also [23]), monographs dealing with the mechanical or engineering background are, for instance, [8], [17], [21], [22], [25]. If $n \geq 3$, then partial regularity of solutions is established in [1]. Here we consider the case $n = 2$ of planar motions and follow the discussions given in [9], [10] and [6].

The existence of a strong solution $u \in \bigcap_{0<\kappa<1} W^2_{2-\kappa,\mathrm{loc}}(\Omega;\mathbb{R}^2)$ to (4.18) is established in [9], [10] under the following growth and ellipticity conditions on the tensor valued function S: $\mathbb{S}^{2\times 2} \times \mathbb{R}^2 \to \mathbb{S}^{2\times 2}$, $S = (S_{ij})$, $S_{ij} = S_{ij}(\varepsilon, H)$, $i,j = 1,2$,

$$\begin{aligned}
&\frac{\partial S_{ij}}{\partial \varepsilon_{\alpha\beta}}(D,E)B_{\alpha\beta}B_{ij} \geq c(1+|D|^2)^{\frac{p(|E|^2)-2}{2}}|B|^2,\\
&S_{ij}(D,E)D_{ij} \geq c(1+|D|^2)^{\frac{p(|E|^2)-2}{2}}|D|^2,\\
&\left|\frac{\partial S}{\partial \varepsilon}(D,E)\right| \leq C(1+|D|^2)^{\frac{p(|E|^2)-2}{2}},\\
&\left|\frac{\partial S}{\partial H}(D,E)\right| \leq C(1+|D|^2)^{\frac{p(|E|^2)-1}{2}}(1+\log(1+|D|^2))
\end{aligned} \tag{4.19}$$

for all D, $B \in \mathbb{S}^{2\times 2}$ with vanishing trace and for all $E \in \mathbb{R}^2$. Here $p \in C^1(\mathbb{R})$ is a given material function with $6/5 < p_\infty \leq p(|E|^2) \leq p_0 < \infty$ for given numbers p_∞ and p_0.
It is proved in [6] that such a solution is actually of class $C^{1,\alpha}$ for any $0 < \alpha < 1$, provided that $p_0 = 2$. Now we show that the result is still true without this restriction.

Theorem 4.2. *Suppose that $6/5 < p_\infty \leq p(|E|^2) \leq p_0 < \infty$ holds. Suppose further that (4.19) is true. Then the solution u described above is of class $C^{1,\alpha}(\Omega;\mathbb{R}^2)$ in the interior of Ω for any $0 < \alpha < 1$.*

The proof of Theorem 4.2 is quite similar to that of Theorem 4.1 and will require only a few minor changes. We will just indicate the differences and leave the details to the reader.
Proof of Theorem 4.2. We fix a small disc $B_{2R}(\bar{x}) \Subset \Omega$ such that $p_1 \leq p(|E(x)|^2) \leq p_2$ for all $x \in B_{2R}(\bar{x})$ with $p_1 \geq p_\infty > 6/5$ and $p_2 < \min(2p_1, p_1+2), p_2 \leq p_0$. This is possible since p and E are smooth functions. Actually, there is a radius $R_0 = R_0(p,E) > 0$ such that this claim holds for any $\bar{x} \in \Omega$ and for any $R \leq R_0$.
Note that the counterpart to (4.5) is valid, i.e. as in (8) of [6] we have

$$\int_{B'} |\nabla(1+|\varepsilon(u)|^2)^{p_1/4}|^2\,dx \leq c(B')\int_{B'} |\nabla\varepsilon(u)|^2(1+|\varepsilon(u)|^2)^{(p_1-2)/2}\,dx < \infty$$

for all $B' \Subset B_{2R}$ with a local constant $c(B')$.
Now let $B_{2r} = B_{2r}(x_0) \subset B_{2R}(\bar{x})$ and η be a cut-off function as before. We use the same notation as in [6] and as before, and introduce the quantities

$$\begin{aligned}
&\sigma = S(\varepsilon(u),E),\\
&H = \left(\frac{\partial S_{ij}}{\partial \varepsilon_{\alpha\beta}}(\varepsilon(u),E)\varepsilon_{ij}(\partial_k u)\varepsilon_{\alpha\beta}(\partial_k u)\right)^{1/2},\\
&\tilde{h} = \max\left\{(1+|\varepsilon(u)|^2)^{\frac{p_2-2}{4}}, (1+|\varepsilon(u)|^2)^{\frac{2-p_1}{4}}\right\},\\
&G = (1+|\nabla u|^2)^t,
\end{aligned} \tag{4.20}$$

where $t > 0$ is sufficiently large. We proceed in the same way as in the proof of Theorem 3 of

[6] and obtain that

$$
\begin{aligned}
\fint_{B_{2r}} \eta^2 \partial_k \sigma : \varepsilon(\partial_k u)\, dx \leq & c \fint_{B_{2r}} \eta |\nabla \sigma||\nabla \eta||\nabla u - Q|\, dx \\
& + \fint_{B_{2r}} \eta |\nabla \eta||\nabla u||\nabla u - Q|\, dx \\
& + \fint_{B_{2r}} \eta^2 |\nabla u||\nabla^2 u|\, dx \\
& + \fint_{B_{2r}} |g| \eta |\nabla \eta||\nabla u - Q|\, dx \\
& + \fint_{B_{2r}} |g| \eta^2 |\nabla^2 u|\, dx,
\end{aligned} \tag{4.21}
$$

from which we will deduce the following estimate similar to (4.9)

$$
\fint_{B_{2r}} \eta^2 H^2\, dx \leq c \left(\fint_{B_{2r}} (H\tilde{h})^\gamma\, dx \right)^{2/\gamma} + c \fint_{B_{2r}} G\, dx. \tag{4.22}
$$

The integrals in (4.21) can be estimated exactly in the same way as in section 4.1, except for the first two integrals involving σ. In the present case $\sigma = \sigma(\varepsilon, E)$ depends not only on ε but also on E. We follow [6] to estimate this two terms. For the first one, we have that

$$
\begin{aligned}
\partial_k \sigma : \varepsilon(\partial_k u) &= H^2 + \frac{\partial S_{ij}}{\partial E}(\varepsilon(u), E) \cdot \partial_k E \varepsilon_{ij}(\partial_k u) \\
&\geq H^2 - c(1 + |\varepsilon(u)|^2)^{\frac{p_2 - 1}{2}} (1 + \log(1 + |\varepsilon(u)|^2)) |\nabla \varepsilon(u)| \\
&\geq H^2 - c(1 + |\varepsilon(u)|^2)^{p_2/2} |\nabla \varepsilon(u)|.
\end{aligned} \tag{4.23}
$$

and that by the Young inequality

$$
\begin{aligned}
& (1 + |\varepsilon(u)|^2)^{p_2/2} |\nabla \varepsilon(u)| \\
\leq & \delta (1 + |\varepsilon(u)|^2)^{\frac{p_1 - 2}{2}} |\nabla \varepsilon(u)|^2 + c(\delta)(1 + |\varepsilon(u)|^2)^{\frac{p_2 - p_1 + 2}{2}} \\
\leq & \delta H^2 + c(\delta)(1 + |\varepsilon(u)|^2)^{\frac{p_2 - p_1 + 2}{2}}
\end{aligned} \tag{4.24}
$$

for any $\delta > 0$. Thus

$$
\partial \sigma : \varepsilon(\partial_k u) \geq \frac{1}{2} H^2 - c(1 + |\varepsilon(u)|^2)^{\frac{p_2 - p_1 + 2}{2}}
$$

and we see that we do not have any problem with the first integral in (4.21). For the second integral in (4.21), we estimate in the same way as (10) and (11) in [6] to get

$$
|\nabla \sigma| \leq c H \tilde{h} + c(1 + |\varepsilon(u)|^2)^{p_2/2},
$$

and then we proceed as in (4.12) and (4.14) (with minor modifications) to get the desired inequality (4.22). We omit the details. Then we finish the proof of Theorem 4.2 as in Section 4.1.

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