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ESSENTIAL SETS AND SUPPORT SETS FOR STOCHASTIC PROCESSES

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Abstract. In recent articles the author used his work in measure and integration to produce a new universal concept of stochastic processes. This concept leads, for the first time, for a stochastic process to a natural notion of *essential subsets* in the path space. But there remained some contrast to the traditional treatment, for example because for the Poisson process the set of càdlàg paths is not an essential subset. The present article is an attempt to harmonize the two approaches, in that it proposes and studies, for a stochastic process, besides the notion of essential sets the somewhat weaker but reasonable notion of *support sets*.

1. Introduction

The present article wants to continue the author's contributions to the fundamentals of stochastic processes [7][8][9][10]. These papers are based on his work in measure and integration [4][6], the aim of which is to build adequate new structures. The outcome is a reformed concept of stochastic processes, which in particular removes notorious deficiencies in case of uncountable time domains. In the sequel we shall make free use of the elements of the author's work in measure theory, of which there are short recollections in [7] section 1, [9] section 2, and [10] section 1.

We start with the precise situation in both the new and the traditional context. Let T be an infinite set called the *time domain*, and Y a nonvoid set called the *state space*, combined to form the product set $X = Y^T$ called the *path space*, the members of which are the paths $x = (x_t)_{t\in T} : T \to Y$. The *new context* assumes in Y a lattice $\mathfrak{K}$ which contains the finite subsets of Y and is τ compact, and forms in X the finite-based product set system

$$(\mathfrak{K} \cup \{Y\})^{[T]} := \{\prod_{t\in T} S_t : S_t \in \mathfrak{K} \cup \{Y\}\ \forall\, t \in T \text{ with } S_t = Y\ \forall\!\forall\, t \in T\},$$

where $\forall\!\forall$ means *for almost all* := for all except finitely many, and the generated lattice $\mathfrak{S} := \big((\mathfrak{K} \cup \{Y\})^{[T]}\big)^{\star}$. Thus $\mathfrak{S}$ contains $\varnothing$ and X and is τ compact after [5] 2.6. Then a *stochastic process* for T and $(Y, \mathfrak{K})$ is defined to be an inner τ prob premeasure $\varphi : \mathfrak{S} \to [0, \infty[$, and its maximal inner τ extension $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ is called the *maximal measure* for the process.

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The *traditional context* assumes in Y a σ algebra $\mathfrak{B}$, and forms in X the finite-based product set system

$$\mathfrak{B}^{[T]} := \{ \prod_{t \in T} B_t : B_t \in \mathfrak{B} \ \forall \, t \in T \text{ with } B_t = Y \ \forall\forall \, t \in T \},$$

and the generated σ algebra $\mathfrak{A} := \mathrm{A}\sigma(\mathfrak{B}^{[T]})$. It is well-known that for uncountable T the formation $\mathfrak{A}$ can be too narrow, because its members $A \in \mathfrak{A}$ are *countably determined* in the sense that $A = \{x \in X : (x_t)_{t \in U} \in M\}$ for some nonvoid countable $U \subset T$ and some $M \subset Y^U$. Then the traditional notion of a *stochastic process* for T and $(Y, \mathfrak{B})$ amounts to a prob measure $\alpha : \mathfrak{A} \to [0, \infty[$, called the *canonical measure* for the process.

In the special case that Y is a Polish topological space with $\mathfrak{K} = \mathrm{Comp}(Y)$ and $\mathfrak{B} = \mathrm{Bor}(Y)$ the two kinds of stochastic processes $\varphi : \mathfrak{S} \to [0, \infty[$ and $\alpha : \mathfrak{A} \to [0, \infty[$ are in one-to-one correspondence. The connection is based on $\mathfrak{S} \subset \mathfrak{A} \subset \mathfrak{C}(\varphi_\tau)$ and reads $\varphi = \alpha | \mathfrak{S}$ and $\alpha = \Phi | \mathfrak{A}$, and hence $\varphi_\tau = (\alpha^\star | \mathfrak{S}_\tau)_\star \leqq \alpha^\star$.

The matter of concern in the present article are the efforts to equip a stochastic process with the collection of those subsets of X which *support* the process in its *essential* features. The most prominent example is the subset of continuous paths $\mathrm{C}(T, \mathbb{R}) \subset X = \mathbb{R}^T$ on $T = [0, \infty[$ for the traditional Wiener measure $\alpha : \mathfrak{A} \to [0, \infty[$, the canonical measure of one-dimensional Brownian motion. Note that this subset is not countably determined and thus not in $\mathfrak{A}$. But above all note that the idea for this example did not come out of mathematics, but from experimental observations. The entire problem is far from obvious, and one could think of several different answers.

In the traditional context it is common to define the *essential* subsets for a stochastic process $\alpha : \mathfrak{A} \to [0, \infty[$ to be the $C \subset X$ of *outer* canonical measure $\alpha^\star(C) = 1$, for example in [1] section 38. Equivalent is that α has a measure extension which lives on C, for example from [8] lemma 2, and also that α has a unique minimal measure extension which lives on C. But the disastrous example in [8] theorem 4 makes clear that this notion is an unnatural one. In sharp contrast, the new context opens the road to a natural definition: One defines the *essential* subsets for a stochastic process $\varphi : \mathfrak{S} \to [0, \infty[$ to be the measurable subsets $C \in \mathfrak{C}(\varphi_\tau)$ of full measure $\Phi(C) = 1$ under the unique maximal extension $\Phi = \varphi_\tau | \mathfrak{C}(\varphi_\tau)$. In particular, the subset $\mathrm{C}(T, \mathbb{R}) \subset X = \mathbb{R}^T$ has been proved in [7] section 6 to be an essential one for the new Wiener measure $\varphi : \mathfrak{S} \to [0, \infty[$, of which the maximal measure Φ has been termed the *true* Wiener measure at that place.

However, after this an additional request came up from the traditional treatment of the Poisson process. Let as above $T = [0, \infty[$ and $Y = \mathbb{R}$, and form the chain $\mathrm{C}(T, \mathbb{R}) \subset D \subset E \subset F \subset X = \mathbb{R}^T$, where F consists of the paths $x : T \to \mathbb{R}$ which have all one-sided limits $x_t^\pm$ for $t \in T$, with the convention $x_0^- := x_0$, while E consists of the $x \in F$ which at each $t \in T$ are either left or right continuous, and D of the $x \in F$ which are right continuous at all $t \in T$, the so-called *càdlàg* ones. One proves that the traditional Poisson process $\alpha : \mathfrak{A} \to [0, \infty[$ fulfils $\alpha^\star(D) = 1$, for example in [1] section 41, and hence $\alpha^\star(E) = \alpha^\star(F) = 1$. The traditional treatment of the process is based on D, that means is in terms of the measure extensions of α which live on D (and of the connected so-called *versions* of the process). On the other side the new Poisson process $\varphi : \mathfrak{S} \to [0, \infty[$ studied in [8] section 5 and [10] reveals that E and F are essential subsets for φ. But D fulfils $\varphi_\tau(D) = 0$ and thus is either nonmeasurable $\mathfrak{C}(\varphi_\tau)$ or in $\mathfrak{C}(\varphi_\tau)$ with $\Phi(D) = 0$, and thus cannot be an essential subset for the process.

Thus one has the problem how to consider and to handle the above contrast, in face of the obvious power of the new concepts vis-à-vis the traditional ones with their decades of practice. In this situation the question arises whether the new context will be able to harmonize: whether it can create a weaker but reasonable sense in that a subset $C \subset X$ can be said to *support* a stochastic process $\varphi : \mathfrak{S} \to [0, \infty[$, a sense which in case of the Poisson process applies to the set D. The desired notion of *support sets* $C \subset X$ for φ should be such that on C there live prob measures which not just reproduce the process φ, but are close companions of its unique maximal extension Φ and of distinctive nature.

The present article has the aim to develop a notion of *support sets* of such kind. It is related to the traditional notion of *modification.* Section 2 will present the definition and basic consequences, in particular assertions of existence and uniqueness. Then section 3 will present three examples, of which the first one is on the Poisson process and the subset $D \subset X$. In this connection we refer to the presentations in [2][3] and to the earlier attempts to improve the basic concepts in Nelson [11] and Tjur [12][13]. The present author is impressed to note that the work of Tjur did not at all find its due attention in the later literature.

In conclusion we recall from [7] section 3 and [8] section 1 the notion of image measures which will be basic for the sequel. Let $H : X \to Y$ be a map between nonvoid sets X and Y. For a σ algebra $\mathfrak{A}$ in X we define the direct image

$$\vec{H}\mathfrak{A} := \{B \subset Y : H^{-1}(B) \in \mathfrak{A}\} \subset \mathfrak{P}(Y),$$

which is a σ algebra in Y. It must not be confused with the set system $H(\mathfrak{A}) := \{H(A) : A \in \mathfrak{A}\}$. Then for a measure $\alpha : \mathfrak{A} \to [0, \infty]$ on $\mathfrak{A}$ we define the direct image $\vec{H}\alpha : \vec{H}\mathfrak{A} \to [0, \infty]$ to be $\vec{H}\alpha(B) = \alpha(H^{-1}(B))$ for $B \in \vec{H}\mathfrak{A}$. Thus $\vec{H}\alpha$ is a measure on $\vec{H}\mathfrak{A}$ and lives on $H(X) \subset Y$. Now if $\mathfrak{B}$ is a σ algebra in Y, then $\mathfrak{B} \subset \vec{H}\mathfrak{A}$ means that the map $H : X \to Y$ is measurable $\mathfrak{A} - \mathfrak{B}$ in the usual sense, and then $\beta := \vec{H}\alpha|\mathfrak{B}$ is the usual image measure $\beta : \mathfrak{B} \to [0, \infty]$ of $\alpha : \mathfrak{A} \to [0, \infty]$ on $\mathfrak{B}$.

2. Modifications and Support Sets

The present section assumes the *new context* as defined above: we fix T and $(Y, \mathfrak{K})$ and form $X = Y^T$ and $\mathfrak{S}$. Let $\varphi : \mathfrak{S} \to [0, \infty[$ be a stochastic process for T and $(Y, \mathfrak{K})$ with $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$. We recall from [7] 1.7 that the subsets $A \subset X$ with $\varphi_\tau(A) = 1$ are members of $\mathfrak{C}(\varphi_\tau)$, and thus are the essential subsets for φ.

We define a map $J : X \to X$ to be a *modification* for φ iff for each $t \in T$ there exists an essential $F(t)$ for φ such that $(Jx)_t = x_t$ for all

$x \in F(t)$. Thus it means that $J : X \to X$ is a map which is a modification of the identity map $\mathrm{Id} : X \to X$ on the measure space $(X, \mathfrak{C}(\varphi_\mathcal{T}), \Phi)$ in the traditional sense, for example in [1] definition 39.1. Equivalent is that for each nonvoid countable $U \subset T$ there exists an essential $F(U)$ for φ such that $Jx|U = x|U$ for all $x \in F(U)$. The modification $J : X \to X$ for φ produces the image prob measure $\overrightarrow{J}\Phi : \overrightarrow{J}\mathfrak{C}(\varphi_\mathcal{T}) \to [0, \infty[$, to be considered as a close companion of Φ. But $\overrightarrow{J}\Phi$ lives on the image set $J(X) \subset X$, and can therefore be quite different from Φ.

After this we define a *support set* for φ to be a subset $C \subset X$ such that there exists a modification $J : X \to X$ for φ with $J(X) \subset C$. Then the image measure $\overrightarrow{J}\Phi$ lives on C.

2.1 Remark. *If $C \subset X$ is an essential set for φ, then C is a support set for φ.* The first example in the next section will show that the converse assertion is not true.

In fact, define $J : X \to X$ to be $Jx = x$ for $x \in C$, while in case $x \notin C$ the image Jx can be an arbitrary member of C. For each $t \in T$ then $(Jx)_t = x_t$ for all $x \in C$, so that J is a modification of φ with $J(X) \subset C$.

2.2 Proposition. *Let $J : X \to X$ be a modification for φ. If $A \in \mathfrak{C}(\varphi_\mathcal{T})$ is countably determined then $A \in \overrightarrow{J}\mathfrak{C}(\varphi_\mathcal{T})$ and $\overrightarrow{J}\Phi(A) = \Phi(A)$.*

Proof. By assumption there exists a nonvoid countable $U \subset T$ and an $M \subset Y^U$ such that $A = \{x \in X : x|U \in M\}$. Let $F(U)$ be essential for φ such that $Jx|U = x|U$ for all $x \in F(U)$. Then

$$\begin{aligned} J^{-1}(A) &= \{x \in X : Jx \in A\} \\ &= \{x \in F(U) : Jx \in A\} \cup \{x \in (F(U))' : Jx \in A\}. \end{aligned}$$

Here the first set is

$$= \{x \in F(U) : Jx|U = x|U \in M\} = \{x \in F(U) : x \in A\} = F(U) \cap A,$$

while the second one is a Φ null set. Likewise $A = \big(F(U) \cap A\big) \cup \big((F(U))' \cap A\big)$ is the union of $F(U) \cap A$ with some Φ null set. It follows that $J^{-1}(A) \in \mathfrak{C}(\varphi_\mathcal{T})$ or $A \in \overrightarrow{J}\mathfrak{C}(\varphi_\mathcal{T})$, and $\Phi(J^{-1}(A)) = \Phi(A)$ or $\overrightarrow{J}\Phi(A) = \Phi(A)$. $\square$

2.3 Consequence. *Assume that Y is a Polish space with $\mathfrak{K} = \mathrm{Comp}(Y)$ and $\mathfrak{B} = \mathrm{Bor}(Y)$, and let $\alpha : \mathfrak{A} \to [0, \infty[$ be the canonical measure connected with φ. Let $J : X \to X$ be a modification for φ. Then the image measure $\overrightarrow{J}\Phi$ is an extension of α which lives on $J(X)$. Thus $\alpha^\star(J(X)) = 1$.*

In the situation of 2.3 the last sentence asserts that each support set $C \subset X$ for φ fulfils $\alpha^\star(C) = 1$. The second example in the next section will show that - aside from pathologies - the converse assertion is not true.

The next results are on the existence of support sets $C \subset X$ for φ and, for a certain class of support sets, on the uniqueness for the image measures $\overrightarrow{J}\Phi$ which live on C.

2.4 Theorem. *Assume that $C \subset X$ is such that there exists an $N \subset T$ with the properties*

i) *each pair $u, v \in C$ fulfils $u|N = v|N \Rightarrow u = v$;*

ii) *for each* $t \in T$ *there exists an essential* $F(t)$ *for* φ *such that* $x|N \cup \{t\} \in C|N \cup \{t\}$ *for all* $x \in F(t)$.

Then C *is a support set for* φ.

Proof. 1) Fix $a \in N$. From ii) then $x|N \in C|N$ for all $x \in F(a)$. Thus i) implies that for each $x \in F(a)$ there exists a unique $Jx \in C$ such that $x|N = Jx|N$. We complete the definition of $J : X \to X$ in that for $x \notin F(a)$ we allow Jx to be an arbitrary member of C. Thus $J(X) \subset C$.

2)We claim that $J : X \to X$ is a modification for φ. In fact, for $t \in T$ the subset $F(a) \cap F(t) \in \mathfrak{C}(\varphi_\tau)$ has full measure $\Phi(F(a) \cap F(t)) = 1$. For $x \in F(a) \cap F(t)$ we have on the one hand $x|N = Jx|N$, and on the other hand $x|N \cup \{t\} = z|N \cup \{t\}$ for some $z \in C$. Then from i) we obtain $z = Jx$. It follows that $x_t = z_t = (Jx)_t$. □

2.5 Theorem. *Let* $C \subset X$ *be a support set for* φ. *Assume that there exists a nonvoid countable* $N \subset T$ *such that each pair* $u, v \in C$ *fulfils* $u|N = v|N \Rightarrow u = v$. *Then all modifications* $J : X \to X$ *for* φ *with* $J(X) \subset C$ *produce the same image measure* $\overrightarrow{J}\Phi$.

Proof. Let $J, K : X \to X$ be modifications for φ with $J(X), K(X) \subset C$. By assumption there exists an essential F for φ such that for all $x \in F$ one has $Jx|N = x|N = Kx|N$ and hence $Jx = Kx$. For $B \subset X$ therefore

$$J^{-1}(B) = \{x \in X : Jx \in B\} = \{x \in F : Jx \in B\} \cup \{x \in F' : Jx \in B\},$$

and the same for K. Here the two first sets are equal, while the two second ones are Φ null sets. It follows that $J^{-1}(B) \in \mathfrak{C}(\varphi_\tau)$ or $B \in \overrightarrow{J}\mathfrak{C}(\varphi_\tau)$ is equivalent to $K^{-1}(B) \in \mathfrak{C}(\varphi_\tau)$ or $B \in \overrightarrow{K}\mathfrak{C}(\varphi_\tau)$, and that in this case $\Phi(J^{-1}(B)) = \Phi(K^{-1}(B))$ or $\overrightarrow{J}\Phi(B) = \overrightarrow{K}\Phi(B)$. □

We conclude with the relevant transformation rule, which reduces integration with respect to an image measure $\overrightarrow{J}\Phi$ to integration with respect to Φ. The most comprehensive version is for the Choquet integral; for this notion we refer to [4] section 11 and [6] section 5.

2.6 Remark. *Let* $J : X \to X$ *be a modification for* φ. *For all functions* $f : X \to [0, \infty]$ *then*

$$⨍ f d\varphi_\tau\big(J^{-1}(\cdot)\big) = ⨍ (f \circ J) d\varphi_\tau.$$

In particular f *is measurable* $\overrightarrow{J}\mathfrak{C}(\varphi_\tau)$ *iff* $f \circ J$ *is measurable* $\mathfrak{C}(\varphi_\tau)$, *and in this case* $\int f d(\overrightarrow{J}\Phi) = \int (f \circ J) d\Phi$.

Proof. The first formula is contained in [7] 3.4. Thus it remains to prove the measurability assertion. Now f measurable $\overrightarrow{J}\mathfrak{C}(\varphi_\tau)$ means that $[f \geqq t] \in \overrightarrow{J}\mathfrak{C}(\varphi_\tau)$ or $J^{-1}\big([f \geqq t]\big) \in \mathfrak{C}(\varphi_\tau)$ for all $t > 0$. We have

$$\begin{aligned} J^{-1}\big([f \geqq t]\big) &= \{x \in X : Jx \in [f \geqq t]\} \\ &= \{x \in X : f(Jx) \geqq t\} = [f \circ J \geqq t]. \end{aligned}$$

Thus the equivalence continues with $[f \circ J \geqq t] \in \mathfrak{C}(\varphi_\tau)$ for all $t > 0$, which means that $f \circ J$ is measurable $\mathfrak{C}(\varphi_\tau)$. □

3. Three Examples

The present section assumes $T = [0, \infty[$ and the Polish space $Y = \mathbb{R}$ with $\mathfrak{K} = \mathrm{Comp}(\mathbb{R})$ and $\mathfrak{B} = \mathrm{Bor}(\mathbb{R})$, and as before the path space $X = \mathbb{R}^T$ with $\mathfrak{S}$ and $\mathfrak{A}$.

First Example. This example is on the Poisson process. We start to recall the pertinent subspace $X^\circ \subset X$ defined in [10] section 5. It consists of the paths $x = (x_t)_{t \in T} : T \to \mathbb{R}$ with the properties

i) x has values in $\mathbb{N}0 := \mathbb{N} \cup \{0\}$ with $x_0 = 0$ and is monotone increasing, and hence has one-sided limits $x_t^{\pm} \in \mathbb{N}0$ for all $t \in T$, with the convention $x_0^- := x_0 = 0$;

ii) $x_t^+ - x_t^- \leqq 1$ for all $t \in T$;

iii) x is unbounded, that is $x_t \uparrow \infty$ for $t \uparrow \infty$.

For $x \in X^\circ$ we form $x^+ = (x_t^+)_{t \in T} \in X$. From [10] 5.1 one deduces the lemma which follows. We recall $D \subset X$ defined in section 1.

3.1 Lemma. *For $x \in X^\circ$ we have $x^+ \in X^\circ \Leftrightarrow x_0^+ = 0$. In this case $x^+ \in D$. Thus $x \mapsto x^+$ defines a map of $X^{\circ\circ} := \{x \in X^\circ : x^+ \in X^\circ\} = \{x \in X^\circ : x_0^+ = 0\}$ onto $X^\circ \cap D = X^{\circ\circ} \cap D$.*

Now let $\varphi : \mathfrak{S} \to [0, \infty[$ with $\alpha : \mathfrak{A} \to [0, \infty[$ be the Poisson process from [8] section 5 and [10] section 6. We recall that $X^\circ \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(X^\circ) = 1$. Moreover $\varphi_\tau(X^\circ \cap D) = \varphi_\tau(D) = 0$, so that $X^\circ \cap D \subset D$ are not essential sets for φ, while $\alpha^\star(X^\circ \cap D) = \alpha^\star(D) = 1$. The present main result reads as follows.

3.2 Theorem. *Define $K : X \to X$ to be $Kx = x^+$ for $x \in X^{\circ\circ}$, while in case $x \notin X^{\circ\circ}$ the image Kx can be an arbitrary member of $X^\circ \cap D$. Then K is a modification for φ with $K(X) = X^\circ \cap D$. Thus the image measure $\overrightarrow{K}\Phi$ lives on $X^\circ \cap D$, so that $X^\circ \cap D \subset D$ are support sets for φ.*

Note from 2.3 that $\overrightarrow{K}\Phi$ is an extension of α, and from 2.5 that all modifications $J : X \to X$ for φ with $J(X) \subset D$ have the same image measure $\overrightarrow{J}\Phi$.

Proof. 1) We recall that the principal actor in [8] section 5 was the subspace $E(T) \subset X$, which likewise is essential for φ. We saw in [8] remark 29 that for each nonvoid countable $U \subset T$ there exists an essential $F(U)$ for φ such that $x|U \in (E(T) \cap D)|U$ for all $x \in F(U)$.

2) This assertion can be fortified as follows: For each nonvoid countable $U \subset T$ there exists an essential $F(U)$ for φ such that $x|U \in (E(T) \cap X^\circ \cap D)|U$ for all $x \in F(U)$. In fact, we can assume that $U \subset T$ is dense, and use 1) with $F(U) \cap X^\circ$ instead of $F(U)$. Then the $x \in F(U) \cap X^\circ$ and hence their $x|U$ are unbounded, so that these $x|U$ are restrictions of unbounded members of $E(T) \cap D$, that is of members of $E(T) \cap X^\circ \cap D$.

3) We fix a countable dense $N \subset T$. In view of 2) the existence assertion 2.4 can be applied to $C := X^\circ \cap D = X^{\circ\circ} \cap D$ and N. Thus there exists a modification $J : X \to X$ for φ with $J(X) \subset C$. Hence there is an essential F for φ such that $Jx|N = x|N$ for all $x \in F$. Here we can use $F \cap X^\circ$ instead of F and thus asssume that $F \subset X^\circ$. It follows that $Jx = x^+$ for all $x \in F$, which in particular implies that $F \subset X^{\circ\circ}$. Therefore $Jx = x^+ = Kx$

for all $x \in F$. Thus the fact that J is a modification for φ implies that K is a modification for φ as well. $\square$

Thus the transition from the traditional canonical measure $\alpha : \mathfrak{A} \to [0, \infty[$ to the new maximal measure $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ means for the benefit of the subset $D \subset X$ that, as to the measure extensions of α which live on D, one has not just the minimal extension as in [8] lemma 2 and/or unspecified and uncontrolled further ones, but in form of $\overrightarrow{K}\Phi : \overrightarrow{K}\mathfrak{C}(\varphi_\tau) \to [0, \infty[$ a unique extension of maximal type with an immense domain.

Second Example. This example assumes the Wiener process $\varphi : \mathfrak{S} \to [0, \infty[$ with $\alpha : \mathfrak{A} \to [0, \infty[$ from [7] section 6. The aim is to exhibit subsets $C \subset X$ with $\alpha^\star(C) = 1$ which are not support sets for φ. We use from [8] theorem 4 for the individual $a = (a_t)_{t \in T} \in X$ the subsets

$$C(a) := \{x \in X : x_t = a_t \text{ for all } t \in T \text{ except countably many ones}\}.$$

However, our result will have to disregard certain pathologies: In order to prove that a modification $J : X \to X$ for φ cannot fulfil $J(X) \subset C(a)$ for an $a \in X$, we shall have to assume that it satisfies a certain measurability condition. The technical reason is that we have to use the Fubini type result [4] 21.19 for the product inner τ premeasure $\vartheta = \varphi \times \lambda$ in the sense of [4] theorem 21.9 for φ and the Lebesgue premeasure $\lambda : \mathrm{Comp}(T) \to [0, \infty[$.

3.3 Proposition. *Assume that $J : X \to X$ is a modification for φ which fulfils $J(X) \subset C(a)$ for an $a \in X$, that is*

$$U(x) := \{t \in T : (Jx)_t \neq a_t\} \subset T \quad \textit{is countable for each } x \in X.$$

Then J must violate the condition that

$$E := \{(x, t) \in X \times T : t \in U(x)\} \quad \textit{is a member of } \mathfrak{C}(\vartheta_\tau).$$

Proof. Assume that $E \in \mathfrak{C}(\vartheta_\tau)$. Then [4] 21.19 asserts that

1) the sections $E(x, \cdot) := \{t \in T : (x, t) \in E\} = U(x) \subset T$ for $x \in X$ are such that the function $x \mapsto \lambda_\tau(E(x, \cdot))$ on X is measurable $\mathfrak{C}(\varphi_\tau)$ and fulfils

$$\Theta(E) = \int \lambda_\tau(E(x, \cdot)) d\Phi(x); \quad \text{and}$$

2) the sections $E(\cdot, t) := \{x \in X : (x, t) \in E\} = \{x \in X : (Jx)_t \neq a_t\} \subset X$ for $t \in T$ are such that the function $t \mapsto \varphi_\tau(E(\cdot, t))$ on T is measurable $\mathfrak{C}(\lambda_\tau)$ and fulfils

$$\Theta(E) = \int \varphi_\tau(E(\cdot, t)) d\Lambda(t);$$

here of course $\Lambda = \lambda_\tau|\mathfrak{C}(\lambda_\tau)$ and $\Theta = \vartheta_\tau|\mathfrak{C}(\vartheta_\tau)$.

Now in 1) we have $\lambda_\tau(E(x, \cdot)) = \lambda_\tau(U(x)) = 0$ for $x \in X$, and it follows that $\Theta(E) = 0$. For 2) we note that for each $t \in T$ there exists an essential $F(t)$ for φ such that $(Jx)_t = x_t$ for all $x \in F(t)$. Therefore

$$E(\cdot, t) \cap F(t) = \{x \in F(t) : x_t \neq a_t\} = [H_t \neq a_t] \cap F(t),$$

with $H_t : X \to Y$ the tth coordinate projection as in [7] section 6. From [7] proposition 6.5 it follows that

$$\begin{aligned}\varphi_\tau(E(\cdot, t)) &= \varphi_\tau\big(E(\cdot, t) \cap F(t)\big) = \varphi_\tau\big([H_t \neq a_t] \cap F(t)\big) \\ &= \varphi_\tau\big([H_t \neq a_t]\big) = (\gamma_t)_\tau(\mathbb{R} \setminus \{a_t\}) = 1,\end{aligned}$$

where $(\gamma_t)_{t\in T}$ is the convolution semigroup of the Gaussian premeasures $\gamma_t : \mathfrak{K} \to [0,\infty[$. Thus we obtain $\Theta(E) = \infty$, and hence the desired contradiction. $\square$

Third Example. The last example is once more intended for the Wiener process, but its frame is more comprehensive: We assume a family $(\gamma_t)_{t\in T}$ of Radon prob premeasures $\gamma_t : \mathfrak{K} \to [0,\infty[$ with $\gamma_0 = \delta_0|\mathfrak{K}$ which under convolution fulfils $\gamma_s \star \gamma_t = \gamma_{s+t}$ for $s,t \in T$. As in [7] section 6 and [8] section 4 we form the resultant projective family $(\varphi_p)_{p\in I}$ of inner τ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ and via the Kolmogorov type projective limit theorem [8] theorem 11 the stochastic process $\varphi : \mathfrak{S} \to [0,\infty[$. The present result wants to point out that there can be more support sets $C \subset X$ for φ than expected or hoped for.

3.4 Proposition. *Assume that γ_s is atomless for some $s > 0$. Then φ has support sets $C \subset X \setminus \mathrm{C}(T,\mathbb{R})$.*

Proof. Fix an $s > 0$ for which γ_s is atomless. 1) For $t \in T$ define $N(t) := \{x \in X : x_s = \pm t\} \subset X$. Then $N(t) \in \mathfrak{S}$, and [7] 6.5 furnishes

$$\varphi(N(t)) = \varphi_\tau\big([H_s - H_0 \in \{t,-t\}]\big) = (\gamma_s)_\tau\big(\{t,-t\}\big) = 0.$$

Thus $F(t) := X \setminus N(t) = \{x \in X : |x_s| \neq t\}$ is essential for φ.

2) We define $J : X \to X$ for $x \in X$ to be $(Jx)_t = x_t$ when $t \neq |x_s|$, and in the point $t = |x_s| \in T$ the value $(Jx)_t$ to be an arbitrary real number such that the function $Jx : T \to \mathbb{R}$ is not continuous at $t = |x_s|$. Thus $J(X) \subset X \setminus \mathrm{C}(T,\mathbb{R})$.

3) It remains to show that J is a modification for φ. In fact, for each $t \in T$ we have the essential subset $F(t)$ for φ which for all $x \in F(t)$ satisfies $t \neq |x_s|$ and hence $(Jx)_t = x_t$. $\square$

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