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two-dimensional domains**

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## Abstract

Let  $u: \mathbb{R}^2 \supset \Omega \rightarrow \mathbb{R}^M$  denote a local minimizer of  $J[w] = \int_{\Omega} f(\nabla^k w) dx$ , where  $k \geq 2$  and  $\nabla^k w$  is the tensor of all  $k^{\text{th}}$  order (weak) partial derivatives. Assuming rather general growth and ellipticity conditions for  $f$ , we prove that  $u$  actually belongs to the class  $C^{k,\alpha}(\Omega; \mathbb{R}^M)$  by the way extending the result of [BF2] to the higher order case by using different methods. A major tool is a lemma on the higher integrability of functions established in [BFZ].

## 1 Introduction

Let  $\Omega$  denote a bounded domain in  $\mathbb{R}^2$  and consider a function  $u: \Omega \rightarrow \mathbb{R}^M$  which locally minimizes the variational integral

$$J[w, \Omega] = \int_{\Omega} f(\nabla^k w) dx,$$

where  $\nabla^k w$  represents the tensor of all  $k^{\text{th}}$  order (weak) partial derivatives. Our main concern is the investigation of the smoothness properties of such local minimizers under suitable assumptions on the energy density  $f$ . For the first order case (i.e.  $k = 1$ ) we have rather general results which can be found for example in the textbooks of Morrey [Mo], Ladyzhenskaya and Ural'tseva [LU], Gilbarg and Trudinger [GT] or Giaquinta [Gi], for an update of the history including recent contributions we refer to [Bi]. In order to keep our exposition simple (and only for this reason) we consider the scalar case (i.e.  $M = 1$ ) and restrict ourselves to variational problems involving the second (generalized) derivative. Then our variational problem is related to the theory of plates: one may think of  $u: \Omega \rightarrow \mathbb{R}$  as the displacement in vertical direction from the flat state of an elastic plate. The classical case of a potential  $f$  with quadratic growth is discussed in the monographs of Ciarlet and Rabier [CR], Necăs and Hláváček [NH], Chudinovich and Constanda [CC] or Friedman [Fr], further references are contained in Zeidler's book [Ze]. We also like to remark that plates with other hardening laws (logarithmic and power growth case) together with an additional obstacle have been studied in the papers [BF1] and [FLM] but not with optimal regularity results. The purpose of this note is to present a rather satisfying regularity theory for a quite large of potentials allowing even anisotropic growth.

To be precise let  $\mathbb{M}$  denote the space of all  $(2 \times 2)$ -matrices and suppose that we are given a function  $f: \mathbb{M} \rightarrow [0, \infty)$  of class  $C^2$  which satisfies with exponents  $1 < p \leq q < \infty$  the anisotropic ellipticity estimate

$$\lambda(1 + |\xi|^2)^{\frac{p-2}{2}} |\sigma|^2 \leq D^2 f(\xi)(\sigma, \sigma) \leq \Lambda(1 + |\xi|^2)^{\frac{q-2}{2}} |\sigma|^2 \quad (1.1)$$

for all  $\xi, \sigma \in \mathbb{M}$  with positive constants  $\lambda, \Lambda$ . Note that (1.1) implies the growth condition

$$a|\xi|^p - b \leq f(\xi) \leq A|\xi|^q + B \quad (1.2)$$

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with suitable constants  $a, A > 0, b, B \geq 0$ . Let

$$J[w, \Omega] = \int_{\Omega} f(\nabla^2 w) \, dx, \quad \nabla^2 w = (\partial_{\alpha} \partial_{\beta} w)_{1 \leq \alpha, \beta \leq 2}.$$

We say that a function  $u \in W_{p,loc}^2(\Omega)$  is a local  $J$ -minimizer iff  $J[u, \Omega'] < \infty$  for any subdomain  $\Omega' \Subset \Omega$  and

$$J[u, \Omega'] \leq J[v, \Omega']$$

for all  $v \in W_{p,loc}^2(\Omega)$  such that  $u - v \in \mathring{W}_p^2(\Omega')$  (here  $W_{p,loc}^k(\Omega)$  etc. denote the standard Sobolev spaces, see [Ad]). Note that (1.1) implies the strict convexity of  $f$ . Therefore, given a function  $u_0 \in W_q^2(\Omega)$ , the direct method ensures the existence of a unique  $J$ -minimizer  $u$  in the class

$$\{v \in W_p^2(\Omega) : J[v, \Omega] < \infty, \quad v - u_0 \in \mathring{W}_p^2(\Omega)\}$$

which motivates the discussion of local  $J$ -minimizers. Our main result reads as follows:

**THEOREM 1.1** *Let  $u$  denote a local  $J$ -minimizer under condition (1.1). Assume further that*

$$q < \min(2p, p + 2) \tag{1.3}$$

*holds. Then  $u$  is of class  $C^{2,\alpha}(\Omega)$  for any  $0 < \alpha < 1$ .*

**REMARK 1.1** i) *Clearly the result of Theorem 1.1 extends to local minimizers of the variational integral*

$$I[w, \Omega] = \int_{\Omega} f(\nabla^2 w) \, dx + \int_{\Omega} g(\nabla w) \, dx,$$

*where  $f$  is as before and where  $g$  denotes a density of class  $C^2$  satisfying*

$$0 \leq D^2 g(\xi)(\eta, \eta) \leq c(1 + |\xi|^2)^{\frac{s-2}{2}} |\eta|^2$$

*for some suitable exponent  $s$ . In case  $p \geq 2$  any finite number is admissible for  $s$ , in case  $p < 2$  we require the bound  $s \leq 2p/(2-p)$ . The details are left to the reader.*

- ii) *W.l.o.g. we may assume that  $q \geq 2$ : if (1.1) holds with some exponent  $q < 2$ , then of course (1.1) is true with  $q$  replaced by  $\bar{q} := 2$  and (1.3) continues to hold for the new exponent.*
- iii) *If we consider the higher order variational integral  $\int_{\Omega} f(\nabla^k w) \, dx$  with  $k \geq 2$  and  $f$  satisfying (1.1), then (1.3) implies that local minimizers  $u \in W_{p,loc}^k(\Omega)$  actually belong to the space  $C^{k,\alpha}(\Omega)$ .*
- iv) *The degree of smoothness of  $u$  can be improved by standard arguments provided  $f$  is sufficiently regular.*
- v) *A typical example of an energy  $J$  satisfying the assumptions of Theorem 1.1 is given by*

$$J[w, \Omega] = \int_{\Omega} |\nabla^2 w|^2 \, dx + \int_{\Omega} (1 + |\partial_1 \partial_2 w|^2)^{\frac{q}{2}} \, dx$$

*with some exponent  $q \in (2, 4)$ .*

vi) *Our arguments can easily be adjusted to prove  $C^{k,\alpha}$ -regularity of local minimizers  $u \in W_{p(x),loc}^k(\Omega)$  of the energy  $\int_{\Omega} (1 + |\nabla^k w|^2)^{p(x)/2} dx$  provided that  $1 < p_* \leq p(x) < p^* < \infty$  for some numbers  $p_*$ ,  $p^*$  and if  $p(x)$  is sufficiently smooth. Another possible extension concerns the logarithmic case, i.e. we now consider the variational integral  $\int_{\Omega} |\nabla^k w| \ln(1 + |\nabla^k w|) dx$  and its local minimizers which have to be taken from the corresponding higher order Orlicz-Sobolev space.*

The proof of Theorem 1.1 is organized as follows: we first introduce some suitable regularization and then prove the existence of higher order weak derivatives for this approximating sequence in Step 2. Here we also derive a Caccioppoli-type inequality using difference quotient methods. In a third step we deduce uniform higher integrability of the second generalized derivatives for any finite exponent. From this together with a lemma established in [BFZ] we finally obtain our regularity result in the last two steps.

## 2 Proof of Theorem 1.1

*Step 1. Approximation*

Let us fix some open domains  $\Omega_1 \Subset \Omega_2 \Subset \Omega$  and denote by  $\bar{u}_m$  the mollification of  $u$  with radius  $1/m$ , in particular

$$\|\bar{u}_m - u\|_{W_p^2(\Omega_2)} \xrightarrow{m \rightarrow \infty} 0.$$

Jensen's inequality implies

$$J[\bar{u}_m, \Omega_2] \leq J[u, \Omega_2] + \tau_m,$$

where  $\tau_m \rightarrow 0$  as  $m \rightarrow \infty$ . This, together with the lower semicontinuity of the functional  $J$ , shows that

$$J[\bar{u}_m, \Omega_2] \xrightarrow{m \rightarrow \infty} J[u, \Omega_2]. \quad (2.1)$$

Next let

$$\rho_m := \|\bar{u}_m - u\|_{W_p^2(\Omega_2)} \left[ \int_{\Omega_2} (1 + |\nabla^2 \bar{u}_m|^2)^{\frac{q}{2}} dx \right]^{-1},$$

which obviously tends to 0 as  $m \rightarrow \infty$ . With these preliminaries we introduce the regularized functional

$$J_m[w, \Omega_2] := \rho_m \int_{\Omega_2} (1 + |\nabla^2 w|^2)^{\frac{q}{2}} dx + J[w, \Omega_2]$$

and the corresponding regularizing sequence  $\{u_m\}$  as the sequence of the unique solutions to the problems

$$J_m[\cdot, \Omega_2] \rightarrow \min \quad \text{in } \bar{u}_m + \mathring{W}_q^2(\Omega_2). \quad (2.2)$$

By (2.1) and (2.2) we have

$$\begin{aligned} J_m[u_m, \Omega_2] &\leq J_m[\bar{u}_m, \Omega_2] \\ &= \|\bar{u}_m - u\|_{W_p^2(\Omega_2)} + J[\bar{u}_m, \Omega_2] \\ &\xrightarrow{m \rightarrow \infty} J[u, \Omega_2], \end{aligned}$$

hence one gets

$$\limsup_{m \rightarrow \infty} J_m[u_m, \Omega_2] \leq J[u, \Omega_2]. \quad (2.3)$$

On account of (2.3) and the growth of  $f$  we may assume

$$u_m \xrightarrow{m \rightarrow \infty} \hat{u} \quad \text{in } W_p^2(\Omega_2).$$

Moreover, lower semicontinuity gives

$$J[\hat{u}, \Omega_2] \leq \liminf_{m \rightarrow \infty} J[u_m, \Omega_2],$$

which together with (2.3) and the strict convexity of  $f$  implies  $\hat{u} = u$  (here we also note that  $\hat{u} - u \in \mathring{W}_p^2(\Omega_2)$ ). Summarizing the results it is shown up to now that (as  $m \rightarrow \infty$ )

$$\begin{aligned} u_m &\rightarrow u \quad \text{in } W_p^2(\Omega_2), \\ J_m[u_m, \Omega_2] &\rightarrow J[u, \Omega_2]. \end{aligned} \quad (2.4)$$

*Step 2. Existence of higher order weak derivatives*

In this second step we will prove that  $(f_m(\xi) := \rho_m(1 + |\xi|^{q/2} + f(\xi)))$

$$\begin{aligned} &\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m) dx \\ &\leq c(\|\nabla \eta\|_\infty^2 + \|\nabla^2 \eta\|_\infty^2) \int_{\text{spt } \nabla \eta} |D^2 f_m(\nabla^2 u_m)| [|\nabla^2 u_m|^2 + |\nabla u_m|^2] dx, \end{aligned} \quad (2.5)$$

where  $\eta \in C_0^\infty(\Omega_2)$ ,  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  on  $\Omega_1$  and where we take the sum over repeated indices. To this purpose let us recall the Euler equation

$$\int_{\Omega_2} Df_m(\nabla^2 u_m) : \nabla^2 \varphi = 0 \quad \text{for all } \varphi \in \mathring{W}_q^2(\Omega_2). \quad (2.6)$$

If  $\Delta_h$  denotes the difference quotient in the coordinate direction  $e_\alpha$ ,  $\alpha = 1, 2$ , then the test function  $\Delta_{-h}(\eta^6 \Delta_h u_m)$  is admissible in (2.6) with the result

$$\int_{\Omega_2} \Delta_h \{Df_m(\nabla^2 u_m)\} : \nabla^2(\eta^6 \Delta_h u_m) dx = 0. \quad (2.7)$$

Now denote by  $\mathcal{B}_x$  the bilinear form

$$\mathcal{B}_x = \int_0^1 D^2 f_m(\nabla^2 u_m(x) + th \nabla^2(\Delta_h u_m)(x)) dt,$$

and observe that

$$\begin{aligned} \Delta_h \{Df_m(\nabla^2 u_m)\}(x) &= \frac{1}{h} \int_0^1 \frac{d}{dt} Df_m(\nabla^2 u_m(x) + t[\nabla^2 u_m(x + he_\alpha) - \nabla^2 u_m(x)]) dt \\ &= \frac{1}{h} \int_0^1 \frac{d}{dt} Df_m(\nabla^2 u_m(x) + ht \nabla^2(\Delta_h u_m)(x)) dt \\ &= \mathcal{B}_x(\nabla^2(\Delta_h u_m)(x), \cdot), \end{aligned}$$



hence (2.7) can be written as

$$\int_{\Omega_2} \mathcal{B}_x(\nabla^2(\Delta_h u_m), \nabla^2(\eta^6 \Delta_h u_m)) \, dx = 0,$$

which means that we have

$$\begin{aligned} \int_{\Omega_2} \eta^6 \mathcal{B}_x(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)) \, dx &= - \int_{\Omega_2} \mathcal{B}_x(\nabla^2(\Delta_h u_m), \nabla^2 \eta^6 \Delta_h u_m) \, dx \\ &\quad - 2 \int_{\Omega_2} \mathcal{B}_x(\nabla^2(\Delta_h u_m), \nabla \eta^6 \odot \nabla(\Delta_h u_m)) \, dx \\ &=: -T_1 - 2T_2. \end{aligned} \quad (2.8)$$

To handle  $T_1$  we just observe  $\partial_\alpha \partial_\beta \eta^6 = 30 \partial_\alpha \eta \partial_\beta \eta \eta^4 + 6 \partial_\alpha \partial_\beta \eta \eta^5$ , for  $T_2$  we use  $\nabla \eta^6 = 6 \eta^5 \nabla \eta$ . The Cauchy-Schwarz inequality for the bilinear form  $\mathcal{B}_x$  implies

$$\begin{aligned} |T_2| &= 6 \left| \int_{\Omega_2} \mathcal{B}_x(\eta^3 \nabla^2(\Delta_h u_m), \eta^2 \nabla \eta \odot \nabla(\Delta_h u_m)) \, dx \right| \\ &\leq 6 \left[ \int_{\Omega_2} \mathcal{B}_x(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)) \eta^6 \, dx \right]^{\frac{1}{2}} \\ &\quad \cdot \left[ \int_{\Omega_2} \mathcal{B}_x(\nabla \eta \odot \nabla(\Delta_h u_m), \nabla \eta \odot \nabla(\Delta_h u_m)) \eta^4 \, dx \right]^{\frac{1}{2}}, \end{aligned}$$

an analogous estimate being valid for  $T_1$ . Absorbing terms, (2.8) turns into

$$\begin{aligned} &\int_{\Omega_2} \eta^6 \mathcal{B}_x(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)) \, dx \\ &\leq c(\|\nabla \eta\|_\infty^2 + \|\nabla^2 \eta\|_\infty^2) \int_{\text{spt } \nabla \eta} |\mathcal{B}_x|(|\nabla(\Delta_h u_m)|^2 + |\Delta_h u_m|^2) \, dx. \end{aligned} \quad (2.9)$$

Next we estimate (note that in the following calculations we always assume w.l.o.g.  $q \geq 2$ , compare Remark 1.1, ii)) for  $h$  sufficiently small

$$\begin{aligned} &\int_{\text{spt } \nabla \eta} |\mathcal{B}_x| |\nabla(\Delta_h u_m)|^2 \, dx \\ &\leq \int_{\text{spt } \nabla \eta} (1 + |\nabla^2 u_m|^2 + h^2 |\nabla^2(\Delta_h u_m)|^2)^{\frac{q-2}{2}} |\nabla(\Delta_h u_m)|^2 \, dx \\ &\leq c \left[ \int_{\text{spt } \nabla \eta} |\nabla(\Delta_h u_m)|^{\frac{q}{2}} \, dx \right. \\ &\quad \left. + \int_{\text{spt } \nabla \eta} (1 + |\nabla^2 u_m|^2 + h^2 |\nabla^2(\Delta_h u_m)|^2)^{\frac{q}{2}} \, dx \right] \\ &\leq c \int_{\text{spt } \nabla \eta} (1 + |\nabla^2 u_m|^2)^{\frac{q}{2}} \, dx. \end{aligned}$$

In a similar way we estimate  $\int_{\text{spt } \nabla \eta} |\mathcal{B}_x| |\Delta_h u_m|^2 dx$  and end up with

$$\begin{aligned} & \limsup_{h \rightarrow 0} \int_{\Omega_2} \eta^6 \mathcal{B}_x(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)) dx \\ & \leq c(\|\nabla \eta\|_\infty^2 + \|\nabla^2 \eta\|_\infty^2) \int_{\text{spt } \nabla \eta} (1 + |\nabla u_m|^2 + |\nabla^2 u_m|^2)^{\frac{q}{2}} dx. \end{aligned} \quad (2.10)$$

Since  $q \geq 2$  is assumed, (2.10) implies that  $\nabla^2 u_m \in W_{2,loc}^1(\Omega_2)$  and

$$\Delta_h(\nabla^2 u_m) \xrightarrow{h \rightarrow 0} \partial_\alpha(\nabla^2 u_m) \quad \text{in } L_{loc}^2(\Omega_2) \quad \text{and a.e.}$$

**REMARK 2.1** *With (2.10) we have*

$$|\Delta_h \{Df_m(\nabla^2 u_m)\}|^{\frac{q}{q-1}} \in L_{loc}^1(\Omega_2) \quad \text{uniformly w.r.t. } h,$$

and, as a consequence,

$$Df_m(\nabla^2 u_m) \in W_{q/(q-1),loc}^1(\Omega_2).$$

*This follows exactly as outlined in the calculations after (3.12) of [BF3].*

With the above convergences and Fatou's lemma we find the lower bound

$$\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m) dx$$

for the l.h.s. of (2.10) which gives using (1.1)

$$\begin{aligned} & \int_{\Omega_2} \eta^6 (1 + |\nabla^2 u_m|^2)^{\frac{p-2}{2}} |\nabla^3 u_m|^2 dx \\ & \leq c(\|\nabla \eta\|_\infty^2 + \|\nabla^2 \eta\|_\infty^2) \int_{\text{spt } \nabla \eta} (1 + |\nabla u_m|^2 + |\nabla^2 u_m|^2)^{\frac{q}{2}} dx < \infty, \end{aligned}$$

in particular

$$h_m := (1 + |\nabla^2 u_m|^2)^{\frac{p}{4}} \in W_{2,loc}^1(\Omega_2). \quad (2.11)$$

But (2.11) implies  $h_m \in L_{loc}^r(\Omega_2)$  for any  $r < \infty$ , i.e.

$$\nabla^2 u_m \in L_{loc}^t(\Omega_2) \quad \text{for any } t < \infty. \quad (2.12)$$

Using Fatou's lemma again we obtain from (2.8)

$$\begin{aligned} & \int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m) dx \\ & \leq \liminf_{h \rightarrow 0} \int_{\Omega_2} \eta^6 \Delta_h \{Df_m(\nabla^2 u_m)\} : \nabla^2(\Delta_h u_m) dx \\ & = \liminf_{h \rightarrow 0} - \int_{\Omega_2} \Delta_h \{Df_m(\nabla^2 u_m)\} : [\nabla^2 \eta^6 \Delta_h u_m + 2\nabla \eta^6 \odot \nabla(\Delta_h u_m)] dx \end{aligned} \quad (2.13)$$

On account of (2.12), Remark 2.1 and Vitali's convergence theorem we may pass to the limit  $h \rightarrow 0$  on the r.h.s. of (2.13) and obtain

$$\begin{aligned} & \int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m) dx \\ & \leq - \int_{\Omega_2} D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \nabla^2 \eta^6 \partial_\alpha u_m + 2\nabla \eta^6 \odot \nabla \partial_\alpha u_m) dx. \end{aligned}$$

This immediately gives (2.5) by repeating the calculations leading from (2.8) to (2.9).

*Step 3. Uniform higher integrability of  $\nabla^2 u_m$*

Let  $\chi$  denote any real number satisfying  $\chi > p/(2p - q)$ , moreover we set  $\alpha = \chi p/2$ . For all discs  $B_r \Subset B_R \Subset \Omega_2$  any  $\eta \in C_0^\infty(B_R)$ ,  $\eta \equiv 1$  on  $B_r$ ,  $|\nabla^k \eta| \leq c/(R - r)^k$ ,  $k = 1, 2$ , we have by Sobolev's inequality

$$\begin{aligned} \int_{B_r} (1 + |\nabla^2 u_m|^2)^\alpha dx &\leq \int_{B_R} (\eta^3 h_m)^{2\chi} dx \\ &\leq c \left[ \int_{B_R} |\nabla(\eta^3 h_m)|^t dx \right]^{\frac{2\chi}{t}}, \end{aligned}$$

where  $t \in (1, 2)$  satisfies  $2\chi = 2t/(2 - t)$ . Hölder's inequality implies

$$\begin{aligned} \int_{B_r} (1 + |\nabla^2 u_m|^2)^\alpha dx &\leq c(r, R) \left[ \int_{B_R} |\nabla(\eta^3 h_m)|^2 dx \right]^\chi \\ &\leq c(r, R) \left[ \int_{B_R} \eta^6 |\nabla h_m|^2 dx + \int_{\text{spt } \nabla \eta} |\nabla \eta^3|^2 h_m^2 dx \right]^\chi. \end{aligned}$$

Observing that obviously

$$\int_{\text{spt } \nabla \eta} |\nabla \eta^3|^2 h_m^2 dx \leq c(r, R) \int_{\text{spt } \nabla \eta} (1 + |\nabla^2 u_m|^2)^{\frac{p}{2}} dx$$

and that by (2.5)

$$\begin{aligned} \int_{B_R} \eta^6 |\nabla h_m|^2 dx &\leq c(r, R) \int_{\text{spt } \nabla \eta} (1 + |\nabla^2 u_m|^2)^{\frac{q-2}{2}} \left[ |\nabla^2 u_m|^2 + |\nabla u_m|^2 \right] dx \\ &\leq c(r, R) \left[ \int_{\text{spt } \nabla \eta} (1 + |\nabla^2 u_m|^2)^{\frac{q}{2}} dx + \int_{\text{spt } \nabla \eta} |\nabla u_m|^q dx \right], \end{aligned}$$

we deduce

$$\int_{B_r} (1 + |\nabla^2 u_m|^2)^\alpha dx \leq c(r, R) \left[ \int_{\text{spt } \nabla \eta} (1 + |\nabla^2 u_m|^2)^{\frac{q}{2}} dx + \int_{\text{spt } \nabla \eta} |\nabla u_m|^q dx \right]^\chi, \quad (2.14)$$

where  $c(r, R) = c(R - r)^{-\beta}$  for some suitable  $\beta > 0$ . For discussing (2.14) we first note that the term  $\int_{\text{spt } \nabla \eta} |\nabla u_m|^q dx$  causes no problems. In fact, since  $\|u_m\|_{W_p^2(\Omega_2)} \leq c < \infty$  we know that  $\nabla u_m \in L_{loc}^t(\Omega_2)$  for any  $t < \infty$  in case  $p \geq 2$ . If  $p < 2$ , then we have local  $L^t$ -integrability of  $\nabla u_m$  provided that  $t < 2p/(2 - p)$ , but  $q < 2p < 2p/(2 - p)$  on account of (1.3). As a consequence, we may argue exactly as in [ELM] or [Bi], p. 60, to derive from (2.14) by interpolation and hole-filling (here  $q < 2p$  enters in an essential way)

$$\nabla^2 u_m \in L_{loc}^t(\Omega_2) \quad \text{for any } t < \infty \text{ and uniformly w.r.t. } m. \quad (2.15)$$

Note that (2.15) implies with Step 2 the uniform bound

$$\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m) dx \leq c(\eta) < \infty, \quad (2.16)$$

in particular (2.16) shows

$$h_m \in W_{2,loc}^1(\Omega_2) \quad \text{uniformly w.r.t. } m. \quad (2.17)$$

**REMARK 2.2** i) If  $u$  is a local  $J$ -minimizer subject to an additional constraint of the form  $u \geq \psi$  a.e. on  $\Omega$  for a sufficiently regular function  $\psi: \Omega \rightarrow \mathbb{R}$ , then it is an easy exercise to adjust the technique used in [BF1] to the present situation which means that we still have (2.15) so that (recall (2.4))  $u \in W_{t,loc}^2(\Omega)$  for any  $t < \infty$ , hence  $u \in C^{1,\alpha}(\Omega)$  for all  $0 < \alpha < 1$ . In [Fr], Theorem 10.6, p. 98, it is shown for the special case  $f(w) = |\Delta w|^2$  that actually  $u \in C^2(\Omega)$  is true, and it would be interesting to see if this result also holds for the energy densities discussed here.

ii) We remark that the proof of (2.15) just needs the inequality  $q < 2p$ , whereas the additional assumption  $q < p + 2$  enters in the next step.

*Step 4.  $C^2$ -regularity*

Now we consider an arbitrary disc  $B_{2R} \Subset \Omega_1$  and  $\eta \in C_0^\infty(B_{2R})$  satisfying  $\eta \equiv 1$  on  $B_R$  and  $|\nabla \eta| \leq c/R$ ,  $|\nabla^2 \eta| \leq c/R^2$ . Moreover we denote by  $T_{2R}$  the annulus  $T_{2R} := B_{2R} - B_R$  and by  $P_m$  a polynomial function of degree less than or equal to 2. Exactly as in Step 2 (replacing  $u_m$  by  $u_m - P_m$ ) we obtain

$$\begin{aligned} & \int_{B_{2R}} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m) dx \\ & \leq - \int_{T_{2R}} D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \nabla^2 \eta^6 \partial_\alpha [u_m - P_m] + 2\nabla \eta^6 \odot \nabla \partial_\alpha (u_m - P_m)) dx. \end{aligned}$$

With the notation

$$H_m := \left[ D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m) \right]^{\frac{1}{2}}, \quad \sigma_m := Df_m(\nabla^2 u_m)$$

we therefore have

$$\int_{B_{2R}} \eta^6 H_m^2 dx \leq c \int_{T_{2R}} |\nabla \sigma_m| [|\nabla^2 \eta^6| |\nabla u_m - \nabla P_m| + |\nabla \eta^6| |\nabla^2 u_m - \nabla^2 P_m|] dx.$$

Moreover, by the Cauchy-Schwarz inequality and (1.1)

$$|\nabla \sigma_m|^2 \leq H_m \left[ D^2 f_m(\nabla^2 u_m)(\partial_\alpha \sigma_m, \partial_\alpha \sigma_m) \right]^{\frac{1}{2}} \leq H_m |\nabla \sigma_m| \Gamma_m^{\frac{q-2}{4}},$$

where  $\Gamma_m := 1 + |\nabla^2 u_m|^2$ . Finally we let

$$\tilde{h}_m := \max \left[ \Gamma_m^{\frac{q-2}{4}}, \Gamma_m^{\frac{2-p}{4}} \right]$$

and obtain

$$|\nabla \sigma_m| \leq c H_m \Gamma_m^{\frac{q-2}{4}} \leq c H_m \tilde{h}_m,$$

hence

$$\int_{B_{2R}} \eta^6 H_m^2 dx \leq c \int_{T_{2R}} H_m \tilde{h}_m [|\nabla^2 \eta^6| |\nabla u_m - \nabla P_m| + |\nabla \eta^6| |\nabla^2 u_m - \nabla^2 P_m|] dx. \quad (2.18)$$

Letting  $\gamma = 4/3$  we discuss the r.h.s. of (2.18):

$$\begin{aligned} & \int_{T_{2R}} H_m \tilde{h}_m |\nabla \eta^6| |\nabla^2 u_m - \nabla^2 P_m| \, dx \\ & \leq \frac{c}{R} \left[ \int_{B_{2R}} (H_m \tilde{h}_m)^\gamma \, dx \right]^{\frac{1}{\gamma}} \left[ \int_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4 \, dx \right]^{\frac{1}{4}}. \end{aligned}$$

Next the choice of  $P_m$  is made more precise by the requirement

$$\nabla^2 P_m = \fint_{B_{2R}} \nabla^2 u_m \, dx. \quad (2.19)$$

Then Sobolev-Poincaré's inequality together with the definition of  $\tilde{h}_m$  gives

$$\left[ \int_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4 \, dx \right]^{\frac{1}{4}} \leq c \left[ \int_{B_{2R}} |\nabla^3 u_m|^\gamma \, dx \right]^{\frac{1}{\gamma}} \leq c \left[ \int_{B_{2R}} (H_m \tilde{h}_m)^\gamma \, dx \right]^{\frac{1}{\gamma}},$$

hence

$$\int_{T_{2R}} H_m \tilde{h}_m |\nabla \eta^6| |\nabla^2 u_m - \nabla^2 P_m| \, dx \leq \frac{c}{R} \left[ \int_{B_{2R}} (H_m \tilde{h}_m)^\gamma \, dx \right]^{\frac{2}{\gamma}}. \quad (2.20)$$

To handle the remaining term on the r.h.s. of (2.18) we need in addition to (2.19)

$$\fint_{B_{2R}} (\nabla u_m - \nabla P_m) \, dx = 0,$$

which can be achieved by adjusting the linear part of  $P_m$ . Then we have by Poincaré's inequality

$$\begin{aligned} & \int_{B_{2R}} H_m \tilde{h}_m |\nabla^2 \eta^6| |\nabla u_m - \nabla P_m| \, dx \\ & \leq \frac{c}{R^2} \left[ \int_{B_{2R}} (H_m \tilde{h}_m)^\gamma \, dx \right]^{\frac{1}{\gamma}} \left[ \int_{B_{2R}} |\nabla u_m - \nabla P_m|^4 \, dx \right]^{\frac{1}{4}} \\ & \leq \frac{c}{R} \left[ \int_{B_{2R}} (H_m \tilde{h}_m)^\gamma \, dx \right]^{\frac{1}{\gamma}} \left[ \int_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4 \, dx \right]^{\frac{1}{4}}, \end{aligned}$$

and the r.h.s. is bounded by the r.h.s. of (2.20). Hence, recalling (2.18) and (2.20), we have established the inequality

$$\left[ \fint_{B_R} H_m^2 \, dx \right]^{\frac{\gamma}{2}} \leq c \fint_{B_{2R}} (H_m \tilde{h}_m)^\gamma \, dx. \quad (2.21)$$

Given this starting inequality we like to apply the following lemma which is proved in [BFZ].

**LEMMA 2.1** *Let  $d > 1$ ,  $\beta > 0$  be two constants. With a slight abuse of notation let  $f$ ,  $g$ ,  $h$  now denote any non-negative functions in  $\Omega \subset \mathbb{R}^n$  satisfying*

$$f \in L_{loc}^d(\Omega), \quad \exp(\beta g^d) \in L_{loc}^1(\Omega), \quad h \in L_{loc}^d(\Omega).$$

*Suppose that there is a constant  $C > 0$  such that*

$$\left[ \int_B f^d dx \right]^{\frac{1}{d}} \leq C \int_{2B} fg dx + C \left[ \int_{2B} h^d dx \right]^{\frac{1}{d}}$$

*holds for all balls  $B = B_r(x)$  with  $2B = B_{2r}(x) \Subset \Omega$ . Then there is a real number  $c_0 = c_0(n, d, C)$  such that if  $h^d \log^{c_0 \beta}(e + h) \in L_{loc}^1(\Omega)$ , then the same is true for  $f$ . Moreover, for all balls  $B$  as above we have*

$$\begin{aligned} \int_B f^d \log^{c_0 \beta} \left[ e + \frac{f}{\|f\|_{d, 2B}} \right] dx &\leq c \left[ \int_{2B} \exp(\beta g^d) dx \right] \left[ \int_{2B} f^d dx \right] \\ &\quad + c \int_{2B} h^d \log^{c_0 \beta} \left[ e + \frac{h}{\|f\|_{d, 2B}} \right] dx, \end{aligned}$$

*where  $c = c(n, d, \beta, C) > 0$  and  $\|f\|_{d, 2B} = (\int_{2B} f^d dx)^{1/d}$ .*

The appropriate choices in the setting at hand are  $d = 2/\gamma = 3/2$ ,  $f = H_m^\gamma$ ,  $g = \tilde{h}_m^\gamma$ ,  $h \equiv 0$ . We claim that

$$\int_{B_{2R}} \exp(\tilde{h}_m^2 \beta) dx \leq c \quad \text{and} \quad \int_{B_{2R}} H_m^2 dx \leq c$$

for a constant being uniform in  $m$ . The uniform bound of the second integral follows from (2.16), thus let us discuss the first one. By (2.17) and Trudinger's inequality (see e.g. Theorem 7.15 of [GT]) we know that for any disc  $B_\rho \Subset \Omega_1$

$$\int_{B_\rho} \exp(\beta_0 h_m^2) dx \leq c(\rho) < \infty,$$

where  $\beta_0$  just depends on the uniformly bounded quantities  $\|h_m\|_{W_2^1(\Omega_1)}$ . This implies for any  $\beta > 0$  and  $\kappa \in (0, 1)$

$$\int_{B_\rho} \exp(\beta h_m^{2-\kappa}) dx \leq c(\rho, \beta, \kappa) < \infty.$$

Moreover, on account of  $q < p + 2$  we have

$$\Gamma_m^{\frac{q-2}{2}} \leq h_m^{2-\kappa} \quad \text{and clearly} \quad \Gamma_m^{\frac{2-p}{2}} \leq h_m^{2-\kappa}$$

for  $\kappa$  sufficiently small, which gives our claim and we may indeed apply the lemma with the result

$$\int_{B_\rho} H_m^2 \log^{c_0 \beta}(e + H_m) dx \leq c(\beta, \rho) < \infty$$

for all discs  $B_\rho \subset \Omega_1$  and all  $\beta > 0$ . Thus we have established the counterparts of (2.7) and (2.10) in [BFZ], and exactly the same arguments as given there lead to (2.11) from [BFZ], thus we deduce the uniform continuity of the sequence  $\{\sigma_m\}$  (see again [BFZ], end of Section 2), hence we have uniform convergence  $\sigma_m \rightarrow \sigma$  for some continuous tensor  $\sigma$ . In order to identify  $\sigma$  with  $Df(\nabla^2 u)$ , we recall the weak convergence stated in (2.4) and also observe that  $\nabla^2 u_m \rightarrow \nabla^2 u$  a.e. which can be deduced along the same lines as in Lemma 4.5 c) of [BF3], we also refer to Proposition 3.29 iii) of [Bi]. Therefore  $Df(\nabla^2 u)$  is a continuous function, i.e.  $\nabla^2 u$  is of class  $C^0$ , and finally  $u \in C^2(\Omega)$  follows.

*Step 5.  $C^{2,\alpha}$ -regularity of  $u$*

To finish the proof of Theorem 1.1 we observe that with Step 4 we get from (2.5) the estimate

$$\int_{\Omega_1} |\nabla^3 u_m|^2 dx \leq c(\Omega_1) < \infty,$$

in particular one has for  $\alpha = 1, 2$

$$U := \partial_\alpha u \in W_{2,loc}^2(\Omega).$$

Moreover we have

$$\int_{\Omega} D^2 f_m(\nabla^2 u_m)(\nabla^2 \partial_\alpha u_m, \nabla^2 \varphi) dx = 0 \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

Together with the convergences (as  $m \rightarrow \infty$ )

$$\begin{aligned} D^2 f_m(\nabla^2 u_m) &\rightarrow D^2 f(\nabla^2 u) \quad \text{in } L_{loc}^\infty(\Omega), \\ \nabla^2 \partial_\alpha u_m &\rightharpoonup \nabla^2 U \quad \text{in } L_{loc}^2(\Omega) \end{aligned}$$

we therefore arrive at the limit equation

$$\int_{\Omega} D^2 f(\nabla^2 u)(\nabla^2 U, \nabla^2 \varphi) dx = 0,$$

hence  $U$  is a weak solution of an equation with continuous coefficients and  $u \in C^{2,\alpha}(\Omega)$  for any  $0 < \alpha < 1$  follows from [GM], Theorem 4.1.  $\square$

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