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Sunghan Bae, Ernst-U. Gekeler and Linsheng Yin

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Sunghan Bae

Department of Mathematics
KAIST
Taejon 305-701
Korea
E-Mail:

Ernst-U. Gekeler

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: gekeler@math.uni-sb.de

Linsheng Yin

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: lsyin@math.uni-sb.de

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Edited by
FR 6.1 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

DISTRIBUTIONS AND Γ-MONOMIALS

1. Introduction.

In the paper [A1], Anderson invented a remarkable method of double complex to compute the sign-cohomology of the universal ordinary distribution. Das [Da] applied Anderson's methods and results to the study of the classical Γ-monomials, and got a series of results greatly illuminating the structure of the Galois group over $\mathbb{Q}$ of the extension of $\mathbb{Q}(\xi_\infty)$ generated by the algebraic Γ-monomials. Using Anderson's methods, he was also able to give elementary proofs of some facts about algebraic Γ-monomials, which previously could only be proved with the aid of Deligne's theory [D] of absolute Hodge cycles on abelian varieties. In Das' paper, he also proposed some questions for further study.

In this paper, we emphasize the applications of distributions in the study of algebraic Γ-monomials. The distribution method introduced by Yin in [Y2] may be applied to any global field. In the largest part of the paper we consider the two cases of the rational number field and of a global function field simultaneously. We explain, among other things, that two criteria for an element to belong to the first or the second sign-cohomology of the universal ordinary distribution are direct consequences of the universality of some distributions constructed from partial zeta functions. This answers Thakur's question 9.7(c) in [Th]. In the rational number field case, this gives new proofs of some results in Das' paper, and also gives an affirmative answer to one of the questions listed in the end of Das' paper. In this paper, we also define a new Γ-function in characteristic p. Let k be a global function field. We fix a place ∞ of k. Let $\mathbb{C}_k$ be the completion of the algebraic closure of k_∞. Thakur [Th] defined a characteristic-p Γ-function from $\mathbb{C}_k$ to $\mathbb{C}_k \cup \{\infty\}$, and studied this Γ-function carefully. Our new Γ-function is defined on the set of non-zero fractional ideals of k and takes values in $\mathbb{C}_k \cup \{\infty\}$. The advantages of the new Γ-function are that we can consider the Galois action on it and that we can apply the distribution method to it. We study the analogues of the properties for the new Γ-function of the reflection and multiplication formulas of the classical Euler gamma function. As a direct consequence, we get the algebraicity of some monomials of this Γ-function. At the same time, we explain the result of Koblitz-Ogus [KO] on the classical Γ-monomials. Finally, we connect algebraic Γ-monomials with the cyclotomic units. In the classical case, this result was first given by Das [Da] by using Anderson's double complex. For further properties of this new Γ-function, one needs to develop Anderson's theory on solitons to show the analogue of Deligne's reciprocity, and to develop Anderson-Das' theory on the double complex to study its Galois theoretic characters. The analogue of Deligne reciprocity [D] in the case of the rational function fields has been given by Sinha [S1] by using Anderson's solitons [A2]. These algebraic Γ-monomials may be a new supply of special units of abelian extensions of a global function field, and may rule out the constant factor in the unit-index formula in [Y1]. Perhaps this gamma function will also give the answer to Thakur's question 9.7(d) in [Th].

In this paper, we also raise two conjectures on the universality of distributions of special values of L-functions and of our Γ-function. The former is a great generalization of the classical Bass theorem [B] on the cyclotomic units (conjectured by Milnor), and the latter is a characteristic-p analogue of Rohrlich's conjecture, which is a major unsolved conjecture in transcendental number theory.

2. Global distributions.

Global distributions here mean distributions of global fields. This concept was introduced by Yin [Y2] recently. In this section, we recall the definitions and some known results in this direction. These definitions and results will be used later.

Let k be a global field, i.e., a number field or a global function field. In the function field case, we fix a place ∞ of k with degree d_∞ and fix a sign function sgn on k_∞, see (Def.4.1, [H2]). Let $\mathbb{A}$ be the Dedekind subring of k consisting of the functions regular outside of ∞. An element $x \in k$ is called (totally) positive and denoted by $x \gg 0$ if $\operatorname{sgn}(x) = 1$. In the number field case, the concept of totally positive is in the ordinary sense. Let $\mathbb{A}$ be the integral closure of $\mathbb{Z}$ in k. In both cases, let T_0 (resp. $\bar{T}_0$) be the set of non-zero integral (resp. fractional) ideals of $\mathbb{A}$. In $\bar{T}_0$, we define an equivalence relation (in the "narrow sense") as follows: $\mathfrak{a} \sim \mathfrak{b}$ if and only if there exists $x \in 1 + \mathfrak{b}^{-1}$ and $x \gg 0$ such that $\mathfrak{a} = x\mathfrak{b}$. If the totally positive condition is removed, we get another equivalence relation $\sim'$ in the "wide sense". The equivalence relation $\sim$ has the following properties, whose proofs we leave to the reader.

Lemma 2.1. *Let* $\mathfrak{a}, \mathfrak{b}, \mathfrak{m} \in T_0$ *and let* $\mathfrak{u}, \mathfrak{v} \in \bar{T}_0$. *We have*
(1) If $\mathfrak{u} \sim \mathfrak{v}$ *then* $\mathfrak{u}\mathfrak{m} \sim \mathfrak{v}\mathfrak{m}$. *Furthermore, if* $\mathfrak{m}$ *is coprime to the fractional parts of* $\mathfrak{u}$ *and* $\mathfrak{v}$, *then the inverse is also valid.*
(2) $\mathfrak{a}\mathfrak{m}^{-1} \sim \mathfrak{b}\mathfrak{m}^{-1}$ *if and only if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are in the same narrow ray ideal class of conductor* $\mathfrak{m}$.
(3) The set of $\sim$*-equivalence classes of fractional ideals* $\mathfrak{v}$ *such that* $\mathfrak{m}\mathfrak{v} \sim \mathfrak{u}$ *is finite.*

Remark: In the function field case one can fix a finite set $\{\infty_1, \cdots, \infty_r\}$ of places of k (not just a single place as above), and fix sign-functions $\{\mathrm{sgn}_1, \cdots, \mathrm{sgn}_r\}$ at each place. Let $\mathbb{A}$ be the Dedekind subring of k of functions regular outside of all ∞_i. Then all the concepts above and below about distributions can be extended to the general case. But we can not define a gamma function, the main object we study in the paper, in this general case. So in this paper we restrict to the simple case of a single place.

An ordinary distribution of k is a function f from $\bar{T}_0$ to an abelian group V that factors through $\bar{T}_0 \to \bar{T}_0/\sim$, and satisfies the following relations, for all $\mathfrak{u} \in \bar{T}_0$ and all $\mathfrak{m} \in T_0$,

$$f(\mathfrak{u}) = \sum_i f(\mathfrak{v}_i), \tag{2.1}$$

where $\mathfrak{v}_i$ runs over a complete set of representatives for $\sim$-equivalence classes in the set of frational ideals $\mathfrak{v}$ such that $\mathfrak{m}\mathfrak{v} \sim \mathfrak{u}$. By Lem.2.1(3) the sum above is a finite sum. This definition fits into Mazur's general framework of distributions on a projective system of finite sets (cf. [Y2]). In this paper we call an ordinary distribution a distribution for simplicity. f is called *even* (*real* in [Y2]) if it factors in the wide sense, and called *punctured* if it is defined only on $\bar{T}_0 \setminus T_0$. The level $\mathfrak{m}$ group of f is the subgroup of V generated by $f(\mathfrak{a}\mathfrak{m}^{-1})$ with $\mathfrak{a} \in T_0$ (and $\mathfrak{m} \nmid \mathfrak{a}$ if f is punctured).

Let $G_\mathfrak{m}$ be the narrow ray class group of k of conductor $\mathfrak{m}$ and let $G = \varprojlim G_\mathfrak{m}$. For $\sigma \in G$ and $\mathfrak{a}\mathfrak{m}^{-1} \in \bar{T}_0$, let $\sigma_\mathfrak{b}$ be the image of σ under the natural map $G \to G_\mathfrak{m}$, where $\sigma_\mathfrak{b}$ is the Artin symbol associated to the integral ideal $\mathfrak{b}$. We define $\sigma f(\mathfrak{a}\mathfrak{m}^{-1}) = f(\mathfrak{a}\mathfrak{b}\mathfrak{m}^{-1})$, then σf is also a distribution by Lem.2.1(1). For any function defined on $\bar{T}_0/\sim$, we can define the action of G on it in the same way. In section 5, we will apply this technique to our Γ-function. Let $J_\mathfrak{m}$ be the classes in $G_\mathfrak{m}$ of the ideals of the form (a) for some $a \in \mathbb{A}$ such that $a \equiv 1 (\mathrm{mod}\, \mathfrak{m})$. We define $J = \varprojlim J_\mathfrak{m} \subset G$ and call it the sign-subgroup of G, following Anderson [A1]. Notice that J is finite. Let $s(J) \in \mathbb{Z}[G]$ be the sum of the elements in J. The distribution f is called *odd* if $s(J)f$ is the zero distribution. Note that in [Y2] we use the term "non-real" for odd, which is easy to misunderstand. The even distributions can also be defined as those on which J acts trivially. Especially when $|J| = 1$ there is no non-trivial odd distribution. This happens in the case of totally imaginary number fields and in the case of function fields with constant field of 2 elements. The V-valued distribution f is called *universal* if for any distribution $g : \bar{T}_0 \longrightarrow W$ there exists a unique homomorphism $h : V \longrightarrow W$ such that $g = hf$. One can construct a universal distribution via the free abelian group generated by $\bar{T}_0/\sim$ modulo the distribution relations. We denote by $A_\mathfrak{m}$ (resp. $A^\pm_\mathfrak{m}$, $A^0_\mathfrak{m}$, $(A^0_\mathfrak{m})^\pm$) the level $\mathfrak{m}$ group of a universal distribution (resp. a universal even (odd), a universal punctured, a universal punctured even (odd) distribution), and call them the universal level $\mathfrak{m}$ groups. They are $G_\mathfrak{m}$-modules via the action of G on f. All the concepts above are the extensions of the corresponding concepts on $\mathbb{Q}/\mathbb{Z}$ (i.e., where $k = \mathbb{Q}$ is the rational number field). In both cases of the rational number field and of the rational function field over a finite field, the structures of the universal level groups are determined completely, see [Thm 12.18, Wa], [GR] and [Ba]. For a general k, we only have the following partial results, see [Sect.3, Y2].

Theorem 2.2. $\mathrm{rank} A_\mathfrak{m} = |G_\mathfrak{m}|$. *If* $p \nmid |G_\mathfrak{m}|$ *for some prime* p, *the* p*-part of* $\mathrm{tor}(A_\mathfrak{m})$ *is zero.*

We can not determine the torsion completely. But in some cases $A_\mathfrak{m}$ is torsion free.

Theorem 2.3. *Assume that* $k = \mathbb{Q}$ *or a function field. Then* $A_\mathfrak{m} \simeq \mathbb{Z}^{|G_\mathfrak{m}|}$.

In the case $k = \mathbb{Q}$, this theorem is due to Kubert [K1]. In the function field case, it is due to Anderson [A1] and Yin [Y2] independently. Let $A = \varinjlim A_\mathfrak{m}$ and let $A^\pm = \varinjlim A^\pm_\mathfrak{m}$ respectively. Let $H^t(J, A)$ be the t-th Tate cohomology of the G-module A. We have

Proposition 2.4. *Assume that k is the rational number field or a function field. Then*

$$\mathrm{tor}(A^+) \simeq H^1(J,A) \qquad \textit{and} \qquad \mathrm{tor}(A^-) \simeq H^2(J,A).$$

Proof. For these two kinds of fields of k, J is cyclic. Let j be a generator of J. We claim that $H^1(J,A) = \mathrm{tor}(A/(1-j)A)$. This implies the first isomorphism since $A/(1-j)A \simeq A^+$. Let $\alpha \in A$ be such that $n\alpha \in (1-j)A$ for some integer n. Then $ns(J)\alpha = 0$. So $s(J)\alpha = 0$ as A is torsion-free. Noting that $H^1(J,A)$ is a torsion group, we get the claim. The proof of the second isomorphism is similar.

We mention that, in the rational number field case, Kubert [K2] computed the cohomology $H^*(J,A)$ building on Sinnott's ideas [Si], and in the function field case, Anderson [A1] computed it by a new method. Let w_k be the number of roots of unity in k. Then w_k annihilates the cohomology group $H^*(J,A)$.

In the end of this section, we recall the definition of Stickelberger distributions. Let $f : \bar{T}_0 \longrightarrow \mathbb{C}$ be a distribution with complex values. Let $\Omega = \varinjlim \mathbb{C}[G_{\mathfrak{m}}]$. The Stickelberger distribution associated to f is defined to be $\mathrm{St}(f) : \bar{T}_0 \longrightarrow \Omega$, where for $\mathfrak{u} = \mathfrak{a}\mathfrak{m}^{-1} \in \bar{T}_0$,

$$\mathrm{St}(f)(\mathfrak{u}) = \sum_{\mathfrak{b}} f(\mathfrak{b}\mathfrak{u})\sigma_{\mathfrak{b}}^{-1},$$

and where $\mathfrak{b} \in T_0$ runs over a complete set of representatives of the classes in $G_{\mathfrak{m}}$.

3. Some universal distributions.

In this section, we construct some universal distributions in the cases of the rational number field and of a function field by using the partial zeta functions.

At the beginning, we still assume that k is a global field. Let $\mathfrak{m}, \mathfrak{a} \in T_0$ be coprime. Let $w_{\mathfrak{m}}$ be the index of the totally positive subgroup in the group of units congruent to 1 modulo $\mathfrak{m}$. Let $\zeta_{\mathfrak{m}}^{\pm}(s,\mathfrak{a})$ be the partial zeta functions of the wide and the narrow ray class of $\mathfrak{a}$ modulo $\mathfrak{m}$, respectively. It is well-known that they can be extended to the whole complex plane and are holomorphic except for a pole at $s = 1$. Let $f^-(\mathfrak{a}\mathfrak{m}^{-1})$ be the first non-zero coefficient in the Taylor expansion of $\zeta_{\mathfrak{m}}^-(s,\mathfrak{a})$ at $s = 0$. Assume that $\mathfrak{m}$ is not the unit ideal $\mathfrak{e}$, let $f^+(\mathfrak{a}\mathfrak{m}^{-1})$ be the first non-zero, say s^r, coefficient in the Taylor expansion of $w_{\mathfrak{m}}\zeta_{\mathfrak{m}}^+(s,\mathfrak{a})$ at $s = 0$. We know that f^- is a distribution and f^+ is a punctured even distribution, with complex values [Sect.2, Y2]. The distribution f^+ is of great arithmetic interest, since the value of L-function at $s = 1$ associated to a real idele class character can be expressed as a finite sum by using this function. On this distribution, Yin raised the following conjecture (unpublished before):

Conjecture 3.1. *Assume that k is a number field. Then f^+ is a universal punctured even distribution with values in torsion-free abelian groups.*

When k is the rational number field, this is the Bass' Theorem [B] (conjectured by Milnor). Let $F^{\pm} = \mathrm{St}(f^{\pm})$ be the Stickelberger distributions associated to $f^{\pm}$, respectively. F^+ is punctured. Let $f_{\mathfrak{e}}(\mathfrak{a})$ be the coefficient of s^r in the Taylor expansion of $w_{\mathfrak{e}}\zeta_{\mathfrak{e}}^+(s,\mathfrak{a})$ at $s = 0$. We define

$$F^+(\mathfrak{a}) = \sum_{\mathfrak{b}} (f_{\mathfrak{e}}(\mathfrak{a}\mathfrak{b}) - f_{\mathfrak{e}}(\mathfrak{b}))\sigma_{\mathfrak{b}}^{-1},$$

where $\mathfrak{b}$ runs over all representatives of the classes in $G_{\mathfrak{m}}$ for some $\mathfrak{m}$. Notice that $F^+(\mathfrak{e}) = 0$. Let $\Omega' = \varinjlim \mathbb{C}[G_{\mathfrak{m}}]/(s(G_{\mathfrak{m}}))$, where $s(G_{\mathfrak{m}})$ is the sum of the elements in $G_{\mathfrak{m}}$. We let $F^{\pm} = \mathrm{St}(f^{\pm})$ take values in Ω' through the natural map $\Omega \to \Omega'$. Then F^+ becomes a non-punctured even distribution [Y2]. Especially, when k is the rational number field or a function field, we claim that F^- is odd. For $\mathfrak{u} = \mathfrak{a}\mathfrak{m}^{-1} \in \bar{T}_0$ with $(\mathfrak{a},\mathfrak{m}) = \mathfrak{e}$, we have

$$S(J)F^-(\mathfrak{u}) = \sum_{\alpha}\sum_{\mathfrak{b}} \zeta_{\mathfrak{m}}^-(0,\alpha\mathfrak{a}\mathfrak{b})\sigma_{\mathfrak{b}}^{-1} = \sum_{\mathfrak{b}}\Big(\sum_{0\neq x 1+\in(\mathfrak{b}\mathfrak{u})^{-1}} N(x)^{-s})|_{s=0}\sigma_{\mathfrak{b}}^{-1},$$

where α runs over all representatives of the principal ideals $\alpha\mathbb{A}$ with $\alpha \in 1+\mathfrak{m}$ modulo the principal ideal $\alpha\mathbb{A}$ with $\alpha \in 1+\mathfrak{m}$ and $\alpha \gg 0$. Further we know that the inner sum in the last equality is 0 if $\mathfrak{m} \neq \mathfrak{e}$ and is -1 if $\mathfrak{m} = \mathfrak{e}$. In the rational number field case, this is Euler's equality $\sum_{i\in\mathbb{Z}} 1 = 0$, and in the function field case, we refer to [Prop.6.1, H2]. We get the claim.

Furthermore, we have the following theorem.

Theorem 3.2. *Assume that k is the rational number field or a global function field. Then F^+ is a universal even distribution of k subject to the condition $F^+(\mathfrak{e})=0$ and F^- is a universal odd distribution with values in abelian groups in which w_k is invertible, where w_k is the number of roots of unity in k.*

Proof. Since $A^\pm$ only have w_k-torsion points, we only need to check the ranks of the level groups of $F^\pm$. Let $F^\pm_\mathfrak{m}$ and $T^\pm_\mathfrak{m}$ be the level $\mathfrak{m}$ groups of $F^\pm$ in Ω' and in Ω, respectively. Then $F^\pm_\mathfrak{m} = T^\pm_\mathfrak{m}/s(G_\mathfrak{m})T^\pm_\mathfrak{m}$. Let $\hat{G}_\mathfrak{m}$ be the group of complex characters of $G_\mathfrak{m}$. A character $\chi\in\hat{G}_\mathfrak{m}$ is called real if $\chi(J_\mathfrak{m})=1$. We extend the definition of χ linearly to $\mathbb{C}[G_\mathfrak{m}]$. By Thm 3.1 in Chap. 1 in [KL] we have

$$\mathrm{rank}(F^\pm_\mathfrak{m}) = \#\{\chi\in\hat{G}_\mathfrak{m}\mid \chi(F^\pm_\mathfrak{m})\neq 0\} = \#\{1\neq\chi\in\hat{G}_\mathfrak{m}\mid \chi(T^\pm_\mathfrak{m})\neq 0\}.$$

Since F^+ is even and F^- is odd, we see that $\chi(T^+_\mathfrak{m})=0$ if χ is not real and $\chi(T^-_\mathfrak{m})=0$ if χ is real. Now let $1\neq\chi\in\hat{G}_\mathfrak{m}$ with conductor $\mathfrak{f}$. Let $L(s,\chi)$ be the L-function associated to χ. In the function field case, it does not contain the Euler factor at ∞. We first assume that χ is not real. Then

$$\chi(F^-(\mathfrak{f}^{-1})) = \sum_\mathfrak{a} f^-(\mathfrak{a}\mathfrak{f}^{-1})\bar{\chi}(\mathfrak{a}) = \frac{|G_\mathfrak{m}|}{|G_\mathfrak{f}|}\sum_\mathfrak{b} f^-(\mathfrak{b}\mathfrak{f}^{-1})\bar{\chi}(\mathfrak{b}) = \frac{|G_\mathfrak{m}|}{|G_\mathfrak{f}|}L(0,\bar{\chi})\neq 0,$$

where $\mathfrak{a}$ and $\mathfrak{b}$ run over the representatives of $G_\mathfrak{m}$ and of $G_\mathfrak{f}$, respectively. Next suppose that χ is real. Then when $\mathfrak{f}\neq\mathfrak{e}$, we have

$$\chi(F^+(\mathfrak{f}^{-1})) = \frac{|G_\mathfrak{m}|}{|G_\mathfrak{f}|}L'(0,\bar{\chi})\neq 0.$$

When $\mathfrak{f}=\mathfrak{e}$, let $\mathfrak{a}\in T_0$ be such that $\chi(\mathfrak{a})\neq 1$. We have

$$\chi(F^+(\mathfrak{a})) = \frac{|G_\mathfrak{m}|}{|G_\mathfrak{e}|}(\chi(\mathfrak{a})-1)L'(0,\bar{\chi})\neq 0.$$

Thus we get

$$\mathrm{rank}(F^+_\mathfrak{m}) = |G_\mathfrak{m}|/|J_\mathfrak{m}|-1 \qquad\text{and}\qquad \mathrm{rank}(F^-_\mathfrak{m}) = (|J_\mathfrak{m}|-1)|G_\mathfrak{m}|/|J_\mathfrak{m}|.$$

Now we consider the distribution $F=F^+ + F^-$. Since F^+ is even and F^- is odd, the level $\mathfrak{m}$ group $F_\mathfrak{m}$ has the same rank as that of $F^+_\mathfrak{m}+F^-_\mathfrak{m}$. The latter is a direct sum. As $A_\mathfrak{m}$ is free of rank $|G_\mathfrak{m}|$, we must have that

$$\mathrm{rank}A^+_\mathfrak{m} = \mathrm{rank}F^+_\mathfrak{m}+1 \qquad\text{and}\qquad \mathrm{rank}A^-_\mathfrak{m} = \mathrm{rank}F^-_\mathfrak{m}.$$

This completes the proof.

We can produce a universal even distribution from F^+ as follows. Define $\bar{F}^+(\mathfrak{e})=\alpha$ for some α. Let $\bar{F}^+(\mathfrak{a}) = F^+(\mathfrak{a})+\alpha$ if $\mathfrak{a}\in T_0$ and $\bar{F}^+(\mathfrak{a}) = F^+(\mathfrak{a})$ if $\mathfrak{a}\in\bar{T}_0\setminus T_0$. Clearly $\bar{F}^+$ is a distribution. Choosing α conveniently, for example $\alpha\in\Omega'\setminus\mathrm{Im}F^+$, we get a universal even distribution with values in abelian groups in which w_k is invertible. For simplicity, we also denote $\bar{F}^+$ and F^- the limits of their level groups, respectively. The theorem above implies

Corollary 3.3. *We have the following exact sequences*

$$0\longrightarrow H^1(J,A)\longrightarrow A^+\xrightarrow{\phi^+}\bar{F}^+\longrightarrow 0,$$

and

$$0\longrightarrow H^2(J,A)\longrightarrow A^-\xrightarrow{\phi^-}F^-\longrightarrow 0,$$

where $\phi^+([\mathfrak{a}])=\bar{F}^+(\mathfrak{a})$ and $\phi^-([\mathfrak{a}])=F^-(\mathfrak{a})$.

4. Two criteria.

From now on, we always assume that k is the rational number field or a global function field. In this section, we give two criteria when an element of A is in $H^1(J, A)$ or in $H^2(J, A)$. In the rational number field case, this gives new proofs of some results in [KO] and in [Da], and also gives an affirmative answer to one of the questions in [Da]. In the function field case, our results are new. Our approach explains the natural reason of the criteria.

We first give a clear description of A and of $A^{\pm}$. Let $\mathcal{A} = \mathcal{A}(k)$ be the free abelian group generated by all the classes $[\mathfrak{a}]$ of $\bar{T}_0/\sim$. We define the action of G on $\mathcal{A}$ in the obvious way via the natural map $G \to G_{\mathfrak{m}}$. We identify A with the quotient of $\mathcal{A}$ by the subgroup generated by all elements in $\mathcal{A}$ of the form

$$[\mathfrak{a}] - \sum_{\mathfrak{b}\sim\mathfrak{a}} [\mathfrak{b}\mathfrak{n}^{-1}], \tag{4.1}$$

where $\mathfrak{n} \in T_0$ and $\mathfrak{a} \in \bar{T}_0$. Let j be a generator of J. We identify $A^{\pm}$ with the quotient of $\mathcal{A}$ by the subgroups generated by all elements of $\mathcal{A}$ of the form (4.1), along with all those of the form $[\mathfrak{a}] - j[\mathfrak{a}]$ and along with all those of the form $s(J)[\mathfrak{a}]$, respectively. We view $H^1(J, A)$ and $H^2(J, A)$ as subgroups of A^+ and of A^-, respectively, through the canonical isomorphisms in Prop. 2.4. Let $\mathbf{a}=\sum_i m_i[\mathfrak{a}_i]$ be an element in $\mathcal{A}$, and let $\mathfrak{m}$ be the lcm of the denominator of the $\mathfrak{a}_i$. For $\sigma \in G$, let $\sigma_{\mathfrak{b}}$ be the image in $G_{\mathfrak{m}}$ of σ under the natural map, where $\sigma_{\mathfrak{b}}$ is the Artin symbol associated to the integral ideal $\mathfrak{b}$. Then the action of σ on $\mathbf{a}$ is defined as

$$\mathbf{a}^{\sigma} = \sum_i m_i[\mathfrak{b}\mathfrak{a}_i].$$

If $\mathfrak{a} \in T_0$, we set $f^+(\mathfrak{a}) = f_{\mathfrak{e}}(\mathfrak{a})$ and define $\sigma f^+(\mathfrak{a}) = f_{\mathfrak{e}}(\mathfrak{a}\mathfrak{b}) - f_{\mathfrak{e}}(\mathfrak{b})$. We have the following criteria.

Theorem 4.1. *(1). If* $\mathbf{a}\in H^1(J, A)$, *we must have* $\sum_i m_i = 0$, *where the summation is over all* i *such that* $\mathfrak{a}_i$ *is integral. Further,* $\mathbf{a}\in H^1(J, A)$ *if and only if* $\sum_i m_i\sigma f^+(\mathfrak{a}_i)$ *is independent of* $\sigma \in G$.

(2). $\mathbf{a}\in H^2(J, A)$ *if and only if* $\sum_i m_i\sigma f^-(\mathfrak{a}_i)$ *is independent of* $\sigma \in G$.

Proof. (1). By Corollary 3.2, $\mathbf{a}\in H^1(J, A)$ if and only if $\mathbf{a}\in \ker\phi^+$. Since $\bar{F}^+(\mathfrak{e})$ can be defined freely in some sense, we must have $\sum_i m_i = 0$, where the summation is over all i such that $\mathfrak{a}_i$ is integral. Clearly $\mathbf{a}\in \ker\phi$ if and only if the condition in the theorem holds. The proof of (2) is also obvious.

We remark that in the criterion (2) we can assume $\mathfrak{a}_i \notin T_0$ since $f^-(\mathfrak{a}) = -1$ for any integral ideal $\mathfrak{a}$. In some cases, we can write out the criteria more concretely. Let $k = \mathbb{Q}$. We speak now of positive rational numbers rather than fractional ideals. We know that $f^-(a/m) =< a/m > -\frac{1}{2}$, where $< a/m >$ is the fractional part of a/m, and $f^+(a/m) = \log|1 - \xi_m^a| = \log(2\sin\pi < a/m >)$ for $m \nmid a$, up to a constant factor. Let $\sigma \in G = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ and let σ_t be the image of σ in $G_m = \mathrm{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})$ under the natural map. The actions of G on $f^{\pm}$ are $\sigma f^+(a/m) = \log(2\sin\pi < at/m >)$ and $\sigma f^-(a/m) =< at/m > -\frac{1}{2}$, respectively. Now let $\mathbf{a}= \sum_i m_i[a_i] \in \mathcal{A}$ and let m be the lcm of the denominators of a_i. The theorem above becomes

Corollary 4.2. *Assume that* $k = \mathbb{Q}$. *We have*

(1). $\mathbf{a}\in H^1(J, A)$ *if and only if* $\prod_i(2\sin\pi < ta_i >)^{m_i}$ *is independent of* $t \in (\mathbb{Z}/m\mathbb{Z})^*$. *In this case we must have* $a_i \notin \mathbb{Z}$.

(2). $\mathbf{a}\in H^2(J, A)$ *if and only if* $\sum_i m_i < ta_i >$ *is independent of* $t \in (\mathbb{Z}/m\mathbb{Z})^*$.

This corollary gives new proofs of Theorems 8 and 14 and Proposition 12 in [Da], and also gives a positive answer to one of the questions listed in the end of that paper. Our approach is simple and brings to light of the essential reasons of the criteria.

Now we consider the function field case. We first consider the case of the rational function field. In this case, the criteria are also simple and clear. Let $k = \mathbb{F}_q(T)$ and $\mathbb{A} = \mathbb{F}_q[T]$. Let $a/m \in k/\mathbb{A}$ with $\deg a < \deg m$. We know that $f^-(a/m) = (q-2)/(q-1)$ if a is monic and $f^-(a/m) = -1/(q-1)$ otherwise. For $a \neq 0$, also $f^+(a/m) = \deg m - \deg a - q/(q-1)$, see [GR]. The actions of G on $f^{\pm}$ are the same as those in the case $k = \mathbb{Q}$. We recall Anderson's [§7.6, Th] notation: $< a/m >= 1$ if a is monic and $< a/m >= 0$ otherwise, where we first normalize, using translation by elements in $\mathbb{A}$, by making m monic and $\deg a < \deg m$. Notice that $< a/m >= f^-(a/m) + 1/(q-1)$. For any $b \in \mathbb{A}$ with $(b, m) = 1$, let $[b]_m$ be the unique polynomial such that $a \equiv [a]_m (\mathrm{mod} m)$ and $\deg[a]_m < \deg m$. Let $\mathbf{a}= \sum_i m_i[a_i/n_i] \in \mathcal{A}$, where $a_i, n_i \in \mathbb{A}$ are coprime, and let m be the lcm of the denominators of a_i/n_i. Now Theorem 4.1 becomes

Corollary 4.3. *Assume that* $k=\mathbb{F}_q(T)$ *and* $\mathbb{A}=\mathbb{F}_q[T]$. *We have*

(1). $\mathbf{a}\in H^1(J,A)$ *if and only if* $\sum_i m_i \deg[ta_i]_{n_i}$ *is independent of* $t\in(\mathbb{A}/m\mathbb{A})^*$. *In this case we must have* $n_i\nmid a_i$.

(2). $\mathbf{a}\in H^2(J,A)$ *if and only if* $\sum_i m_i < ta_i/n_i >$ *is independent of* $t\in(\mathbb{A}/m\mathbb{A})^*$.

The two corollaries above give an explanation of the question 9.7(c) in [Th].

Finally we consider a global function field k. In this general case, we have not found an obvious expression for the function f^-. But Hayes gave a simple formula at least for the function f^+. We now recall this result.

Let $\mathbb{C}_k$ be the completion of the algebraic closure of k_∞. For an ideal $\mathfrak{u}\in\bar{T}_0$, let $\xi(\mathfrak{u})\in\mathbb{C}_k$ be the ξ-invariant associated to $\mathfrak{u}$, which is characterized by the condition that the lattice ($\mathbb{A}$-submodule of $\mathbb{C}_k$) $\xi(\mathfrak{u})\mathfrak{u}$ corresponds to some sgn-normalized rank one Drinfeld module. Thus it is determined up to a non-zero factor in $\mathbb{F}_\infty$, the residue field of k at ∞. We fix a value of $\xi(\mathbb{A})$. Then by the technique in [Sect.2, Y3], the value $\xi(\mathfrak{u})$ for any $\mathfrak{u}\in\bar{T}_0$ is fixed completely. In the paper [Yu], Yu showed that $\xi(\mathfrak{u})$ is transcendental over k. Let

$$e_{\mathfrak{u}}(s)=s\prod_{0\neq x\in\mathfrak{u}}(1-\frac{s}{x}) \tag{4.2}$$

be the Drinfeld exponential function associated to $\mathfrak{u}$. Let v_∞ be the extension to $\mathbb{C}_k$ of the normalized valuation at ∞. We have $f^+(\mathfrak{u})=v_\infty(\xi(\mathfrak{u}^{-1})e_{\mathfrak{u}^{-1}}(1))$ up to a constant if $\mathfrak{u}\notin T_0$ and $f_{\mathfrak{e}}(\mathfrak{u})=v_\infty(\xi(\mathfrak{u}^{-1}))+\deg\mathfrak{u}$ up to the same constant when $\mathfrak{u}\in T_0$, see [§6, H2] or [§7, GrR]. Let $\mathbf{a}=\sum_i m_i[\mathfrak{a}_i n_i^{-1}]\in\mathcal{A}$, where $a_i,\mathfrak{n}_i\in T_0$, and let $\mathfrak{m}$ be the lcm of the fractional parts of $\mathfrak{a}_i\mathfrak{n}_i^{-1}$. Let $<\mathfrak{u}>=f^-(\mathfrak{u})+\frac{1}{q-1}$, which is an integral number in the case $\deg\infty=1$ by [Sect.6, H2].

Corollary 4.4. *Assume that* k *is a global function field. Let* $\mathfrak{b}$ *run over a complete set of integral representatives of* $G_{\mathfrak{m}}$. *We have:*

(1). Assume that $\mathbf{a}=\sum_s m_s[\mathfrak{a}_s\mathfrak{n}_s^{-1}]+\sum_t m_t[\mathfrak{a}_t]$, *where* $\mathfrak{n}_s\nmid\mathfrak{a}_s$. *Then* $\mathbf{a}\in H^1(J,A)$ *if and only if* $\sum_s m_s v_\infty(\xi(\mathfrak{b}^{-1}\mathfrak{a}_s^{-1}\mathfrak{n}_s)e_{\mathfrak{b}^{-1}\mathfrak{a}_s^{-1}\mathfrak{n}_s}(1))+\sum_t m_t v_\infty(\xi(\mathfrak{b}^{-1}\mathfrak{a}_t^{-1}))$ *is independent of* $\mathfrak{b}$. *In this case we must have* $\sum_t m_t=0$.

(2). $\mathbf{a}\in H^2(J,A)$ *if and only if* $\sum_i m_i<\mathfrak{b}\mathfrak{a}_i\mathfrak{n}_i^{-1}>$ *is independent of* $\mathfrak{b}$.

5. Γ-function in characteristic p.

We introduce a new characteristic-p Γ-function and study its basic properties in this section. Our new Γ-function has the advantage that it is provided with a Galois action. The classical Euler gamma function has the following well-known functional equations:

$$\begin{aligned}\Gamma(s+1)&=s\Gamma(s)\\ \Gamma(s)\Gamma(1-s)&=\pi/\sin\pi s\\ (2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-s}\Gamma(s)&=\prod_{i=0}^{n-1}\Gamma(\frac{s+i}{n}).\end{aligned} \tag{5.1 – 5.3}$$

We will give the analogous properties for the new Γ-function.

Let k be a global function field with constant field $\mathbb{F}_q$ of q elements. Our Γ-function is defined on the set $\bar{T}_0$ of non-zero fractional ideals of k. Assume $\deg\infty=1$. We will give a remark later how to deal with general $\deg\infty$. We define $\Gamma:\bar{T}_0\longrightarrow\mathbb{C}_k\cup\{\infty\}$ as follows: for $\mathfrak{u}\in\bar{T}_0$,

$$\Gamma(\mathfrak{u})=\prod_{a\in\mathfrak{u}^{-1},a\gg0}(1-\frac{1}{a})^{-1},$$

and we call it the gamma function of k. The reader can compare the definition with that of the Drinfeld exponential function (4.2). Notice that $\Gamma(\mathfrak{a})=\infty$ if $\mathfrak{a}\in T_0$. Thus we only consider the Γ-function on $\bar{T}_0\setminus T_0$. We mention that there are many ways to define such a "gamma" function, for example, we could replace the minus in the products by plus, multiply this gamma function by a factor, or do not take the reciprocal and so on. But they are the same essentially. We mainly wish that it has properties analogous with the properties (5.1–5.3) of the classical Euler gamma function. We also point out that when $k=\mathbb{F}_q(T)$, our gamma function, via the action of the sign-subgroup (see below), can be identified to Thakur's gamma function on k. There are natural extensions of Γ to larger sets, for example, to $\mathbb{C}_k^*\times\bar{T}_0$. But the present function is enough for our purposes to the study of the arithmetic of k. We now give some properties of this Γ-function. The following result may be viewed as an analogue of Eq.(5.1).

Proposition 5.1. *Let $\mathfrak{a}, \mathfrak{b} \in \bar{T}_0 \setminus T_0$. Suppose that $\mathfrak{a} \sim \mathfrak{b}$. Then* $\Gamma(\mathfrak{a}) \equiv \Gamma(\mathfrak{b}) \mod k^*$.

Proof. Choose $x \in 1 + \mathfrak{a}^{-1}$ so that $x \gg 0$ and $\mathfrak{b} = x\mathfrak{a}$. We have

$$\begin{aligned}\Gamma(\mathfrak{b}) &= \prod_{0 \ll b \in \mathfrak{b}^{-1}} (1 - \frac{1}{b})^{-1} \\ &= \prod_{0 \ll a \in \mathfrak{a}^{-1}} (1 - \frac{1}{x^{-1}a})^{-1} \\ &= \prod_{0 \ll a \in \mathfrak{a}^{-1}} \frac{a}{a - x} \\ &= \prod_{0 \ll a \in \mathfrak{a}^{-1}} (\frac{a-1}{a})^{-1} \prod_{0 \ll a \in \mathfrak{a}^{-1}, \deg a \leq \deg x} \frac{a-1}{a-x} \\ &= \Gamma(\mathfrak{a}) \prod_{0 \ll a \in \mathfrak{a}^{-1}, \deg a \leq \deg x} \frac{a-1}{a-x}.\end{aligned}$$

Thus we get the result.

By this result, Γ is defined up to a factor in k^* on the set $(\bar{T}_0 \setminus T_0)/\sim$ of equivalence classes, and thus we can define the action of G on Γ as we mentioned in section 2. More precisely, let $\sigma_{\mathfrak{b}} \in G_{\mathfrak{m}}$ be the image of $\sigma \in G$ under the natural projection. We define

$$\Gamma^\sigma(\mathfrak{u}) \equiv \Gamma(\mathfrak{b}\mathfrak{u}) \bmod k^*,$$

where we take $\mathfrak{m}$ to be the denominator of $\mathfrak{u}$. We also define the action of G on $\xi(\mathfrak{u})$ in the same way, i.e., $\xi^\sigma(\mathfrak{u}) = \xi(\mathfrak{b}\mathfrak{u})$ up to a factor in k^*. The following result gives the analogue of the reflection formula (5.2) for the classical Γ-function. Recall that J is the sign-subgroup of G and $s(J)$ is the sum of the elements in J.

Proposition 5.2. *Let $\mathfrak{a} \in \bar{T}_0 \setminus T_0$. Then* $\Gamma^{s(J)}(\mathfrak{a}) \equiv e_{\mathfrak{a}^{-1}}(1)^{-1} (\bmod k^*)$.

Proof. We have

$$\Gamma^{s(J)}(\mathfrak{a}) \equiv \prod_{\alpha} \prod_{0 \ll a \in (\alpha\mathfrak{a})^{-1}} (1 - \frac{1}{a})^{-1} \equiv \prod_{0 \neq a \in \mathfrak{a}^{-1}} (1 - \frac{1}{a})^{-1} \equiv e_{\mathfrak{a}^{-1}}(1)^{-1},$$

where α runs over all representatives of the principal ideals $\alpha\mathbb{A}$ with $\alpha \in 1 + \mathfrak{m}$ modulo the principal ideal $\alpha\mathbb{A}$ with $\alpha \in 1 + \mathfrak{m}$ and $\alpha \gg 0$. Here $\mathfrak{m}$ is the denominator of $\mathfrak{a}$. This completes the proof.

Since $\xi(\mathfrak{a}^{-1})e_{\mathfrak{a}^{-1}}(1)$ is a division point of a rank 1 Drinfeld module, Yu's results [Yu] imply that $e_{\mathfrak{a}^{-1}}(1)$, thus $\Gamma^{s(J)}(\mathfrak{a})$, is transcendental over k^*.

Finally we consider the analogue of the multiplication formula (5.3) for the classical Γ-function. For $\mathfrak{u} \in \bar{T}_0$ and $\mathfrak{n} \in T_0$, let $w = w(\mathfrak{u}/\mathfrak{u}\mathfrak{n})$ be a complete set of representatives of $\mathfrak{u}/\mathfrak{u}\mathfrak{n}$. We can assume that the elements in $1 + w$ are totally positive. Up to a factor in $\mathbb{F}_q^*$, we set $\bar{\Gamma}(\mathfrak{u}) = (\xi(\mathbb{A}))^{\frac{1}{q-1}}/\Gamma(\mathfrak{u})$. Let $K_{\mathfrak{e}}$ be the normalizing field (i.e., the narrow Hilbert class field) of $(k, \infty, \mathrm{sgn})$, which is generated over k by the coefficients of sgn-normalized rank 1 Drinfeld modules, see [Part 2, H1].

Proposition 5.3. *Let $\mathfrak{a} \in \bar{T}_0 \setminus T_0$ and $\mathfrak{n} \in T_0$. Then*

$$\bar{\Gamma}(\mathfrak{a}) \equiv \prod_j \bar{\Gamma}(\mathfrak{b}_j) \mod (K_{\mathfrak{e}}^*)^{1/(q-1)},$$

where $\{\mathfrak{b}_j\}$ is a complete set of representatives for $\sim$-equivalence classes in the set of fractional ideals $\mathfrak{b}$ such that $\mathfrak{n}\mathfrak{b} \sim \mathfrak{a}$.

Proof. Let $w = w(\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{n})$. For each $\mathfrak{b}_j$ in the product there is one and only one $x \in 1 + w$ such that $\mathfrak{b}_j = x\mathfrak{a}$. For a fractional ideal $\mathfrak{u}$ and an integral number i, we write $\mathfrak{u}_i^+ = \{u \in \mathfrak{u} | \deg u = i, u \gg 0\}$. Set

$$R = \frac{\prod_{i=0}^{t} \prod_{a \in (\mathfrak{a}^{-1})_i^+} (a - 1)}{\prod_{x \in 1+w} \prod_{i=0}^{t - \deg \mathfrak{n}} \prod_{a \in (\mathfrak{a}^{-1}\mathfrak{n})_i^+} (a - x)},$$

where t is any integer with $t > \max\{\deg x \mid x \in 1+w\} + 2g + \deg \mathfrak{n}$ and g =genus of k. We abbreviate $d_{\mathfrak{a}} = \deg \mathfrak{a}$ and $u = \deg \mathfrak{n} - \deg \mathfrak{a}$. Then

$$\begin{aligned}
\prod_j \Gamma(\mathfrak{b}_j) &= \prod_{x \in 1+w} \Gamma(x\mathfrak{a}\mathfrak{n}^{-1}) \\
&= \prod_{x \in 1+w} \prod_{0 \ll a \in \mathfrak{a}^{-1}\mathfrak{n}} (1 - \frac{1}{x^{-1}a})^{-1} \\
&= \lim_{j \to \infty} \prod_{i=u}^{j} \prod_{x \in w} \prod_{a \in (\mathfrak{a}^{-1}\mathfrak{n})_i^+} \frac{a}{a-x} \\
&= R \cdot \lim_j \frac{\prod_{i=u}^{j} \prod_{x \in w} \prod_{a \in (\mathfrak{a}^{-1}\mathfrak{n})_i^+} (a-1)}{\prod_{i=-d_{\mathfrak{a}}}^{j+d} \prod_{a \in (\mathfrak{a}^{-1})_i^+} a} \\
&= R \cdot \Gamma(\mathfrak{a}) \cdot \lim_j \frac{\prod_{i=u}^{j} \prod_{a \in (\mathfrak{a}^{-1}\mathfrak{n})_i^+} a^{|w|}}{\prod_{i=-d_{\mathfrak{a}}}^{j+d} \prod_{a \in (\mathfrak{a}^{-1})_i^+} a} \\
&= R \cdot \Gamma(\mathfrak{a}) \cdot \frac{\xi(\mathfrak{a}^{-1}\mathfrak{n})^{\frac{|w|}{q-1}}}{\xi(\mathfrak{a}^{-1})^{\frac{1}{q-1}}}.
\end{aligned}$$

The last equality comes from the following fact obtained by Gekeler ([Ge], P.36) that up to roots of unity, see also ([Th], §3),

$$\xi(\mathfrak{u}) = \lim_{j \to \infty} \prod_{i=0}^{j} \prod_{a \in (\mathfrak{u})_i^+} a^{q-1}.$$

Since $\xi(\mathfrak{u})/\xi(\mathbb{A}) \in K_{\mathfrak{e}}$ for $\mathfrak{u} \in \bar{T}_0$, we get the result.

Remark. For $\deg \infty > 1$, write $\mathbb{F}_\infty$ for the residue class field at ∞. Choose a set S of representatives of the classes $\mathbb{F}_\infty^*/\mathbb{F}_q^*$, and denote $\operatorname{sgn}(a) \in S$ by "$a > 0$". We define

$$\Gamma(\mathfrak{a}) = \prod_{a \in \mathfrak{a}^{-1}, a > 0} (1 - \frac{1}{a})^{-1}.$$

Then essentially the same proofs work by replacing $(\mathfrak{u})_i^+$ with $\{a \in \mathfrak{a} | \deg a = i, a > 0\}$.

6. Γ-monomials.

In this section, we explain the algebraicity of some Γ-monomials, which is an analogue of Koblitz-Ogus's result [KO] for the classical Euler Γ-function. Our approach works also in the classical case.

We let $\bar{\Gamma}$ take values in $\mathbb{C}_k^*/\bar{k}^*$ through the natural map, where $\bar{k}$ is the algebraic closure of k in $\mathbb{C}_k$. Prop.5.1 says that $\bar{\Gamma}$ is a function on $\bar{T}_0 \setminus T_0/\sim$, and Prop.5.3 tells us that $\bar{\Gamma}$ is a punctured distribution. Furthermore, it is an odd distribution by Prop.5.2. Thus we get

Theorem 6.1. $\bar{\Gamma} : \bar{T}_0 \setminus T_0 \to \mathbb{C}_k^*/\bar{k}^*$ *is a punctured odd distribution.*

When $k = \mathbb{Q}$, the theorem is also valid after replacing $\bar{\Gamma}$ here by $\bar{\Gamma} = \sqrt{2\pi}/\Gamma$.

In [Th. 3.1, Y4] Yin gives a sufficient and necessary condition when a punctured distribution can be completed to a non-punctured distribution. As a corollary we have that a punctured odd distribution with values in a group on which $|J|$ and $|G_{\mathfrak{e}}|$ are invertible can be completed to a unique non-punctured odd distribution with value 0 at the unit ideal $\mathfrak{e}$ (see [Cor.3.3, Y4]). Here an integer is invertible on a group means that multiplying by the integer is an ismorphism of the group. The uniqueness can be seen easily from the distribution relations (2.1).

Since the group $\mathbb{C}_k^*/\bar{k}^*$ is torsion free, the punctured distribution $\bar{\Gamma}$ can be completed to a unique odd distribution, denoted also by $\bar{\Gamma}$, with $\bar{\Gamma}(\mathfrak{e}) = 1$. The same procedure may be applied to the classical Γ-function $\bar{\Gamma} = \sqrt{2\pi}/\Gamma$.

Let S be a complete set of representatives of $\bar{T}_0/\sim$. For $\mathfrak{a} \in \bar{T}_0$, we denote by $\mathfrak{a}^{(S)}$ the unique element in S such that $\mathfrak{a}^{(S)} \sim \mathfrak{a}$. Let $\mathbf{a} = \sum_i m_i[\mathfrak{a}_i] \in \mathcal{A}$. We define

$$\bar{\Gamma}(\mathbf{a}) = \prod_i \bar{\Gamma}(\mathfrak{a}_i^{(S)})^{m_i}.$$

Note that by Prop.5.1, $\bar{\Gamma}(\mathbf{a})$ is independent of the choice of S up to a factor in k^*. Let $\mathfrak{m}$ be the lcm of the denominators of the $\mathfrak{a}_i$. We have

Theorem 6.2. *Let the integral ideal $\mathfrak{b}$ run over all representatives of the classes in $G_{\mathfrak{m}}$. If $\sum_i m_i < \mathfrak{b}\mathfrak{a}_i >$ is independent of $\mathfrak{b}$, then $\bar{\Gamma}(\mathbf{a})$ is algebraic over k.*

As we remarked behind Theorem 4.1, we can assume that the $\mathfrak{a}_i$ are not integral. In the rational number field case, it is Koblitz-Ogus' theorem, by combining with Corollary 4.2(2). In the rational function field case, this result, linked with Corollary 4.3(2), first appeared in [§7.8, Th] without proof, but Thakur mentioned that it can be proved in Koblitz-Ogus' way.

The proof of the theorem is trivial. Namely, if $\mathbf{a}$ satisfies the condition in the theorem, it is a torsion element of $A_{\mathfrak{m}}^-$ by Theorem 4.1. Thus its image $\bar{\Gamma}(\mathbf{a})$ in $\mathbb{C}_k^*/\bar{k}^*$ is also torsion. Noting that $\mathbb{C}_k^*/\bar{k}^*$ is torsion free, we must have $\bar{\Gamma}(\mathbf{a}) \in \bar{k}^*$. Similarly, we get Koblitz-Ogus's theorem from Corollary 4.2. Essentially, the algebraicity of some Γ-monomial comes from the distribution properties of the Γ-functions. We conjecture that the inverse of the theorem is also valid. By Theorem 4.1 and Corollary 3.3, it is equivalent to the following conjecture.

Conjecture 6.3. $\bar{\Gamma} : \bar{T}_0 \to \mathbb{C}_k^*/\bar{k}^*$ *is a universal odd distribution with values in abelian groups in which w_k is invertible.*

This is a characteristic-p analogue of Rohrlich's conjecture on the classical Γ-function. The latter is a major unsolved conjecture in transcendental number theory. The conjecture is trivial when $q = 2$, since there is only the trivial odd distribution in this case. The constant factor in the unit-index formula in [Theorem B, Y1] is also trivial in this case. In addition, in the case of the rational function field, Sinha [Thm 6.2.4, S2] has proved the transcendence of many special values of Thakur's Γ-function by connecting the special values with the periods of t-motives.

7. Connection with cyclotomic units.

When k is the rational number field or a global function field, the cyclotomic extensions of k are the narrow ray class fields of k. They can be described in the obvious manner. The classical cyclotomic theory is well-known. In the function field case, the cyclotomic extensions are generated by torsion points of Hayes' sgn-normalized rank one Drinfeld modules, for which we refer to [Part 2, H1]. The units generated by these torsion points are called the cyclotomic units [Y1]. In this section, we connect the algebraic Γ-monomials with the cyclotomic units over k.

Let $\mathfrak{u}$ be a non-integral ideal of k. When $k = \mathbb{Q}$, we set $F(\mathfrak{u}) = |1 - \exp(2\pi i u)| = 2\sin\pi < u >$, where $\mathfrak{u} = u\mathbb{Z}$. It is a punctured even distribution. When k is a global function field, we set $F(\mathfrak{u}) = \xi(\mathfrak{u}^{-1})e_{\mathfrak{u}^{-1}}(1)$. The following result tells us that F with values in $\mathbb{C}_k^*/\mathbb{F}_\infty^*$ is also a punctured even distribution.

Proposition 7.1. *Let $\mathfrak{u} \in \bar{T}_0$ and $\mathfrak{m} \in T_0$. For $y \in \mathbb{C}_k$, we have*

$$\prod_{x \in \mathfrak{u}/\mathfrak{u}\mathfrak{m}} \xi(\mathfrak{u}\mathfrak{m})e_{\mathfrak{u}\mathfrak{m}}(y + x) = \xi(\mathfrak{u})e_{\mathfrak{u}}(y)$$

and

$$\prod_{0 \neq x \in \mathfrak{u}/\mathfrak{u}\mathfrak{m}} \xi(\mathfrak{u}\mathfrak{m})e_{\mathfrak{u}\mathfrak{m}}(x) = \xi(\mathfrak{u})/\xi(\mathfrak{u}\mathfrak{m}).$$

Proof. We use the basic fact in non-archimedean analysis that an entire function on $\mathbb{C}_k$ is determined, up to a factor, by its roots, multiplicities being counted [Th.2.14, Go].

Both $\prod_x e_{\mathfrak{um}}(y+x)$ and $e_{\mathfrak{u}}(y)$ have the simple root set $\mathfrak{u}$. Thus the second equality implies the first. Let ρ' correspond to the lattice $\xi(\mathfrak{um})\mathfrak{um}$ and let $D(\rho'_{\mathfrak{m}})$ be the constant coefficient of the isogeny $\rho_{\mathfrak{m}}$. By [Sect.2, Y3], we have

$$\frac{\xi(\mathfrak{u})}{\xi(\mathfrak{um})} = D(\rho'_{\mathfrak{m}}) = \prod_{0\neq\alpha\in\Lambda^{\rho}_{\mathfrak{m}}} \alpha = \prod_{0\neq x\in\mathfrak{u}/\mathfrak{um}} \xi(\mathfrak{um})e_{\mathfrak{um}}(x).$$

This completes the proof.

Let w_k be the number of roots of unity in k. In both cases, we set $I = \bar{\Gamma}^{w_k}/F$. We let I take values in $\mathbb{C}^*/(\mathbb{Q}^*)^{1/\infty}$ and in $\mathbb{C}^*_k/K^*_{\mathfrak{e}}$, respectively. Here $(\mathbb{Q}^*)^{1/\infty}$ is the subgroup of elements of $\mathbb{C}^*$ for which some power is in $\mathbb{Q}^*$. Then I is a punctured distribution. Since $I(\mathfrak{a})^{s(J)} \equiv 1(\mod K^*_{\mathfrak{e}})$ for $\mathfrak{a} \in \bar{T}_0 \setminus T_0$ by Eq.5.2 and Prop.5.2, I is odd. Let $\mathbf{a}=\sum_i m_i[\mathfrak{a}_i] \in H^2(J,A)$. Then $w_k\mathbf{a} = 0$ in A^-. Thus $I(\mathbf{a})^{w_k} \in (\mathbb{Q}^*)^{1/\infty}$ and $\in K^*_{\mathfrak{e}}$, respectively. Let $\mathfrak{m}$ be the lcm of the denominators of $\mathfrak{a}_i$. Notice that in the case $k=\mathbb{Q}$, the level $\mathfrak{m}$ group associated to I actually takes values in $\mathbb{C}^*/(\mathbb{Q}^*)^{1/m}$ by Eq.5.3, where $m = |\mathbb{Z}/\mathfrak{m}|$. Thus we get

Theorem 7.2. *With notations as above, we have $\bar{\Gamma}(\mathbf{a})^2 = r^{1/2m}F(\mathbf{a})$ in the case $k=\mathbb{Q}$, where r is some element of $\mathbb{Q}^*$, and $\bar{\Gamma}(\mathbf{a})^{w_k} = r^{1/w_k}F(\mathbf{a})$ in the function field case, with some $r \in K^*_{\mathfrak{e}}$.*

The result in the rational number field case was first given by Das [Th.6, Da] by using the method of the double complex. Only using distributions, we can give no further information about the constant r. To get further properties of the new Γ-monomials, we need to construct the double complex for global function fields, and also to develop Anderson's theory of solitons to show the analogue of Deligne's reciprocity for the new Γ-function.

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References

[A1] G.Anderson, *A double complex for computing the sign-cohomology of the universal ordinary distribution*, Contemp. Math. **224** (1999), 1-27.
[A2] G.Anderson, *A two-dimensional analogue of Stickelberger's theorem, in* "The Arithmetic of Function Field", Proc. of the workshop at the Ohio state University (D.Goss et al. Eds.), (1992), de Gruyter.
[B] H.Bass, *Generators and relations of cyclotomic units*, Nagoya Math. J. **27** (1966), 401-407.
[Ba] S.Bae, *Punctured distributions in the rational function fields*, J. Number Theory (to appear).
[Da] P.Das, *Algebraic gamma monomials and double covering of cyclotomic fields*, Tran. Amer. Math. Soc. **352** (2000), 3557-3594.
[D] P.Deligne *et al*, *Hodge Cycles, Motives and Shimura Varieties, LNM 900*, Springer-Verlag, New York/Berlin, 1982.
[G] E.Gekeler, *Drinfeld Modular Curves, LNM 1231*, Springer-Verlag, 1987.
[Go] D.Goss, *Basic Structures of Function Field Arithmetic*, Springer-Verlag, Berlin-Heidelberg, 1996.
[GR] S.Galovich and M.Rosen, *Distributions on the rational function fields*, Math. Ann. **256** (1981), 541-560.
[GrR] B.Gross and M.Rosen, *Fourier series and the special values of L-functions*, Adv. in Math. **69** (1988), 1-31.
[H1] D.Hayes, *A brief introduction to Drinfeld modules, in* "The Arithmetic of Function Field", Proc. of the workshop at the Ohio state University (D.Goss et al. Eds.), (1992), de Gruyter.
[H2] D.Hayes, *Stickelberger elements in function fields*, Compositio Math. **55** (1985), 209-235.
[KL] D.Kubert and S.Lang, *Modular Units*, Springer-Verlag, New York, 1981.
[KO] N.Koblitz and A.Ogus, *Algebraicity of some products of values of the Γ-function, Appendix: valeursde fonctions L by P.Deligne*, Proc. of Symp. Pure Math. **33** (1979), 343-346.
[K1] D.Kubert, *The universal ordinary distribution*, Bull. Soc. Math. France **107** (1979), 179-202.
[K2] D.Kubert, *The $\mathbb{Z}/2\mathbb{Z}$ cohomology of the universal ordinary distribution*, Bull. Soc. Math. France **107** (1979), 203-224.
[S1] S.Sinha, *Deligne's reciprocity for function fields*, J. Number Theory **63** (1997), 65-88.
[S2] S.Sinha, *Periods of t-motives and transcendence*, Duke Math. J. **88** (1997), 465-535.
[Si] W.Sinnott, *On the Stickelberger ideals and the circular units of a cyclotomic field*, Ann. of Math. **108** (1978), 107-134.
[Th] D.Thakur, *Gamma functions for function fields and Drinfeld modules*, Ann. of Math. **134** (1991), 25-64.
[Wa] L.Washington, *Introduction to Cyclotomic Fields, GTM 83*, Springer-Verlag, New York, 1996.
[Y1] L.Yin, *Index-class number formulas over global function fields*, Compositio Math. **109** (1997), 49-66.
[Y2] L.Yin, *Distributions on a global field*, J. of Number Theory **80** (2000), 154-167.
[Y3] L.Yin, *On the index of cyclotomic units in characteristic p and its applications*, J. Number Theory **63** (1997), 302-324.
[Y4] L.Yin, *Global distributions and special zeta values*, Submitted for publication.
[Yu] J.Yu, *Transcendence and Drinfeld modules*, Invent. Math. **83** (1986), 507-517.