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Abstract

We consider local minimizers $u\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}^N$ of variational integrals like $\int_\Omega [(1+|\partial_1 u|^2)^{p/2} + (1+|\partial_2 u|^2)^{q/2}]\,dx$ or its degenerate variant $\int_\Omega [|\partial_1 u|^p + |\partial_2 u|^q]\,dx$ with exponents $2 \le p < q < \infty$ which do not fall in the category studied in [BF2]. We prove interior $C^{1,\alpha}$- respectively C^1-regularity of u under the condition that $q < 2p$. For decomposable variational integrals of arbitrary order a similar result is established by the way extending the work [BF3].

1 Introduction

This paper is devoted to the study of the interior regularity of local minimizers $u\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}^N$ of anisotropic variational integrals of the form

$$J[u,\Omega] = \int_\Omega f(\nabla u)\,dx, \tag{1.1}$$

where Ω denotes a bounded open set in the plane and where the energy density $f\colon \mathbb{R}^{2N} \to \mathbb{R}$ satisfies the estimate

$$a|Z|^p - b \le f(Z) \le A|Z|^q + B \quad \text{for all} \quad Z \in \mathbb{R}^{2N} \tag{1.2}$$

with exponents $2 \le p \le q < \infty$ and constants a, $A > 0$, b, $B \ge 0$. Due to (1.2) it is natural to discuss J on the local Sobolev space $W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ (see, e.g., [Ad] for a definition of these spaces) and to call a function u from this class a local J-minimizer iff $J[u,\Omega'] < \infty$ and $J[u,\Omega'] \le J[v,\Omega']$ for all $v \in W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ such that $\mathrm{spt}(u-v) \subset \Omega'$, where Ω' is any subdomain of Ω with compact closure in Ω. As a matter of fact, (1.2) is not sufficient for building up a regularity theory for locally J-minimizing functions, in place of (1.2) a suitable ellipticity condition is needed: for example, the validity of

$$\lambda(1+|X|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2 f(X)(Y,Y) \le \Lambda(1+|X|^2)^{\frac{q-2}{2}}|Y|^2 \tag{1.3}$$

for all X, $Y \in \mathbb{R}^{2N}$ with constants λ, $\Lambda > 0$ guarantees the strict convexity of f and clearly implies (1.2). Then, if u is a local J-minimizer and if for the moment Ω is a domain in some $\mathbb{R}^n$, $n \ge 2$, (1.3) ensures the following regularity results:

i.) (full interior regularity in the scalar case) If $N = 1$, then u is of class $C^{1,\alpha}(\Omega)$ for any $\alpha < 1$.

ii.) (partial regularity in the vector case) If $N > 1$, then there is an open subset Ω_0 of Ω such that $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^N)$ for any $0 < \alpha < 1$. Moreover, $\Omega - \Omega_0$ is of Lebesgue measure 0.

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We refer the reader, for instance, to the papers of Esposito, Leonetti and Mingione [ELM1]–[ELM3], of Marcellini [Ma1]–[Ma3], of Acerbi and Fusco [AF], of Fusco and Sbordone [FS] and of the authors [BF1]. We also mention the monograph [Bi], where one can find further references. We wish to emphasize that all these results are valid either under a condition of the form

$$q < c(n)p, \quad c(n) \to 1 \quad \text{as } n \to \infty, \tag{1.4}$$

or they require bounds like

$$q < p + 2 \tag{1.5}$$

together with the assumption $u \in L^{\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and with additional structural hypothesis imposed on f. It is also important to remark that counterexamples of Giaquinta [Gi2] and (later) Hong [Ho] show that the smoothness of local minimizers can only be expected if q and p are not too far apart, i.e. some variant of (1.4) is necessary for local regularity. Of course the "two-dimensional vector case" (i.e. $n = 2$, $N > 1$) is included in ii.) but for this particular situation we proved in [BF2]:

iii.) If $n = 2$ and $N \geq 1$, then (1.3) together with $q < 2p$ implies $u \in C^{1,\alpha}(\Omega; \mathbb{R}^N)$, $0 < \alpha < 1$.

The counterexamples of Giaquinta [Gi2] and Hong [Ho] as well as the papers of Acerbi and Fusco [AF] and of Fusco and Sbordone [FS] also suggest to study classes of anisotropic integrands, which are in some sense decomposable, which means that in our two-dimensional case we have $f(\nabla u) = F(\partial_1 u) + G(\partial_2 u)$ for functions F, G: $\mathbb{R}^N \to \mathbb{R}$ of class C^2 which satisfy separately the isotropic ellipticity conditions

$$\lambda(1 + |X|^2)^{\frac{p-2}{2}}|Y|^2 \leq D^2F(X)(Y,Y) \leq \Lambda(1 + |X|^2)^{\frac{p-2}{2}}|Y|^2, \tag{1.6}$$

$$\lambda(1 + |X|^2)^{\frac{q-2}{2}}|Y|^2 \leq D^2G(X)(Y,Y) \leq \Lambda(1 + |X|^2)^{\frac{q-2}{2}}|Y|^2 \tag{1.7}$$

for all X, $Y \in \mathbb{R}^N$. Note that (1.6) and (1.7) imply the (p, q)-growth of f stated in (1.2). Clearly (1.3) does not give (1.6), (1.7), we just get the anisotropic versions of (1.6), (1.7) with exponent p on the l.h. sides and exponent q the r.h. sides. If we start from (1.6) and (1.7), then we arrive at (1.3) *but with exponent* 2 *instead of* p *on the l.h.s.*, and iii.) implies the weak result:

iv.) If (1.6), (1.7) hold with exponents $2 \leq p \leq q < 4$, then any local minimizer has Hölder continuous first derivatives in the interior of Ω.

The first goal of our paper is to improve iv.) in the spirit of iii.), i.e. we like to show that even under the new hypothesis on f the condition $q < 2p$ gives the regularity of local minimizers, more precisely:

THEOREM 1.1. *Suppose that* $u \in W^1_{p,\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *locally minimizes the energy* J *defined in (1.1) (with* $\Omega \subset \mathbb{R}^2$*) and let*

$$f(X_1 X_2) = F(X_1) + G(X_2), \quad X_1,\ X_2 \in \mathbb{R}^N,$$

with functions F and G satisfying (1.6) and (1.7). Then, if $2 \le p \le q < \infty$ and if in addition

$$q < 2p \tag{1.8}$$

holds, we have $u \in C^{1,\alpha}(\Omega;\mathbb{R}^N)$ for all $0 < \alpha < 1$.

REMARK 1.1. *In [BFZ2] we recently showed that this result holds in the scalar case even if $q = 2p$, and that the statement also can be extended to domains $\Omega \subset \mathbb{R}^n$, $n \ge 3$, provided we know $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Earlier results in this spirit are due to Ural'tseva and Urdaletova [UU].*

REMARK 1.2. *It is not hard to prove Theorem 1.1 in the subquadratic case, we leave the details to the reader.*

REMARK 1.3. *Of course it would also be possible to replace (1.6) as well as (1.7) by anisotropic conditions with exponents $p_1 < q_1$ in (1.6) and $p_2 < q_2$ in (1.7). Then appropriate relations between p_i and q_i will imply regularity.*

REMARK 1.4. *In [Ma1], Theorem A, Marcellini considers a class of decomposable integrals defined for scalar functions. Then, if $p = 2$ and $\Omega = \mathbb{R}^2$, he obtains regularity without any restriction on q. It would be interesting to see if this result can be extended to two-dimensional vector problems.*

Next we formulate an extension of Theorem 1.1 to the higher order case, i.e. we replace (1.1) by the functional

$$\widetilde{J}[u,\Omega] := \int_\Omega \tilde{f}(\nabla^k u)\,dx \tag{1.9}$$

for functions u: $\mathbb{R}^2 \supset \Omega \to \mathbb{R}^N$. Here $k \ge 2$ is a fixed integer and $\nabla^k u$ denotes the tensor of all weak partial derivatives of order k. In [BF3] we showed: if $\tilde{f}$ satisfies an ellipticity condition analogous to (1.3) and if u is a local $\widetilde{J}$-minimizer (from the natural class $W^k_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$), then we have $u \in C^{k,\alpha}(\Omega;\mathbb{R}^N)$ for all $\alpha \in (0,1)$ provided

$$q < \min\{p+2, 2p\}. \tag{1.10}$$

As in [BF3] it is easy to check that it is sufficient to study the case $k = 2$ together with $N = 1$. Then $\nabla^2 u(x)$ can be seen as an element of $\mathbb{R}^4$, and we will select l fixed entries, $1 \le l \le 3$, of $E \in \mathbb{R}^4$ and denote this vector in $\mathbb{R}^l$ by E_I, whereas $E_{II} \in \mathbb{R}^{4-l}$ denotes the vector of the remaining components. Then we assume that

$$\tilde{f}(E) = \widetilde{F}(E_I) + \widetilde{G}(E_{II}), \quad E \in \mathbb{R}^4, \tag{1.11}$$

with functions $\widetilde{F}$: $\mathbb{R}^l \to \mathbb{R}$, $\widetilde{G}$: $\mathbb{R}^{4-l}$ of class C^2 satisfying

$$\lambda(1+|X|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2\widetilde{F}(X)(Y,Y) \le \Lambda(1+|X|^2)^{\frac{p-2}{2}}|Y|^2, \tag{1.12}$$

$$\lambda(1+|U|^2)^{\frac{q-2}{2}}|V|^2 \le D^2\widetilde{G}(U)(V,V) \le \Lambda(1+|U|^2)^{\frac{q-2}{2}}|Y|^2 \tag{1.13}$$

for all X, $Y \in \mathbb{R}^l$, U, $V \in \mathbb{R}^{4-l}$ with constants λ, $\Lambda > 0$.

THEOREM 1.2. *Suppose that $\tilde{f}$ satisfies (1.11)–(1.13) for exponents $2 \le p < q < \infty$, and let $u \in W^2_{p,\mathrm{loc}}(\Omega)$ denote a local $\tilde{J}$-minimizer. Then u is of class $C^{2,\alpha}(\Omega)$ for any $\alpha \in (0,1)$ provided*

$$q < 2p. \tag{1.14}$$

REMARK 1.5. *If $k \ge 2$, then under comparable conditions on the decomposition of $\tilde{f}$, we get $u \in C^{k,\alpha}(\Omega)$ if again (1.14) is satisfied.*

REMARK 1.6. *In contrast to (1.10), (1.14) does not require the additional bound $q < p+2$.*

Our paper is organized as follows: in Section 2 we introduce a suitable local regularization and recall some results on uniform local higher integrability and higher weak differentiability, where we can follow the lines of, e.g., [BF1], [BF2] with minor modifications. Then it is no longer possible to benefit from the paper [BF2]: the approach towards regularity based on techniques introduced by Frehse and Seregin [FrS], which was carried out in [BF2], does not work if (1.3) is replaced by (1.6) and (1.7). In Section 3 we apply a new tool, namely a lemma on the higher integrability of functions established in [BFZ1], to overcome this difficulty and to complete the proof of Theorem 1.1. In Section 4 we briefly indicate how to adjust the foregoing arguments in order to handle the situation described in Theorem 1.2, and in Section 5 we give some comments concerning the degenerate case. In the appendix we state the above mentioned (Gehring-type) lemma in a form valid for any dimension.

2 Preparations for the proof of Theorem 1.1

Suppose that the assumptions of Theorem 1.1 are satisfied and consider a local J-minimizer u. Fix two subdomains Ω_1, Ω_2 s.t. $\Omega_1 \Subset \Omega_2 \Subset \Omega$, and denote by $\overline{u}_m$, $m \in \mathbb{N}$, the mollification of u with radius $1/m$, in particular $\|\overline{u}_m - u\|_{W^1_p(\Omega_2)} \to 0$ as $m \to \infty$. We let

$$\rho_m := \|\overline{u}_m - u\|_{W^1_p(\Omega_2)} \left[\int_{\Omega_2} (1 + |\nabla \overline{u}_m|^2)^{q/2}\, dx \right]^{-1}$$

and introduce the functional

$$J_m[w,\Omega_2] := \rho_m \int_{\Omega_2} (1 + |\nabla w|^2)^{q/2}\, dx + J[w,\Omega_2].$$

Finally, we consider the sequence $u_m \in W^1_q(\Omega_2;\mathbb{R}^N)$ of solutions of the minimization problem

$$J_m[\cdot,\Omega_2] \to \min \quad \text{in} \quad \overline{u}_m + \overset{\circ}{W}{}^1_q(\Omega_2;\mathbb{R}^N).$$

The following facts have been established for example in [BF1]–[BF3]:

LEMMA 2.1. *We have as $m \to \infty$:*

i) $u_m \rightharpoondown u$ *in* $W^1_p(\Omega_2;\mathbb{R}^N)$,

ii) $\rho_m \int_{\Omega_2} (1+|\nabla u_m|^2)^{q/2}\,dx \to 0$,

iii) $\int_{\Omega_2} f(\nabla u_m)\,dx \to \int_{\Omega_2} f(\nabla u)\,dx$.

From [BF1], Lemma 2.3, we deduce:

LEMMA 2.2. *Let* $P \in \mathbb{R}^{2N}$ *and define* $u^*_m(x) := u_m(x) - Px$. *Then, for any* $\eta \in C^\infty_0(\Omega_2)$ *and for* $\gamma = 1,\,2$, *it holds that*

$$\begin{aligned} &\int_{\Omega_2} D^2 f_m(\nabla u_m)(\partial_\gamma \nabla u_m, \partial_\gamma \nabla u_m)\eta^2\,dx \\ &\leq\; c\int_{\Omega_2} D^2 f_m(\nabla u_m)(\nabla\eta\otimes\partial_\gamma u^*_m, \nabla\eta\otimes\partial_\gamma u^*_m)\,dx, \end{aligned} \tag{2.1}$$

c being a positive constant independent of m.

In (2.1) $\otimes$ denotes the tensor product of vectors. We use (2.1) to prove

LEMMA 2.3. *For any finite t we have that* $\nabla u_m \in L^t_{\mathrm{loc}}(\Omega_2;\mathbb{R}^{2N})$ *uniformly w.r.t. to m.*

Proof. We use the interpolation and hole-filling trick originating in [ELM1]. Let $\tilde h_{1,m} := (1+|\partial_1 u_m|^2)^{p/4}$, $\tilde h_{2,m} := (1+|\partial_2 u_m|^2)^{q/4}$, fix a disc $B_{2R} = B_{2R}(x_0) \Subset \Omega_2$, select radii $r \in (R, \frac{3}{2}R)$, $\rho \in (0, R/2)$ and choose $\eta \in C^\infty_0(B_{r+\rho/2})$, $\eta \equiv 1$ on B_r, $|\nabla\eta| \leq c/\rho$, $0 \leq \eta \leq 1$. Finally, we let $\alpha := \frac{p}{2}\chi$ with χ sufficiently large. Then, if we take the sum w.r.t. γ in (2.1) and choose $P = 0$, we get (by Sobolev's inequality with $t \in (1,2)$ defined through $2\chi = \frac{2t}{2-t}$)

$$\begin{aligned} &\int_{B_r} (1+|\partial_1 u_m|^2)^\alpha\,dx + \int_{B_r} (1+|\partial_2 u_m|^2)^\alpha\,dx \\ &\leq\; \int_{B_{2R}} (\eta\tilde h_{1,m})^{2\chi}\,dx + \int_{B_{2R}} (\eta\tilde h_{2,m})^{2\chi}\,dx \\ &\leq\; c\left[\left(\int_{B_{2R}} |\nabla(\eta\tilde h_{1,m})|^t\,dx\right)^{\frac{2\chi}{t}} + \left(\int_{B_{2R}} |\nabla(\eta\tilde h_{2,m})|^t\,dx\right)^{\frac{2\chi}{t}}\right] \\ &\leq\; c\left[\int_{B_{2R}} |\nabla(\eta\tilde h_{1,m})|^2\,dx + \int_{B_{2R}} |\nabla(\eta\tilde h_{2,m})|^2\,dx\right]^\chi \\ &\leq\; c\Bigg[\int_{B_{2R}} |\nabla\eta|^2\tilde h^2_{1,m}\,dx + \int_{B_{2R}} |\nabla\eta|^2\tilde h^2_{2,m}\,dx \\ &\qquad + \int_{B_{2R}} \eta^2|\nabla\tilde h_{1,m}|^2\,dx + \int_{B_{2R}} \eta^2|\nabla\tilde h_{2,m}|^2\,dx\Bigg]^\chi \\ &\leq\; c\left[\frac{1}{\rho^2}\int_{B_{2R}} (\tilde h^2_{1,m} + \tilde h^2_{2,m})\,dx + \int_{B_{r+\rho}-B_r} |D^2 f_m(\nabla u_m)(\nabla\eta\otimes\partial_\gamma u_m, \nabla\eta\otimes\partial_\gamma u_m)|\,dx\right]^\chi. \end{aligned}$$

If we estimate $\int_{B_{r+\rho}-B_\rho} \ldots$ roughly through $\frac{1}{\rho^2}\int_{B_{r+\rho}-B_r}(1+|\nabla u_m|^2)^{q/2}\,dx$, then we have shown that

$$\begin{aligned}&\int_{B_r}(1+|\nabla u_m|^2)^\alpha\,dx\\ (2.2)\qquad &\leq\; c\frac{1}{\rho^2}\Bigg[\int_{B_{2R}}(\tilde h_{1,m}^2+\tilde h_{2,m}^2)\,dx+\int_{B_{r+\rho}-B_r}(1+|\nabla u_m|^2)^{q/2}\,dx\Bigg]^\chi.\end{aligned}$$

By Lemma 2.1 the first integral on the r.h.s. of (2.2) can be estimated by a local constant independent of m. If we choose χ to satisfy $p\chi>q$, then with $\Theta\in(0,1)$ we can write $\frac{1}{q}=\frac{\Theta}{p}+\frac{1-\Theta}{p\chi}$, hence

$$\|\nabla u_m\|_{L^q}\leq\|\nabla u_m\|_{L^p}^\Theta\|\nabla u_m\|_{L^{p\chi}}^{1-\Theta},$$

where the norms are calculated w.r.t. $T_{r,\rho}:=B_{r+\rho}-B_r$, and therefore

$$(2.3)\qquad \frac{1}{\rho^2}\int_{T_{r,\rho}}|\nabla u_m|^q\,dx\leq\frac{1}{\rho^2}\Big(\int_{T_{r,\rho}}|\nabla u_m|^p\,dx\Big)^{\Theta q/p}\Big(\int_{T_{r,\rho}}|\nabla u_m|^{p\chi}\,dx\Big)^{(1-\Theta)\frac{q}{p\chi}}.$$

Now from (1.8) it follows that $(1-\Theta)\frac{q}{p}<1$, provided we choose $\chi>p/(2p-q)$. Then we can apply Young's inequality on the r.h.s. of (2.3) with the result (s_1,s_2 denoting positive exponents)

$$(2.4)\qquad \frac{1}{\rho^2}\int_{T_{r,\rho}}|\nabla u_m|^q\,dx\leq c\rho^{-s_1}\Bigg[\int_{B_{2R}}|\nabla u_m|^p\,dx\Bigg]^{s_2}+c\Bigg[\int_{T_{r,\rho}}|\nabla u_m|^{p\chi}\,dx\Bigg]^{1/\chi}.$$

Using (2.4) in inequality (2.2) and "filling the hole", it follows that $\nabla u_m\in L^{2\alpha}_{\mathrm{loc}}(\Omega_2;\mathbb{R}^{2N})$ uniformly in m. But α can be chosen arbitrary large, and Lemma 2.3 is established. □

From Lemma 2.3 combined with (2.1) (and the choice $P=0$) we immediately deduce that

$$(2.5)\qquad \tilde h_{1,m},\ \tilde h_{2,m}\in W^1_{2,\mathrm{loc}}(\Omega_2)\quad\text{uniformly w.r.t. } m,$$

since by (2.1)

$$\begin{aligned}&\int_{\Omega_2}\eta^2\Big[|\nabla\tilde h_{1,m}|^2+|\nabla\tilde h_{2,m}|^2\Big]\,dx\\ &\leq\; c\|\nabla\eta\|_\infty^2\Bigg[\rho_m\int_{\Omega_2}(1+|\nabla u_m|^2)^{\frac{q}{2}}\,dx+\int_{\operatorname{spt}\eta}|D^2F(\partial_1 u_m)||\nabla u_m|^2\,dx\\ &\qquad+\int_{\operatorname{spt}\eta}|D^2G(\partial_2 u_m)|\,|\nabla u_m|^2\,dx\Bigg]\leq c(\eta)<\infty.\end{aligned}$$

Clearly the same argument gives in addition to (2.5)

$$(2.6)\qquad \rho_m^{\frac{1}{2}}(1+|\nabla u_m|^2)^{\frac{q}{4}}=:\tilde h_{3,m}\in W^1_{2,\mathrm{loc}}(\Omega_2)\quad\text{uniformly w.r.t. } m.$$

Since we assume $p \geq 2$, the ellipticity estimates (1.6) and (1.7) imply that $\lambda \int_{\Omega_2} \eta^2(|\nabla\partial_1 u_m|^2 + |\nabla\partial_2 u_m|^2)\,dx$ is bounded from above by the l.h.s. of (2.1), thus with a repetition of the above argument we get as a further consequence of (2.1)

$$u_m \in W^2_{2,\mathrm{loc}}(\Omega_2;\mathbb{R}^N) \quad \text{uniformly w.r.t. } m. \tag{2.7}$$

Since we already know $u_m \rightharpoondown u$ in $W^1_p(\Omega_2;\mathbb{R}^N)$, we may pass to a subsequence to deduce from (2.7)

$$\nabla u_m \to \nabla u \quad \text{a.e. on } \Omega_2. \tag{2.8}$$

We wish to remark that (2.8) extends to the case that $p < 2$. The reader will find the necessary adjustments in [BF1].

3 Proof of Theorem 1.1

We continue to use the notation introduced in the previous section and recall from [BF1] the inequality

$$\begin{aligned} &\int_{\Omega_2} D^2 f_m(\nabla u_m)(\partial_\gamma \nabla u_m, \partial_\gamma \nabla u_m)\eta^2\,dx \\ \leq\ & -2\int_{\Omega_2} \eta D^2 f_m(\nabla u_m)(\partial_\gamma \nabla u_m, \partial_\gamma u_m^* \otimes \nabla\eta)\,dx, \quad \eta \in C_0^\infty(\Omega_2), \end{aligned} \tag{3.1}$$

where from now on summation w.r.t. to γ is used. Note that (3.1) implies (2.1) with the help of the Cauchy-Schwarz inequality applied to the bilinear form $D^2 f_m(\nabla u_m)$. Let $B_{2R} = B_{2R}(x_0) \Subset \Omega_2$ and choose $\eta \in C_0^\infty(B_{2R})$ according to $\eta \equiv 1$ on B_R, $|\nabla\eta| \leq c/R$, $0 \leq \eta \leq 1$. We further introduce the following auxiliary functions:

$$\begin{aligned} H_m^2 &:= D^2 f_m(\nabla u_m)(\partial_\gamma \nabla u_m, \partial_\gamma \nabla u_m) \\ &= \rho_m D^2 g(\nabla u_m)(\partial_\gamma \nabla u_m, \partial_\gamma \nabla u_m) + D^2 F(\partial_1 u_m)(\partial_\gamma\partial_1 u_m, \partial_\gamma\partial_1 u_m) \\ &\quad + D^2 G(\partial_2 u_m)(\partial_\gamma\partial_2 u_m), \partial_\gamma\partial_2 u_m), \end{aligned}$$

where $g(Z) := (1+|Z|^2)^{q/2}$ for $Z \in \mathbb{R}^{2N}$, moreover

$$\begin{aligned} h_{1,m} &:= (1+|\partial_1 u_m|^2)^{\frac{p-2}{4}}, \\ h_{2,m} &:= (1+|\partial_2 u_m|^2)^{\frac{q-2}{4}}, \\ h_{3,m} &:= (1+|\nabla u_m|^2)^{\frac{q-2}{4}}\sqrt{\rho_m}. \end{aligned}$$

Recalling (2.1) and Lemma 2.3 one more time we get

$$H_m \in L^2_{\mathrm{loc}}(\Omega_2) \quad \text{uniform w.r.t. } m, \tag{3.2}$$

moreover, the ellipticity estimates (1.6) and (1.7) show that

$$c_1\Bigg[\rho_m(1+|\nabla u_m|^2)^{\frac{q-2}{2}}|\nabla^2 u_m|^2+(1+|\partial_1 u_m|^2)^{\frac{p-2}{2}}|\nabla\partial_1 u_m|^2$$

$$+(1+|\partial_2 u_m|^2)^{\frac{q-2}{2}}|\nabla\partial_2 u_m|^2\Bigg]\leq H_m^2\leq c_2[\dots] \tag{3.3}$$

holds with constants c_1, $c_2>0$ being independent of m. With this observation we deduce from (3.1)

$$\begin{aligned}
\int_{B_R} H_m^2\,dx \;&\leq\; -2\int_{B_{2R}}\eta\big[\rho_m D^2g(\nabla u_m)(\partial_\gamma\nabla u_m,\partial_\gamma u_m^*\otimes\nabla\eta)\\
&\quad +D^2F(\partial_1 u_m)(\partial_\gamma\partial_1 u_m,\partial_1\eta\partial_\gamma u_m^*)+D^2G(\partial_2 u_m)(\partial_\gamma\partial_2 u_m,\partial_2\eta\partial_\gamma u_m^*)\big]\,dx\\
&\leq\; \frac{c}{R}\int_{B_{2R}}\big[\rho_m(1+|\nabla u_m|^2)^{\frac{q-2}{2}}|\nabla^2 u_m||\nabla u_m-P|\\
&\quad +(1+|\partial_1 u_m|^2)^{\frac{p-2}{2}}|\nabla\partial_1 u_m||\nabla u_m-P|\\
&\quad +(1+|\partial_2 u_m|^2)^{\frac{q-2}{2}}|\nabla\partial_2 u_m||\nabla u_m-P|\big]\,dx\\
&\underset{(3.3)}{\leq}\; \frac{c}{R}\int_{B_{2R}} H_m|\nabla u_m-P|\{h_{1,m}+h_{2,m}+h_{3,m}\}\,dx\\
&\leq\; \frac{c}{R}\int_{B_{2R}} H_m h_m|\nabla u_m-P|\,dx,
\end{aligned}\tag{3.4}$$

where $h_m:=(h_{1,m}^2+h_{2,m}^2+h_{3,m}^2)^{1/2}$. Let $s=4/3$ and apply Hölder's inequality as well as the Sobolev-Poincaré inequality to the last line of (3.4) in order to deduce from (3.4)

$$\fint_{B_R} H_m^2\,dx\leq c\Bigg[\fint_{B_{2R}}(H_m h_m)^s\,dx\Bigg]^{\frac{1}{s}}\Bigg[\fint_{B_{2R}}|\nabla^2 u_m|^s\,dx\Bigg]^{\frac{1}{s}}. \tag{3.5}$$

Here $\fint_{B_R}$ etc. denotes the mean value, and in (3.4) we take $P:=\fint_{B_{2R}}\nabla u_m\,dx$. Finally we observe using $p\geq 2$ and (3.3)

$$|\nabla^2 u_m|=\big(|\partial_1\nabla u_m|^2+|\partial_2\nabla u_m|^2\big)^{1/2}\leq cH_m\leq cH_m h_m,$$

thus (3.5) implies

$$\Bigg[\fint_{B_R} H_m^2\,dx\Bigg]^{\frac{1}{2}}\leq c\Bigg[\fint_{B_{2R}}(h_m H_m)^s\,dx\Bigg]^{\frac{1}{s}}, \tag{3.6}$$

and if for example we require $B_{2R}\subset\Omega_1$, then c is uniform in B_{2R} and also in m. In order to apply Lemma A.1 we let $d:=2/s=3/2$, $\overline{f}:=H_m^s$, $\overline{g}:=h_m^s$ in this lemma, so that

(3.6) can be rewritten as

$$\left[\fint_{B_R} \overline{f}^d\,dx\right]^{1/d} \le c \fint_{B_{2R}} \overline{f}\,\overline{g}\,dx.$$

From (3.2) we get $\overline{f} \in L^d_{\mathrm{loc}}(\Omega_2)$, and it remains to check if $\exp(\beta\overline{g}^d) \in L^1_{\mathrm{loc}}(\Omega_2)$ for arbitrary $\beta > 0$, i.e. if

$$\exp(\beta h_m^2) \in L^1_{\mathrm{loc}}(\Omega_2) \tag{3.7}$$

(of course everything is meant uniform in m). To prove (3.7) we let $\tilde{h}_m := (\tilde{h}^2_{1,m} + \tilde{h}^2_{2,m} + \tilde{h}^2_{3,m})^{1/2}$ and observe that

$$\begin{aligned} |\nabla\tilde{h}_m| &\le \frac{1}{\tilde{h}_m}\Big(\tilde{h}_{1,m}|\nabla\tilde{h}_{1,m}| + \tilde{h}_{2,m}|\nabla\tilde{h}_{2,m}| + \tilde{h}_{3,m}|\nabla\tilde{h}_{3,m}|\Big) \\ &\le |\nabla\tilde{h}_{1,m}| + |\nabla\tilde{h}_{2,m}| + |\nabla\tilde{h}_{3,m}|, \end{aligned}$$

and (2.5), (2.6) give $|\nabla\tilde{h}_m| \in L^2_{\mathrm{loc}}(\Omega_2)$ uniformly w.r.t. m. This implies by Trudinger's inequality (see [GT], Theorem 7.15)

$$\int_{B_\rho} \exp\left(\beta_0\tilde{h}_m^2\right) dx \le c(\rho) < \infty \tag{3.8}$$

for disks $B_\rho \Subset \Omega_2$ with β_0 depending on the $W^1_2(B_\rho)$-norm of $\tilde{h}_m$. From the definition of the function h_m it is immediate that

$$h_m^2 \le c\,\tilde{h}_m^{2(1-2/q)},$$

so that by (3.8) for any $\beta > 0$

$$\begin{aligned} \int_{B_\rho} \exp(\beta h_m^2)\,dx &\le \int_{B_\rho} \exp\Big(c\beta\tilde{h}_m^{2(1-2/q)}\Big)\,dx \\ &\le \int_{B_\rho} \exp\Big(\beta_0\tilde{h}_m^2 + c(\beta)\Big)\,dx < \infty, \end{aligned}$$

and (3.7) follows. Lemma A.1 implies

$$\int_{B_\rho} H_m^2 \log^{c_0\beta}(e + H_m)\,dx \le c(\beta,\rho)\,. \tag{3.9}$$

Let $\sigma_{1,m} := DF(\partial_1 u_m)$. Then

$$\begin{aligned} |\nabla\sigma_{1,m}|^2 &= \partial_\gamma\big(DF(\partial_1 u_m)\big)\cdot\partial_\gamma\sigma_{1,m} \\ &= D^2F(\partial_1 u_m)(\partial_\gamma\partial_1 u_m, \partial_\gamma\sigma_{1,m}) \\ &\le c(1 + |\partial_1 u_m|^2)^{\frac{p-2}{2}}|\nabla\partial_1 u_m|\,|\nabla\sigma_{1,m}| \\ &\le c\,H_m h_{1,m}|\nabla\sigma_{1,m}|, \end{aligned}$$

and we get $|\nabla\sigma_{1,m}| \le cH_m h_{1,m} \le cH_m h_m$. But as demonstrated in [BFZ1] (compare the calculations after inequality (2.11)) the latter estimate together with (3.9) and the inequality $\int_{B_\rho} \exp(\beta h_m^2)\, dx \le c(\beta,\rho)$ implies

$$\int_{B_\rho} |\nabla\sigma_{1,m}|^2 \log^\alpha \big(e + |\nabla\sigma_{1,m}|\big)\, dx \le c(\alpha,\rho), \tag{3.10}$$

and (3.10) also holds with $\sigma_{1,m}$ replaced by $\sigma_{2,m} := DG(\partial_2 u_m)$, where α is arbitrary large. If $\alpha > 1$, (3.10) shows that the vectors $\sigma_{1,m}$, $\sigma_{2,m}$ are continuous uniformly w.r.t. m, see, e.g., [KKM], Example 5.3. Alternatively, we may use Lemma A.2 (choose E as a disc of radius ρ and apply a scaled version of (A3)) combined with the variant of the Dirichlet-growth theorem given by Frehse [Fr], p.287, to deduce the uniform continuity of $\sigma_{1,m}$ and $\sigma_{2,m}$. Since DF and DG are isomorphisms $\mathbb{R}^N \to \mathbb{R}^N$, we get the uniform continuity of $\partial_1 u_m$, $\partial_2 u_m$, hence the sequence $\{\nabla u_m\}$ is uniformly continuous. Recalling (2.8) and using Arcela's theorem, we have shown that u is in the space $C^1(\Omega_2;\mathbb{R}^N)$. If we let $\overline{u} = \partial_\gamma u$, $\gamma = 1,2$, then

$$0 = \int_\Omega D^2 f(\nabla u)(\nabla\overline{u}, \nabla\varphi)\, dx \quad \text{for all}\, \varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$$

is an elliptic system for $\overline{u}$ with coefficients $D^2 f(\nabla u)$ of class C^0, thus $\overline{u} \in C^{0,\alpha}(\Omega;\mathbb{R}^N)$, $0 < \alpha < 1$, follows from classical results (see e.g. [Gi1]). □

4 Proof of Theorem 1.2

In accordance with [BF3] we now let

$$\begin{aligned} \rho_m &:= \|\overline{u}_m - u\|_{W_p^2(\Omega_2)} \left[\int_{\Omega_2} \big(1 + |\nabla^2\overline{u}_m|^2\big)^{q/2}\right]^{-1}, \\ \widetilde{J}_m[w,\Omega_2] &:= \rho_m \int_{\Omega_2} \big(1 + |\nabla^2 w|^2\big)^{\frac{q}{2}}\, dx + \widetilde{J}[w,\Omega_2] \end{aligned}$$

for functions $w \in W_q^2(\Omega_2)$, and denote by u_m the $\widetilde{J}_m[\cdot,\Omega_2]$-minimizer in $\overline{u}_m + \overset{\circ}{W}{}_q^2(\Omega_2)$, where $\overline{u}_m$ is defined as in Section 2. Lemma 2.1 remains valid with obvious modifications and as a substitute for (2.1) we get (compare the inequality stated in Step 4 of Section 2 of [BF3])

$$\begin{aligned} &\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\gamma\nabla^2 u_m, \partial_\gamma\nabla^2 u_m)\, dx \\ &\quad \le\ -\int_{\Omega_2} D^2 f_m(\nabla^2 u_m)\big(\partial_\gamma\nabla^2 u_m, \nabla^2\eta^6\partial_\gamma u_m^* + 2\nabla\eta^6 \otimes \nabla\partial_\gamma u_m^*\big). \end{aligned} \tag{4.1}$$

Here $\eta \in C_0^\infty(\Omega_2)$ is arbitrary and $u_m^*(x) := u(x) - k(x)$, where $k(x)$ is any polynomial of degree ≤ 2. Choosing $k = 0$ in (2.1) we can adjust Step 3 in Section 2 of [BF3] along the lines of Section 2 to deduce $\nabla^2 u_m \in L^t_{\mathrm{loc}}(\Omega_2)$ uniformly w.r.t. m for any $t < \infty$. During this procedure the quantities $\partial_1 u_m$, $\partial_2 u_m$ have to be replaced by $(\nabla^2 u_m)_I$, $(\nabla^2 u_m)_{II}$, respectively, for example we now have $\tilde{h}_{1,m} = (1 + |(\nabla^2 u_m)_I|^2)^{p/4}$, etc. In the same spirit we deduce (2.5) and (2.6), (2.7) has to be replaced by $u_m \in W^3_{2,\mathrm{loc}}(\Omega_2)$ uniformly w.r.t. to m, and (2.8) now reads $\nabla^2 u_m \to \nabla^2 u$ a.e. on Ω_2. In Section 3 we replace the old function H_m by

$$H^2_m := D^2 \tilde{f}_m(\nabla^2 u_m)(\partial_\gamma \nabla^2 u_m, \partial_\gamma \nabla^2 u_m),$$

and get from (4.1) (with an obvious new meaning of $h_{1,m}$, $h_{2,m}$, $h_{3,m}$, h_m)

$$\int_{B_R} H^2_m \, dx \le c \int_{B_{2R}} H_m h_m \big[|\nabla^2 \eta^6| |\nabla u_m - \nabla k| + |\nabla \eta^6| \, |\nabla^2 u_m - \nabla^2 k| \big] \, dx. \tag{4.2}$$

This is exactly (2.18) in [BF3], and with the same calculations as in this paper we get from (4.2) after appropriate choice of k the validity of (3.6). The hypothesis of Lemma A.1 are still valid, so that we can deduce (3.9). Next we let $\sigma_{I,m} := D\widetilde{F}\big((\nabla^2 u_m)_I\big), \sigma_{II,m} := D\widetilde{G}\big((\nabla^2 u_m)_{II}\big)$ and get the uniform continuity of $\sigma_{I,m}$, $\sigma_{II,m}$, from which now the continuity of $\nabla^2 u$ follows. For the higher regularity of u we can quote Section 2, Step 5, of [BF3].

5 Remarks on the degenerate case

In order to simplify our exposition and to benefit from our earlier work we have stated our results for the non-degenerate case by the way excluding the example $\int_\Omega [|\partial_1 u|^p + |\partial_2 u|^q] dx$, $2 \le p < q < \infty$, or more general densities $f(\nabla u) = F(\partial_1 u) + G(\partial_2 u)$ for which

$$\lambda |X|^{p-2} |Y|^2 \le D^2 F(X)(Y,Y) \le \Lambda (1 + |X|^2)^{\frac{p-2}{2}} |Y|^2 \quad , \tag{5.1}$$

$$\lambda |X|^{q-2} |Y|^2 \le D^2 G(X)(Y,Y) \le \Lambda (1 + |X|^2)^{\frac{q-2}{2}} |Y|^2 \tag{5.2}$$

is true with constants λ, $\Lambda > 0$ and for all X, $Y \in \mathbb{R}^N$. Under these assumptions we have a regularity result which is slightly weaker than the conclusion formulated in Theorem 1.1:

THEOREM 5.1. *Suppose that $u \in W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ locally minimizes the energy J from (1.1) and let $f(X_1 X_2) = F(X_1) + G(X_2)$, X_1, $X_2 \in \mathbb{R}^N$, with F and G satisfying (5.1) and (5.2) for exponents $2 \le p \le q < \infty$. Then, if (1.8) holds, u is continuously differentiable in Ω.*

REMARK 5.1. *Of course a corresponding version of Theorem 1.2 is valid, if we replace (1.12) and (1.13) by their degenerate variants.*

Sketch of the proof of Theorem 5.1. The following calculations have to be made precise by approximation, which we leave to the reader. We have (compare (3.1))

$$\int_\Omega D^2 f(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma \nabla u) \eta^2 \, dx \le -2 \int_\Omega D^2 f(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma u^* \otimes \nabla \eta) \, dx \tag{5.3}$$

for any $\eta \in C_0^\infty(\Omega)$. Again we use summation w.r.t. γ. In (5.3) u^* denotes the function $u - Px$ for a matrix $P \in \mathbb{R}^{2N}$. We let

$$\begin{aligned} H^2 &:= D^2 f(\nabla u)(\partial_\gamma \nabla u, \partial_\gamma \nabla u), \\ h_1 &:= (1 + |\partial_1 u|^2)^{\frac{p-2}{4}}, \\ h_2 &:= (1 + |\partial_2 u|^2)^{\frac{q-2}{4}}, \\ h &:= (h_1^2 + h_2^2)^{\frac{1}{2}} \end{aligned}$$

and get from (5.3), if $\eta \equiv 1$ on a disc $B_R = B_R(x_0)$, $\eta \equiv 0$ outside of $B_{2R} \Subset \Omega$, $0 \le \eta \le 1$ and $|\nabla \eta| \le c/R$ (see (3.4))

$$\int_{B_R} H^2 \, dx \le \frac{c}{R} \int_{B_{2R}} Hh|\nabla u - P| \, dx. \tag{5.4}$$

Clearly (5.4) implies the "starting inequality" (compare (3.6))

$$\left[\fint_{B_R} H^2 \, dx \right]^{\frac{1}{2}} \le c \left[\fint_{B_{2R}} (hH)^{\frac{4}{3}} \, dx \right]^{\frac{3}{4}}, \tag{5.5}$$

and in order to combine (5.5) with the lemma from the appendix we have to check the validity of (3.7) for the function h in place of h_m. Introducing $\tilde{h}_1 := |\partial_1 u|^{p/2}$, $\tilde{h}_2 := |\partial_2 u|^{q/2}$ and $\tilde{h} := (\tilde{h}_1^2 + \tilde{h}_2^2)^{1/2}$ we have as before $|\nabla \tilde{h}| \le |\nabla \tilde{h}_1| + |\nabla \tilde{h}_2|$, and since the functions $\tilde{h}_1$, $\tilde{h}_2$ are of class $W^1_{2,\mathrm{loc}}$, we arrive at (3.8) for the function $\tilde{h}$, which implies (3.7) with minor changes in the calculation. The same arguments as used in Section 3 then give continuity of $\partial_1 u$ and $\partial_2 u$, so that we deduce $u \in C^1(\Omega; \mathbb{R}^N)$. □

REMARK 5.2. *Due to the degeneracy of the problem we cannot use the hole-filling argument originating in [FrS] and successfully applied in [BF4] in order to deduce from* $\nabla u \in C^0(\Omega; \mathbb{R}^{2N})$ *the local Hölder continuity of the gradient for some exponent* $0 < \alpha < 1$.

Appendix. A lemma on the higher integrability of functions

The following result has been established in [BFZ1], Lemma 1.2.

LEMMA A.1. *Let* $d > 1$, $\beta > 0$ *be given numbers. Consider functions* $\overline{f}$, $\overline{g}$, $\overline{h}$ *from a domain* $G \subset \mathbb{R}^n$, $n \ge 2$, *being non-negative and satisfying*

$$\overline{f} \in L^d_{\mathrm{loc}}(G), \quad \exp(\beta \overline{g}^d) \in L^1_{\mathrm{loc}}(G), \quad \overline{h} \in L^d_{\mathrm{loc}}(G).$$

Suppose further that there is a constant $C > 0$ *such that*

$$\left[\fint_{B_R} \overline{f}^d \, dx \right]^{\frac{1}{d}} \le C \fint_{B_{2R}} \overline{f}\overline{g} \, dx + C \left[\fint_{B_{2R}} \overline{h}^d \, dx \right]^{\frac{1}{d}} \tag{A.1}$$

holds for all balls $B_{2R} = B_{2R}(x_0) \Subset G$. Then there exists a real number $c_0 = c_0(n, d, C)$ as follows: if

$$\overline{h}^d \log^{c_0\beta}(e + \overline{h}) \in L^1_{\mathrm{loc}}(G), \tag{A.2}$$

then the same is true for $\overline{f}$.

It follows from Lemma A.1 (see Corollary 1.3 in [BFZ1])

LEMMA A.2. *Suppose that $\overline{f}$, $\overline{g}$, $\overline{h}$ are the same as in Lemma A.1, and that (A.1) is true for all balls $B_{2R} = B_{2R}(x_0) \Subset B_1(0) \subset \mathbb{R}^n$. Suppose also that $\overline{h}^d \log^{c_0\beta}(e + \overline{h}) \in L^1_{\mathrm{loc}}(B_1(0))$, where c_0 is as in Lemma A.1. Then*

$$\int_E \overline{f}^d \, dx \le c \log^{-c_0\beta}\Big(e + \frac{1}{\mathcal{L}^n(E)}\Big) \tag{A.3}$$

for all measurable sets $E \subset B_{1/2}(0)$, where the constant c depends only on n, d, C, β, $\overline{f}$, $\overline{g}$ and $\overline{h}$ but not on the set E, and $\mathcal{L}^n(E)$ denotes the n-dimensional Lebesgue measure of the set E.

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