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Abstract

A parabolic obstacle-type problem without sigh restriction on a solution is considered. An exact representation of the global solutions (i.e., solutions in the entire half-space $\{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$) is found. It is proved, without any additional assumptions on a free boundary, that near the fixed boundary where the homogeneous Dirichlet condition is fulfilled, the boundary of the "noncoincidence set" is the graph of a Lipschitz function.

1 Introduction

In this paper, the regularity properties of a free boundary in a neighborhood of the fixed boundary of a domain are studied for a parabolic obstacle-type problem with no restriction on the sign of the solution. Mathematically the problem is formulated as follows.
Let a function u and an open set $\Omega \subset \mathbb{R}^{n+1}_+$ solve the problem:

$$\begin{gathered} H(u) = \chi_\Omega \quad \text{in} \quad Q_1^+, \\ u = |Du| = 0 \quad \text{in} \quad Q_1^+ \setminus \Omega, \qquad u = 0 \quad \text{on} \quad \Pi \cap Q_1, \end{gathered} \tag{1.1}$$

where $H = \Delta - \partial_t$ is the heat operator, χ_Ω denotes the characteristic function of Ω, Q_1 is the unit cylinder in $\mathbb{R}^{n+1}$, $Q_1^+ = Q_1 \cap \{x_1 > 0\}$, $\Pi = \{(x,t) : x_1 = 0\}$, and the first equation in (1.1) is satisfied in the sense of distributions.
The regularity of the free boundary for the problem (1.1) was investigated earlier only in the special case of the parabolic obstacle problem (see [ASU3]), where the additional information $u \geqslant 0$ permits to establish the Lipschitz regularity of a free boundary in a neighborhood of Π as well as $C^{1,\alpha}$-regularity for parts of a free boundary lying inside Q_1^+.
Results for an elliptic problem related to (1.1) were obtained in [SU]. It should be mentioned also the paper [CPS] where the parabolic problem (1.1) was considered in the whole unit cylinder, without conditions on the fixed boundary Π. In papers [SU, CPS] the optimal regularity of solutions were studied in addition to the regularity properties of a free boundary. For corresponding optimal regularity result for solutions of the problem (1.1) we refer the reader to [ASU1].

1.1 Notations and definitions.

Throughout the paper we will use the following notations:
$z = (x,t)$ are points in $\mathbb{R}^{n+1}$, where $x = (x_1, x') = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \geqslant 2$, and $t \in \mathbb{R}^1$;

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$;
$\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$;
$\mathbb{R}_- = (-\infty, 0]$;
$\Pi = \{(x,t) \in \mathbb{R}^{n+1} : x_1 = 0\}$;
$e_1, \dots, e_n$ is the standard basis in $\mathbb{R}^n_x$;
e_0 is the standard basis in $\mathbb{R}^1_t$;
χ_Ω denotes the characteristic function of the set Ω ($\Omega \subset \mathbb{R}^{n+1}$);
$v_+ = \max\{v, 0\}$; $\qquad v_- = \max\{-v, 0\}$;
$B_r(x^0)$ denotes the open ball in $\mathbb{R}^n$ with center x^0 and radius r;
$B^+_r(x^0) = B_r(x^0) \cap \mathbb{R}^n_+$;
$B_r = B_r(0)$;
$Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0]$ is the cylinder in $\mathbb{R}^{n+1}$;
$Q^+_r(z^0) = Q^+_r(x^0, t^0) = Q_r(x^0, t^0) \cap \mathbb{R}^{n+1}_+$;
$Q_r = Q_r(0,0)$;
$Q^+_r = Q^+_r(0,0)$.
We emphasize that in this paper the top of the cylinder $Q_r(z^0)$ is included in the set $Q_r(z^0)$. If $Q = \mathbb{R}^{n+1}_+ \cap Q_r(z^0)$, then $\partial' Q$ is the parabolic boundary of Q, i.e., $\partial' Q = \overline{Q} \setminus Q$. For a point $z = (x,t) \in \mathbb{R}^{n+1}_+$ we define its *parabolic distance* to a set $E \subset \mathbb{R}^{n+1}$ as $\text{dist}\{z, E\} := \sup\{r > 0 : Q_r(z) \cap E = \emptyset\}$.
D_i denotes the differential operator with respect to x_i; $\partial_t = \frac{\partial}{\partial t}$;
$D = (D_1, D') = (D_1, D_2, \dots, D_n)$ denotes the spatial gradient; D_ν stands for the operator of differentiation along the direction $\nu \in \mathbb{R}^{n+1}$, i.e., $|\nu| = 1$ and

$$D_\nu u = \sum_{i=1}^n \nu_i D_i u + \nu_0 \partial_t u;$$

$D^2 u = D(Du)$ denotes the Hessian of u;
$H = \Delta - \partial_t$ is the heat operator.
We adopt the convention that the index τ runs from 2 to n. We also adopt the convention regarding summation with respect to repeated indices.

$\| \cdot \|_{p,E}$ denotes the norm in $L_p(E)$, $1 < p \leqslant \infty$;
$W^{2,1}_p(E)$ is the anisotropic Sobolev space with the norm

$$\|u\|_{W^{2,1}_p(E)} = \|\partial_t u\|_{p,E} + \|D(Du)\|_{p,E} + \|u\|_{p,E};$$

We use letters M, N, A, and C (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in the parentheses: $C(\dots)$.

Let M be a constant, $M \geqslant 1$.

Definition 1. We say a function $u \in W^{2,1}_{\infty}(Q^+_R)$ (not identically zero) belongs to the class $P^+_R(M)$ if u satisfies:

(a) $H[u] = \chi_\Omega$ in Q^+_R, for some open set $\Omega = \Omega(u) \subset Q^+_R$;

(b) $u = |Du| = 0$ in $Q^+_R \setminus \Omega$;

(c) $u = 0$ on $\Pi \cap Q_R$;

(d) $\operatorname*{ess\,sup}_{Q^+_R}(|\partial_t u| + |D^2 u|) \leqslant M$

and the equation in (a) is understood in the sense of distributions. The elements of $P^+_R(M)$ will be called *local solutions*.

Definition 2. Let $P^+_\infty(M)$ be the class of functions $u \in W^{2,1}_{\infty}(\mathbb{R}^n_+ \times \mathbb{R}_-)$ such that

(a') $H[u] = \chi_\Omega$ in $\mathbb{R}^{n+1}_+$ for some open set $\Omega = \Omega(u) \subset \mathbb{R}^n_+ \times \mathbb{R}_-$;

(b') $u = |Du| = 0$ in $\mathbb{R}^{n+1}_+ \setminus \Omega(u)$;

(c') $u = 0$ on Π;

(d') $\operatorname*{ess\,sup}_{\mathbb{R}^n_+ \times \mathbb{R}_-}(|\partial_t u| + |D^2 u|) \leqslant M$,

where the equation in (a') is understood in the sense of distributions. The elements of $P^+_\infty(M)$ will be called *global solutions*.
In both cases we shall use the following notation:

- $\Lambda(u) = \{(x,t) : u(x,t) = |Du(x,t)| = 0\}$;
- $\Gamma(u) = \{z = (x,t) \in \Lambda(u) : Q_\rho(z) \cap \Omega(u) \neq \emptyset \quad \forall \rho > 0\}$ is the free boundary;
- $\Gamma(u) \cap \Pi$ is the set of contact points.

It is assumed that $\Gamma(u) \neq \emptyset$.
We also introduce the class $P_\infty(M)$. In this case the half-space $\mathbb{R}^{n+1} \cap \{t \leqslant 0\}$ is considered instead of $\mathbb{R}^{n+1}_+ \cap \{t \leqslant 0\}$, and we omit the condition $u|_\Pi = 0$.

Definition 3. Let $z^* = (x^*, t^*)$ be a point in $\mathbb{R}^{n+1}$. We say that a function v is *parabolic homogeneous of degree 2 w.r.t.* z^* if either of the following statements is satisfied:

(i) a function v is defined in $\mathbb{R}^{n+1} \cap \{t \leqslant t^*\}$ and the identity

$$v(\lambda x + x^*, \lambda^2 t + t^*) = \lambda^2 v(x + x^*, t + t^*) \tag{1.2}$$

holds for all $\lambda > 0$, $t \leqslant 0$, and for all $x \in \mathbb{R}^n$ such that $x_1 + x_1^* \geqslant 0$, $\lambda x_1 + x_1^* \geqslant 0$.

(ii) $x_1^* \geqslant 0$, a function v is defined in $\{(x,t) \in \mathbb{R}^{n+1} : x_1 \geqslant 0, t \leqslant t^*\}$, and the identity (1.2) holds for all $\lambda > 0$, and for all $(x,t) \in \mathbb{R}^{n+1} \cap \{\lambda x_1 + x_1^* \geqslant 0, t \leqslant 0\}$.

1.2 Main results

Our prime goal in this paper is to obtain the following results:

Theorem I *Let $u \in P_\infty^+(M)$, and let $z^0 = (x^0, t^0) \in \Gamma(u)$. Then u is independent of t and of the variables $x_3, \dots, x_n$. More precisely, for $(x,t) \in \mathbb{R}_+^{n+1} \cap \{t \leqslant t^0\}$ we have*

$$x_1^0 = 0 \quad \Longrightarrow \quad u(x,t) = \frac{x_1^2}{2} + a x_1 x_2 \tag{1.3}$$

in some suitable rotated coordinate system that leaves e_1 fixed, and for some real number a,

$$x_1^0 > 0 \quad \Longrightarrow \quad u(x,t) = \frac{((x_1 - x_1^0)_+)^2}{2}. \tag{1.4}$$

Theorem II *Let $u \in P_2^+(M)$, let $\delta = \delta(n, M)$ be the same constant as in Lemma 5.5, and let $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_{1/8}^+ \cap \{x_1 < \delta/2\}$. There exists a Lipschitz continuous nonnegative function f, defined on $\Pi \cap Q_{2r}(z^0)$, and such that*

$$\Omega(u) \cap Q_{2r}(z^0) = \{(x,t) \in Q_{2r}(z^0) : x_1 > f(x', t)\}.$$

Here $r = x_1^0$, and the Lipschitz constant of f w.r.t. the x-variables depends on n and M only, while the Lipschitz constant of f w.r.t. t equals $C(n,M)r^{-1}$.

2 Useful facts.

For the reader's convenience and for future references, we recall and explain some facts.

2.1 Nondegeneracy

Fact 1. *Let* $u \in P_R^+(M)$ *with* $0 < R \leqslant +\infty$. *Then there exists a constant* $C(n) > 0$ *such that*

(i) *for all* $z^0 \in \{u > 0\} \cup \Gamma(u)$, *and for all* ρ *satisfying* $Q_\rho(z^0) \subset Q_R$ *we have*

$$\sup_{Q_\rho^+(z^0)} u \geqslant u(z^0) + C(n)\rho^2;$$

(ii) *for all* $z^0 \in \Lambda(u)$, *and for all* ρ *satisfying* $Q_\rho(z^0) \subset Q_R^+$ *we have either*

$$\sup_{Q_\rho(z^0)} u \geqslant C(n)\rho^2$$

or $u \equiv 0$ *in* $Q_{\rho/2}(z^0)$;

(iii) *if* $u(x,t) < 0$ *for all* $(x,t) \in Q_\rho(z^0) \subset Q_R^+$ *then we have*

$$u(z^0) \leqslant -C(n)\rho^2.$$

Proof. The proof of case (i) with $C(n) = \frac{1}{2n+1}$ is just the same as the proof of Fact 2 in [ASU2]. Case (ii) was considered in [CPS] (see Lemma 5.1).
It remains only to verify case (iii). It is clear that $H[u] = 1$ in $Q_\rho(z^0)$. Further, we consider the function

$$w(x,t) = u(x,t) - u(x^0,t^0) - \frac{1}{2n+1}(|x-x^0|^2 - (t-t^0)).$$

Then w is caloric in $Q_\rho(z^0)$, and $w(x^0,t^0) = 0$. Moreover, the maximum principle yields

$$0 \leqslant \sup_{Q_\rho(z^0)} w = \sup_{\partial' Q_\rho(z^0)} w \leqslant \sup_{\partial' Q_\rho(z^0)} u - u(x^0,t^0) - \inf_{\partial' Q_\rho(z^0)} \frac{1}{2n+1}(|x-x^0|^2 - (t-t^0))$$
$$< -u(x^0,t^0) - \frac{\rho^2}{2n+1}.$$

And we are done with $C(n) = \frac{1}{2n+1}$. This completes the proof. □

2.2 Monotonicity formulas.

So-called monotonicity formulas will play an essential role in this paper and will appear in almost every section.

We will use two different kind of monotonicity formulas, the first due to G.Weiss [W] and the second due to L.Caffarelli (see [C], [CK] and [ASU2], Lemma 2.1).
To formulate the first monotonicity formula, we define Weiss's functional as follows:

$$W(r,x^0,t^0,v):=\frac{1}{r^4}\int\limits_{t^0-4r^2}^{t^0-r^2}\int\limits_{\mathbb{R}^n}\left(|Dv|^2+2v+\frac{v^2}{t-t^0}\right)G(x-x^0,t^0-t)dxdt,$$

where (x^0,t^0) is a point in $\mathbb{R}^{n+1}_+$, r is a positive constant, the heat kernel $G(x,t)$ is defined by

$$G(x,t)=\frac{\exp\left(-|x|^2/4t\right)}{(4\pi t)^{n/2}}\quad\text{for } t>0\quad\text{and}\quad G(x,t)=0\quad\text{for } t\leqslant 0,\tag{2.1}$$

and v is a continuous function defined on $\mathcal{Q}:=\mathbb{R}^n\times[t^0-4\mathcal{R}^2,t^0]$, $\mathcal{R}\geqslant r$. We also suppose that $D_iv\in L_{2,loc}(\mathcal{Q})$ and Dv have at most polynomial growth w.r.t x as $|x|\to\infty$.
It is easy to check that for any $\lambda\in]0,\mathcal{R}/r]$ the functional W has the following scaling property:

$$W(\lambda r,x^0,t^0,v)=W(\lambda,0,0,v_r),\tag{2.2}$$

where

$$v_r(x,t)=\frac{v(rx+x^0,r^2t+t^0)}{r^2}\tag{2.3}$$

is the parabolic scaling of v around $z^0=(x^0,t^0)$.
Suppose now that a function $u\in P^+_\infty(M)$ and $z^0=(x^0,t^0)\in\Gamma(u)$. Then we extend u by zero across the plane Π to the set $\mathbb{R}^{n+1}\cap\{x_1<0,t\leqslant 0\}$; we preserve the notation u for this extension. From Lemma 1.2 [ASU3] it follows that

$$\begin{aligned}\frac{dW(r,x^0,t^0,u)}{dr}&=\frac{1}{r}\int\limits_{-4}^{-1}\int\limits_{\mathbb{R}^n}\frac{|\mathcal{L}u_r|^2}{-t}G(x,-t)dxdt\\&+\frac{x_1^0}{r^2}\int\limits_{-4}^{-1}\int\limits_{x_1=\frac{-x_1^0}{r}}|D_1u_r|^2G(x,-t)dx'dt\geqslant 0,\end{aligned}\tag{2.4}$$

where u_r is as in (2.3), and

$$\mathcal{L}u_r(x,t):=x\cdot Du_r(x,t)+2t\partial_tu_r(x,t)-2u_r(x,t).$$

Relation (2.4) guaranties that the functional W is monotone nondecreasing with respect to r. In particular, the equality $\frac{dW}{dr} = 0$ for all $r > 0$ is equivalent to

$$\begin{aligned} \mathcal{L}u_r(x,t) &= 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times]-4,-1[, \\ D_1 u_r &= 0 \quad \text{on} \quad \{x_1 = \frac{-x_1^0}{r}\}. \end{aligned} \tag{2.5}$$

It is evident that the first equality in (2.5) gives the degree 2 parabolic homogeneity of the function u w.r.t. z^0.
For our purposes it is also essential to introduce a local version of the Weiss functional. In particular, this permits us to make a conclusion about the homogeneity of the blow-up and blow-down limits. Similarly to [ASU3] we define the local Weiss functional as follows:

$$W_b(r, x^0, t^0, v) := \frac{1}{r^4} \int\limits_{t^0-4r^2}^{t^0-r^2} \int\limits_{B_b(x^0)} \left(|Dv|^2 + 2v + \frac{v^2}{t-t^0} \right) G(x-x^0, t^0-t)\,dxdt,$$

where b and r are positive constants, $z^0 = (x^0, t^0)$ is a point in $\mathbb{R}^{n+1}$, the function G is the same as in (2.1), and v is a continuous function defined on $\mathcal{Q}_b(z^0) := B_b(z^0) \times]t^0 - 4\mathcal{R}^2, t^0[$, $\mathcal{R} \geqslant r$ and satisfying $|Dv| \in L_2(\mathcal{Q}_b(z^0))$.
It should be mentioned that Lemma 1.1 [ASU3] guarantees for the local Weiss functional and for any $\lambda \in]0, \mathcal{R}/r]$ the scaling property

$$W_b(\lambda r, x^0, t^0, v) = W_{b/r}(\lambda, 0, 0, v_r), \tag{2.6}$$

with v_r defined by (2.3).
To apply the local Weiss functional W_b to $u \in P_b^+(M)$ we always assume that u is extended by zero across the plane Π to the set $Q_b \cap \{x_1 < 0\}$ and preserve the notation u for this extension.
In [ASU3], see Lemma 1.2 and Remark after that, it was proved the following estimate: if $u \in P_2^+(M)$ and $(x^0, t^0) \in \Gamma(u) \cap Q_1$ then for arbitrary ρ and α satisfying $\rho > \alpha > 0$ we have

$$W_1(\rho, x^0, t^0, u) - W_1(\alpha, x^0, t^0, u) \geqslant -C_0(n, M)(\rho - \alpha) \exp(-1/20\rho^2). \tag{2.7}$$

From here, see Corollary 1.3 [ASU3], it follows that the function $W_1(r, x^0, t^0, u)$ has a limit as $r \to 0^+$. The corresponding limit

$$\omega(x^0, t^0, u) := \lim_{r \to 0^+} W_1(r, x^0, t^0, u) \tag{2.8}$$

will be called the *balanced energy* of the function u at the point (x^0, t^0) of the free boundary.

To formulate the second monotonicity formula we denote

$$I(r,v,z^0)=\int\limits_{t^0-r^2}^{t^0}\int\limits_{\mathbb{R}^n}|Dv(x,t)|^2G(x-x^0,t^0-t)dxdt,$$

where $r\in]0,R]$, $z^0=(x^0,t^0)$ is a point in $\mathbb{R}^{n+1}$, a function v is defined in the strip $E=\mathbb{R}^n\times[t^0-R^2,t^0]$, and the function $G(x,t)$ is defined in (2.1).
Suppose now that h_1 and h_2 are nonnegative sub-caloric functions in the strip E, with a at most polynomial growth at infinity, such that

$$h_1(z^0)=h_2(z^0)=0 \qquad \text{and} \qquad h_1\cdot h_2\equiv 0.$$

Then the functional

$$\Phi(r)=\Phi(r,h_1,h_2,z^0)=\frac{1}{r^4}I(r,h_1,z^0)I(r,h_2,z^0) \tag{2.9}$$

is monotone nondecreasing in r. More precisely, if the supports of $h_1(\cdot,t^0-r^2)$ and $h_2(\cdot,t^0-r^2)$ are not complementary halfspaces in $\mathbb{R}^n$ containing x^0 on their boundaries, then either $\Phi'(r)>0$, or $\Phi(\rho)\equiv 0$ for $\rho\in(0,r]$.

2.3 Blow-up and blow-down

For a function $u\in P_R^+(M)$ with $0<R\leqslant+\infty$ and for a point $z^0=(x^0,t^0)\in\Gamma(u)$ we consider the parabolic scaling u_r defined by (2.3).
By the standard compactness arguments, we may pass to the limit along a subsequence $r_k\to 0$; as a result we obtain a global solution $u_0\in P_\infty(M)$. More precisely, this will be true if $x_1^0>0$. If $x_1^0=0$, then the function u_0 belongs to the class $P_\infty^+(M)$. Moreover, in both cases the point $(0,0)\in\Gamma(u_0)$.
Usually, such a process is referred to as the blow-up limit passage, and any global solution u_0 thus obtained is called a *blow-up* of the function u at the point z^0.
Similarly, if u is a global solution, i.e., $u\in P_\infty^+(M)$, and $z^0=(x^0,t^0)\in\Gamma(u)$ we can consider the scaled functions u_r around z^0 and let $r\to+\infty$. Then the u_r converge (for a subsequence) to a function u_∞ uniformly on the compact subsets of $(\mathbb{R}^n_+\cup\Pi)\times\mathbb{R}_-$. It is easy to see that $u_\infty\in P_\infty^+(M)$ and $(0,0)\in\Gamma(u_\infty)$. The limit function u_∞ is called a *blow-down* of the global solution u at the point z^0.
In general, possible different blow-up and blow-down limits may be obtained at the same point if we choose different subsequences r_k.

Fact 2. *The functions* u_0 and u_∞ *are degree 2 parabolic homogeneous w.r.t. the origin.*

Proof. We prove Fact 2 for the blow-up of u at the point z^0. The case of u_∞ is treated in the same way.
To prove the statement for u_0, it suffices to observe that for any $0 < \lambda$ we have

$$W(\lambda, 0, 0, u_0) = \lim_{r \to 0+} W_{1/r}(\lambda, 0, 0, u_r) = \lim_{r \to 0+} W_1(\lambda r, x^0, t^0, u)$$
$$= \text{const} = \omega(x^0, t^0, u).$$

The second equality in the above relation follows from scaling property (2.6), while the monotonicity of the local Weiss functional and regularity properties of u give the third equality. □

3 Global solutions

Lemma 3.1 *Let $u \in P_\infty^+(M)$ be a degree 2 parabolic homogeneous function w.r.t. the origin, and let $(0,0) \in \Gamma(u)$. Then for some $a \in \mathbb{R}$ we have*

$$u(x,t) = \frac{x_1^2}{2} + a x_1 x_2 \quad in \quad \mathbb{R}_+^n \times \mathbb{R}_-. \tag{3.1}$$

Proof. Let e be a unit spatial direction orthogonal to e_1. We claim that the function $v := D_e u$ does not change its sign. To prove this, first we extend v by zero across the plane Π to the entire space $\mathbb{R}^n \times \mathbb{R}_-$ and keep the notation v for the extension. From homogeneity of u w.r.t. the origin it follows that

$$\Phi(\lambda, v_+, v_-, 0, 0) = C(e) \tag{3.2}$$

where Φ is as in (2.9). However, in our case the equality (3.2) is possible only if $C(e) = 0$, see item 2.2. This implies $v \geqslant 0$ or $v \leqslant 0$ for all points of $\mathbb{R}_+^n \times \mathbb{R}_-$. So, we have proved that $D_e u$ preserves its sign. Since this is true for all spatial directions e orthogonal to e_1, it follows that $u(x,t)$ is two-space dimensional, i.e., in suitable spatial coordinate

$$u(x,t) = u(x_1, x_2, t). \tag{3.3}$$

For definiteness, we assume in the rest of the proof that

$$D_2 u \geqslant 0 \tag{3.4}$$

(otherwise we replace e_2 by $-e_2$).

Further, we consider the case where the interior of $\Lambda(u)$ is empty. For $(x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_-$ we define the function w by the formula

$$w(x,t) = u(x,t) - \frac{x_1^2}{2}.$$

It is easy to see that w is caloric in $\mathbb{R}^n_+ \times \mathbb{R}_-$, and has at most quadratic growth with respect to x and at most linear growth with respect to t. Then we extend w by the odd reflection to the whole space $\mathbb{R}^{n+1} \cap \{t \leqslant 0\}$ and preserve the notation w for the extended function. By the Liouville theorem (see Lemma 2.1 in [ASU1]), the function w, and, consequently, the function u, is a polynomial of degree 2. Taking into account the homogeneity of u, the equality (3.3) and the condition $u|_{\Pi} = 0$ we get the exact representation

$$u(x,t) = \frac{x_1^2}{2} + ax_1x_2, \qquad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}_-$$

with some constant a.
Now we claim that the interior of $\Lambda(u)$ is always empty. Suppose, towards a contradiction, that we may fix a cylinder $Q_{2r}(z^0)$ in the interior of $\Lambda(u)$. In this part our arguments are similar to that of the proof of Theorem B [SU]. Due to (3.4) we must have $u \leqslant 0$ in

$$K_{2r}(z^0) := \{(x_1, x_2 - s, t) : (x_1, x_2, t) \in Q_{2r}(z^0), s \geqslant 0\}.$$

From here one infers that for the smaller set $K_r(z^0)$ we have

$$\partial\Omega(u) \cap K_r(z^0) = \emptyset. \tag{3.5}$$

Indeed, if there exists $z^* \in \partial\Omega(u) \cap K_r(z^0)$, then the maximum principle applied to the sub-caloric function u in $Q_r(z^*)$ gives that $u \equiv 0$ in $Q_r(z^*)$. Then, by homogeneity of u, it vanishes also in $B_{\lambda r}(\lambda x^*) \times \{\lambda^2 t^*\}$ for any $\lambda > 0$. Hence $z^* \notin \partial\Omega(u)$.
Combining (3.5) and the fact $Q_r(z^0) \subset \Lambda(u)$, we conclude that $K_r(z^0) \subset \Lambda(u)$. Hence we can translate u in the x_2-direction by considering the functions $u_m(x,t) = u(x_1, x_2 - m, t)$ in $\mathbb{R}^n_+ \times \mathbb{R}_-$. Since u is homogeneous w.r.t. the origin, each element of $\{u_m\}$ satisfies

$$u_m(\lambda x, \lambda^2 t) = \lambda^2 u(x_1, x_2 - \lambda^{-1} m, t) = \lambda^2 u_{m/\lambda}(x_1, x_2, t). \tag{3.6}$$

In addition, for all $x_2 \leqslant x_2^0$ we have

$$|u(x,t)| \leqslant M(|x_1 - x_1^0|^2 + |t - t^0|),$$

and, hence

$$|u_m(x,t)| \leqslant C(1 + x_1^2 + |t|). \tag{3.7}$$

Due to (3.4) and (3.7), the sequence $\{u_m\}$ is non-increasing and bounded for any fixed x_1 and t. Therefore, by compactness, it converges to a limit function $\widetilde{u}$, which is a global solution independent of x_2. It should be mentioned also that $\widetilde{u}$ is parabolic homogeneous function degree 2 w.r.t. the origin provided by (3.6). In other words, $\widetilde{u}$ is one-space dimensional homogeneous global solution with $Q_r(z^0) \subset \Lambda(\widetilde{u})$.
If $\widetilde{u}$ does not vanish identically in $\mathbb{R}^n_+ \times \mathbb{R}_-$ then, in view of the rest of the proof of Lemma 6.3 [CPS], we get the representation $\widetilde{u}(x,t) = (x_1)^2/2$. But the latter contradicts the fact $Q_r(z^0) \subset \Lambda(\widetilde{u})$.
Thus, the only possible case is $\widetilde{u} \equiv 0$ in $\mathbb{R}^n_+ \times \mathbb{R}_-$. Since convergence of u_m is monotone, we may conclude that $u \geqslant 0$ in $\mathbb{R}^n_+ \times \mathbb{R}_-$. Now we can apply Theorem 2 [ASU2] to the function u. This gives the exact representation $u(x,t) = (x_1)^2/2$, which contradicts our assumption about interior of $\Lambda(u)$. □

Lemma 3.2 *Let $u \in P^+_\infty(M)$, and let $z^0 = (x^0, t^0) \in \Gamma(u)$. Then the function u is a parabolic homogeneous function of degree 2 w.r.t. z^0, and*

$$W(r, x^0, t^0, u) = \frac{15}{4} =: A \quad \textit{for any} \quad r > 0. \tag{3.8}$$

Proof. We only need to prove (3.8). Then the first statement of our lemma follows immediately (see item 2.2).
Due to scaling property (2.2) and the monotonicity of the Weiss functional we have

$$W(1,0,0,u_0) \leqslant W(1,0,0,u_r) = W(r, x^0, t^0, u) \leqslant W(1,0,0,u_\infty), \tag{3.9}$$

where u_0 and u_∞ are blow-up and blow-down limits of the function u at the point z^0, respectively. We recall that according to Fact 2 the functions u_0 and u_∞ are degree 2 parabolic homogeneous w.r.t. the origin.
If $x_1^0 > 0$ then $u_0 \in P_\infty(M)$. Hence we can apply successively Lemma 6.3 and Lemma 6.2 [CPS] to the function u_0, which gives the bound

$$W(1,0,0,u_0) \geqslant A. \tag{3.10}$$

Otherwise, the function u_0 belongs to the class $P^+_\infty(M)$ and $(0,0) \in \Gamma(u_0)$. In this case Lemma 3.1 guarantees the exact representation $u_0(x,t) = \frac{x_1^2}{2} + a x_1 x_2$ where a is a constant. Direct computations (see, for example, Lemma 6.2 [CPS]) now give

$$W(1,0,0,u_0) = A. \tag{3.11}$$

Similarly, application of Lemma 3.1 to the function $u_\infty \in P_\infty^+(M)$ with $(0,0) \in \Gamma(u_\infty)$ and direct computations provide the identity

$$W(1,0,0,u_\infty) = A. \tag{3.12}$$

Finally, combining together the inequalities (3.9)-(3.12) we get the desired result. □

Proof of Theorem I. Let us suppose first that $x_1^0 = 0$. In this case the successive application of Lemma 3.2 and Lemma 3.1 to the function u gives the desired representation.

We now turn to the case $x_1^0 > 0$. By Lemma 3.2 the function u is parabolic homogeneous degree 2 w.r.t. z^0. This together with the assumption $z^0 \in \Gamma(u)$ guarantees that $u(x,t) \equiv 0$ in the infinite set $\{(x,t) : 0 < x_1 \leqslant x_1^0, t \leqslant t^0\}$. From here it follows that the function $\widetilde{u}(x,t) = u(x+x^0, t+t^0)$ belongs to the class $P_\infty^+(M)$ and $(0,0) \in \Gamma(\widetilde{u})$. It is obvious also that $\widetilde{u}$ is parabolic homogeneous degree 2 w.r.t. the origin. Therefore, applying Lemma 3.1 to $\widetilde{u}$ we get in some suitable rotated coordinate system that leaves e_1 fixed,the following exact representation

$$\widetilde{u}(x,t) = \frac{x_1^2}{2} + a x_1 x_2 \qquad \text{as} \quad x_1 \geqslant 0,$$

and, consequently,

$$u(x,t) = \begin{cases} \dfrac{(x_1 - x_1^0)^2}{2} + a(x_1 - x_1^0)(x_2 - x_2^0), & \text{if} \quad x_1 \geqslant x_1^0, \\ 0, \qquad \text{otherwise.} \end{cases} \tag{3.13}$$

Since Du is a continuous function we can conclude that in (3.13) parameter $a = 0$. This finishes the proof. □

4 Characterization of the free boundary points near Π

The information about global solutions that we obtained in Theorem I can be applied to the study of the behavior of the free boundary near points where it contacts the fixed boundary Π. We present first a preliminary result of this kind.

Lemma 4.1 *For any $\varepsilon > 0$ there exists $\rho = \rho_\varepsilon$ such that if $u \in P_1^+(M)$ and $(0,0) \in \Gamma(u)$, then for $z^* = (x^*, t^*) \in \Gamma(u) \cap Q_{\rho_\varepsilon}^+(0, \rho_\varepsilon^2/2)$ we have*

$$z^* \in Q_{\rho_\varepsilon}^+(0, \rho_\varepsilon^2/2) \setminus K_\varepsilon, \tag{4.1}$$

where

$$K_\varepsilon := \left\{ (x,t) : x_1 > \varepsilon\sqrt{|x'|^2+|t|} \right\}.$$

Proof. The proof of this statement is similar to (that of) Lemma 3.1 [ASU2]. For the reader's convenience and for the completeness, we provide it here.
Suppose, towards a contradiction, that for every $j \in \mathbb{N}$ there exist $u_j \in P_1^+(M)$ and $z^j = (x^j, t^j) \in \Gamma(u_j)$ such that $r_j := \sqrt{|x^j|^2+|t^j|} \leqslant j^{-1}$ and (4.1) with r_j instead of ρ_ε fails for z^j.
Considering the functions

$$\widetilde{u}_j(x,t) = \frac{u_j(r_j x, r_j^2 t)}{r_j^2},$$

we observe that for each of the functions $\widetilde{u}_j$ we have a point $\widetilde{z}^j = (\widetilde{x}^j, \widetilde{t}^j) \in \Gamma(\widetilde{u}_j)$ with $|\widetilde{x}^j|^2 + |\widetilde{t}^j| = 1$ and

$$\widetilde{x}_1^j \geqslant \varepsilon\sqrt{|(\widetilde{x}^j)'|^2 + |\widetilde{t}^j|} = \varepsilon\sqrt{1-(\widetilde{x}_1^j)^2}.$$

Then, for a subsequence, the sequences $\widetilde{u}_j$ and $\widetilde{z}^j$ converge to a global solution $u_0 \in P_\infty^+(M)$ and $z^* = (x^*, t^*)$ with

$$x_1^* \geqslant \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} > 0,$$

respectively, and the points $z^* = (x^*, t^*)$ and $(0,0)$ lie on the free boundary $\Gamma(u_0)$.
We see that two situations may arise: $t^* \leqslant 0$ or $t^* > 0$. In the first case, applying Theorem I to the function u_0 and the point $z^0 = (0,0)$, we get for $(x,t) \in \mathbb{R}_+^{n+1} \cap \{t \leqslant 0\}$ the exact representation (1.3), which is not possible since $z^* \in \Gamma(u_0)$. In the case $t^* > 0$, we apply Theorem I to the function u_0 and the point $z^0 = z^*$. This leads to the exact representation (1.4), which is also not possible since $(0,0) \in \Gamma(u_0)$. Thus, in both cases we are led to a contradiction. □

Corollary 4.2 *There is a universal constant $r_0 = r_0(n,M)$ and a modulus of continuity σ ($\sigma(0^+) = 0$) such that if $u \in P_1^+(M)$ and $(0,0) \in \Gamma(u)$, then*

$$\Gamma(u) \cap Q_{r_0}(0, r_0^2/2) \subset \left\{ (x,t) : x_1 \leqslant \sigma(\sqrt{|x|^2+|t|}) \cdot \sqrt{|x|^2+|t|} \right\}.$$

Proof. It suffices to consider the modulus of continuity $\sigma(\rho)$ given by the inverse of the function $\varepsilon \to \rho_\varepsilon$ provided by Lemma 4.1 and to put $r_0 = \rho_{\varepsilon=1}$. □

Lemma 4.3 *Let $u \in P_2^+(M)$. There exists $\delta_0 = \delta_0(n, M) > 0$ such that if $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_1$ and $x_1^0 \leqslant \delta_0$, then for the balanced energy at the point z^0 (see (2.8)) we have*

$$\omega(x^0, t^0, u) = \frac{15}{4} = A. \tag{4.2}$$

Proof. From (2.8) and (2.6) it follows that $\omega(x^0, t^0, u) = W(1, 0, 0, u_0)$, where u_0 is an arbitrary blow-up limit of the solution u at the point z^0. As we have mentioned in item 2.3, there are only two possibilities: $u_0 \in P_\infty^+(M)$ or $u_0 \in P_\infty(M)$. In the first case $z^0 \in \Pi$ and Lemma 3.2 immediately provides (4.2). Whereas Lemmas 6.3 and 6.2 [CPS] say for the second case that either $W(1, 0, 0, u_0) = A$ and we are done, or $W(1, 0, 0, u_0) = 2A$.
Now we claim that $\omega(x^0, t^0, u) \neq 2A$ if δ_0 is small enough. Indeed, suppose, towards a contradiction, that there exists a sequence $z^k = (x^k, t^k) \in \Gamma(u) \cap Q_1$ such that $z^k \to z^* = (x^*, t^*) \in \Gamma(u) \cap Q_1 \cap \Pi$ and $\omega(x^k, t^k, u) = 2A$.
Considering the functions

$$v_k(x, t) = \frac{u(r_k x + x^k - r_k e_1, r_k^2 t + t^k)}{r_k^2}$$

for $(x, t) \in Q^+_{1/r_k}$ with $r_k := x_1^k \to 0^+$ as $k \to \infty$, we observe that the sequence $\{v_k\}$ converges, for a subsequence, to a global solution $v \in P_\infty^+(M)$.
It is evident that $(e_1, 0) = (1, 0, \dots, 0) \in \Gamma(v)$. By Lemma 3.2 we have

$$W(\rho, e_1, 0, v) = A. \tag{4.3}$$

On the other hand, for arbitrary $\rho > 0$ elementary computation combined with estimate (2.7) and scaling property (2.6) gives

$$\begin{aligned} 2A = \omega(x^k, t^k, u) &\leqslant W_1(\rho r_k, x^k, t^k, u) + C_0 \rho r_k \exp\left(-1/20\rho^2 r_k^2\right) \\ &= W_{1/r_k}(\rho, e_1, 0, v_k) + C_0 \rho r_k \exp\left(-1/20\rho^2 r_k^2\right) \\ &= W(\rho, e_1, 0, v) + \vartheta_k(\rho) + C_0 \rho r_k \exp\left(-1/20\rho^2 r_k^2\right), \end{aligned} \tag{4.4}$$

where

$$\vartheta_k(\rho) := \frac{1}{\rho^4} \int\limits_{-4\rho^2}^{-\rho^2} \int\limits_{B_{1/r_k}(e_1)} \left(|Dv_k|^2 - |Dv|^2 + 2(v_k - v) + \frac{v_k^2 - v^2}{t} \right) G(x - e_1, -t) dxdt$$
$$- \frac{1}{\rho^4} \int\limits_{-4\rho^2}^{-\rho^2} \int\limits_{\mathbb{R}^n \setminus B_{1/r_k}(e_1)} \left(|Dv|^2 + 2v + \frac{v^2}{t} \right) G(x - e_1, -t) dxdt.$$

It is evident that for fixed ρ the sequence $\{\vartheta_k(\rho)\}$ converges to zero as $k \to \infty$.
Thus, (4.4) contradicts (4.3) for large k. The proof is complete. $\square$

5 Regularity properties of solutions

Lemma 5.1 *Let $u \in P_2^+(M)$. Then the time derivative $\partial_t u$ is continuous on the set $Q_{1/2} \cap \{0 \leqslant x_1 < \delta_0\}$, where $\delta_0 = \delta_0(n, M)$ is the same constant as in Lemma 4.3.*

Proof. Since we have Lemma 4.3 at our disposal, the proof of this statement follows by verbatim repetition the corresponding part of the proof of Lemma 7.7 [CPS]. □

Lemma 5.2 *For any $\varepsilon > 0$ there exists $\delta_1 = \delta_1(n, \varepsilon, M) > 0$ such that if $u \in P_2^+(M)$ and $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_{1/8}^+ \cap \{x_1 < \delta_1\}$ then for $r := x_1^0$ and $\psi(x) = \frac{((x_1 - x_1^0)_+)^2}{2}$ we have*

$$\sup_{Q_{8r}^+(z^0)} |u(x,t) - \psi(x)| \leqslant \varepsilon r^2,$$

$$\sup_{Q_{8r}^+(z^0)} |Du(x,t) - D\psi(x)| \leqslant \varepsilon r.$$

Proof. The statement is proved along the same lines as Lemma 2.3 [ASU3] or Lemma 5.2 [SU]. □

Remark. It should be mentioned that the additional assumption $u \geqslant 0$ made possible to prove in Lemma 2.3 [ASU3] more general statement in comparison with Lemma 5.2 in this paper.

Lemma 5.3 *Let $u \in P_2^+(M)$, let $\varepsilon > 0$ be an arbitrary sufficiently small number, and let $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_{1/8}^+ \cap \{x_1 < \delta_1\}$, where $\delta_1 = \delta_1(n, \varepsilon, M)$ is the constant occurring in Lemma 5.2.*
Then, there exists a positive number $N = N(n)$ such that for $r := x_1^0$ and $\Sigma := Q_{7r}(z^0) \cap \{0 \leqslant x_1 < r(1 - N\sqrt{\varepsilon})\}$ we have

$$u(x,t) \leqslant 0 \quad in \quad \Sigma. \tag{5.1}$$

Moreover, if there exists a point $\widehat{z} = (\widehat{x}, \widehat{t}) \in \Sigma$ such that $u(\widehat{z}) = 0$, then

$$u \equiv 0 \quad in \quad \Sigma \cap \{t \leqslant \widehat{t}\}. \tag{5.2}$$

Proof. Suppose that there is a point $z^{(1)} = (x^{(1)}, t^{(1)}) \in Q_{7r}^+(z^0) \cap \{x_1 < r\}$ such that $u(z^{(1)}) > 0$; otherwise we already have (5.1) with $N(n) = 0$.

Then for $\rho := r - x_1^{(1)}$ we deduce from Fact 1 and Lemma 5.2 the inequalities

$$\frac{\rho^2}{2n+1} \leqslant \sup_{Q_\rho^+(z^{(1)})} u - u(z^{(1)}) \leqslant \sup_{Q_{8r}^+(z^0)\cap\{x_1<r\}} |u| \leqslant \varepsilon r^2,$$

which are impossible if $\rho > r\sqrt{(2n+1)\varepsilon}$. Therefore, for all $z = (x,t) \in Q_{7r}^+(z^0)$ with $x_1 < r(1 - \sqrt{(2n+1)\varepsilon})$ we have $u(z) \leqslant 0$. Choosing $N(n) = \sqrt{2n+1}$ we arrive at (5.1).
Finally, with (5.1) at the hand, application the maximum principle implies (5.2), since $u(\widehat{z}) = 0$. $\square$

Lemma 5.4 *Let all the assumptions of Lemma 5.3 hold. Then* $t^* := \sup\{t : z = (x,t) \in \Sigma$ *and* $u(z) = 0\}$ *satisfies the inequality*

$$t^* \geqslant t^0 - \varepsilon(2n+1)r^2, \tag{5.3}$$

where parameters ε,r, *and the set* Σ *are the same as in Lemma 5.3.*

Proof. If $t^* = t^0$ then (5.3) is true. Otherwise, there exists $\rho > 0$ such that for $z^{(1)} := (x_1^0/2, (x^0)', t^0)$ we have $u < 0$ in $Q_\rho(z^{(1)})$ and $Q_\rho(z^{(1)}) \subset \Sigma$. To prove (5.3) it suffices to show that

$$\rho \leqslant r\sqrt{\varepsilon(2n+1)}. \tag{5.4}$$

It is easy to see that Fact 1 (see item (iii)) and Lemma 5.2 imply the inequalities

$$\frac{\rho^2}{2n+1} \leqslant |u(z^{(1)}| \leqslant \varepsilon r^2,$$

which are only possible if (5.4) holds. $\square$

Lemma 5.5 *Let* $u \in P_2^+(M)$, *and let* N_0 *and* N_τ *(with* $\tau = 2, \dots, n$*) be some constants satisfying*

$$|N_0| \leqslant \frac{1}{32(2n+1)M}, \qquad \sum_{\tau=2}^{n} |N_\tau| \leqslant 1. \tag{5.5}$$

There exists $\delta = \delta(n, M) > 0$ *such that if* $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_{1/8}^+ \cap \{x_1 < \delta\}$ *then for* $r = x_1^0$ and $v := rD_1u + r\sum_{\tau=2}^{n} N_\tau D_\tau u + r^2 N_0 \partial_t u - u$ *we have*

$$\text{(i)} \qquad v \geqslant 0 \quad in \quad Q_{r/2}(z^0); \tag{5.6}$$

$$\text{(ii)} \qquad v \geqslant 0 \quad in \quad Q_{2r}^+(z^0) \cap \{t \leqslant t^*\}, \tag{5.7}$$

where t^* is the same as in Lemma 5.4.

Proof. We take $\varepsilon := \frac{1}{32(2n+1)}$ and set $\delta = \min\{\delta_0, \delta_1\}$, where $\delta_0 = \delta_0(n, M)$ and $\delta_1 = \delta_1(n, \varepsilon, M)$ are the constants defined in Lemmas 4.3 and 5.2, respectively.
In view of Lemma 5.2 (with fixed $\varepsilon := \frac{1}{32(2n+1)}$) and Lemma 5.1 we have in $Q^+_{8r}(z^0)$ the estimate

$$v \geqslant -r^2/8(2n+1). \tag{5.8}$$

Since in the case (i) the zero condition on Π is not used, we can proceed analogously to the proof of Lemma 7.6 [CPS] and deduce (5.6) from (5.8) and the maximum principle.
Consider now case (ii). Suppose that (5.7) fails. Then there is a function $u \in P^+_2(M)$ and a point $z^0 \in \Gamma(u) \cap Q^+_{1/8}$ such that the assumptions of lemma are satisfied, but there is a point $z^{(1)} = (x^{(1)}, t^{(1)}) \in Q^+_{2r}(z^0) \cap \{t \leqslant t^*\}$ with

$$v(z^{(1)}) := rD_1u(z^{(1)}) + r\sum_{\tau=2}^{n} N_\tau D_\tau u(z^{(1)}) + r^2 N_0 \partial_t u(z^{(1)}) - u(z^{(1)}) < 0. \tag{5.9}$$

Let

$$w(x,t) = v(x,t) + \frac{|x - x^{(1)}|^2 + t^{(1)} - t}{2n+1}.$$

Then w is caloric in $Q^+_r(z^{(1)}) \cap \Omega(u)$, and, by (5.9), $w(x^{(1)}, t^{(1)}) < 0$. It is evident that $w(x,t) \geqslant 0$ on $\Gamma(u) \cap Q^+_r(z^{(1)})$. Observe also that the equality (5.2) implies

$$w(x,t)\big|_{\Pi \cap Q_r(z^{(1)})} = \frac{1}{2n+1}(|x - x^{(1)}|^2 + t^{(1)} - t)\big|_{\Pi \cap Q_r(z^{(1)})} \geqslant 0.$$

Finally, from (5.8) it follows that

$$w(x,t)\big|_{\Omega(u) \cap \partial' Q^+_r(z^{(1)})} \geqslant -\frac{r^2}{8(2n+1)} + \frac{r^2}{(2n+1)} \geqslant 0.$$

Hence by the maximum principle the infimum of w in $\Omega(u) \cap Q^+_r(z^{(1)})$ is nonnegative which is a contradiction with $w(z^{(1)}) < 0$. This give us the desired estimate (5.7) and completes the proof of the lemma. □

Corollary 5.6 *Let all the assumptions of Lemma 5.5 hold. Then*

$$v \geqslant 0 \quad in \quad Q^+_{2r}(z^0), \tag{5.10}$$

and

$$u \geqslant 0 \quad in \quad Q^+_{2r}(z^0). \tag{5.11}$$

Proof. To verify (5.10) it suffices to show that $t^* = t^0$, where t^* is the parameter occurring in Lemma 5.4. Suppose, towards a contradiction, that $t^* < t^0$, and consider the point $z^{(2)} := (r/2, (z^0)', t^*) \in \Sigma$, where Σ is the set defined in Lemma 5.3.
In view of (5.3), it is evident that for sufficiently small ε the whole segment $]z^{(2)}, z^0] \subset Q_{r/2}(z^0)$ and $N_0 := 2(t^0 - t^*)r^{-2}$ satisfies (5.5). With such a number N_0 and $N_\tau \equiv 0$, $(\tau = 2, \dots, n)$ consider the direction

$$e = (1/\sqrt{1 + r^2 N_0^2}, 0, \dots, 0, rN_0/\sqrt{1 + r^2 N_0^2}) = \frac{z^0 - z^{(2)}}{|z^0 - z^{(2)}|}.$$

Then the corresponding inequality (5.6) can be written in the following form

$$D_e \left(u \exp\left\{ -(z, e)/r\sqrt{1 + r^2 N_0^2} \right\} \right) \geqslant 0 \quad \text{in} \quad Q_{r/2}(z^0).$$

Observe also that according to our choice of z^0 and the definition of t^* we have $u(z^{(2)}) = u(z^0) = 0$.
Integrating the last inequality along the segment $[z^{(2)}, z^0]$ we get $u \equiv 0$ on this segment. Since $[z^{(2)}, z^0]$ lays above the set $\{t = t^*\}$ and partially in Σ, we come to the contradiction with the definition of t^*. Thus, we have proved (5.10). It is easy to see that (5.11) follows from (5.10) combining with the condition $u = 0$ on Π. $\square$

6 Regularity properties of the free boundary

Lemma 6.1 *Let $u \in P_2^+(M)$, and let $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q^+_{1/8} \cap \{x_1 < \delta\}$, where $\delta = \delta(n, M)$ is the constant occurring in Lemma 5.5.*
There exist absolute positive constants $C_1 = C_1(n)$ and $C_2 = C_2(n, M)$ such that for a cone

$$\mathcal{K} := \left\{ (x, t) \in \mathbb{R}^{n+1}_+ : x_1 > \sqrt{C_1 |x'|^2 + C_2 (x_1^0)^2 t^2} \right\}$$

we have

$$Q^+_{2x_1^0}(z^0) \cap \{z^0 - \mathcal{K}\} \subset \Lambda(u). \tag{6.1}$$

Proof. Let us set $r := x_1^0$, and take $C_1 = (n-1)^2$ and $C_2 = \big(32(2n+1)M\big)^2$. From Lemma 5.5 and Corollary 5.6 it follows that for arbitrary unit vector $e \in \mathcal{K}$ the inequality $D_e u \geqslant 0$ holds true in $Q^+_{2r}(z^0)$. Then, evidently,

$D_{-e}u \leqslant 0$ in $Q^+_{2r}(z^0)$. The latter together with the assumptions $z^0 \in \Gamma(u)$ and (5.11) gives the desired inclusion (6.1). $\square$

Proof of Theorem II. We set $r = x_1^0$ and define f as follows:

$$f(x', t) = \sup\{x_1 \in [0, 2r] : u(x_1, x', t) = 0\}.$$

We claim that f is a Lipschitz function. Indeed, Lemma 6.1 guarantees the existence of space-time cones lying in $\Lambda(u)$ at every point on $\Gamma(u) \cap Q^+_{1/8} \cap \{x_1 < \delta\}$, which implies the corresponding space-time Lipschitz regularity of $\Gamma(u) \cap Q_{2r}(z^0)$. The theorem is proved. $\square$

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