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Abstract

In this note, we prove that an operator between reproducing kernel Hilbert spaces is a multiplication operator if and only if it leaves invariant zero sets. To be more precise, it is shown that an operator T between reproducing kernel Hilbert spaces is a multiplication operator if and only if $(Tf)(z) = 0$ holds for all f and z satisfying $f(z) = 0$. As possible applications, we deduce a general reflexivity result for multiplier algebras, and furthermore prove fully vector-valued generalizations of mulitplier lifting results of Beatrous and Burbea.

A Characterization of Multiplication Operators on Reproducing Kernel Hilbert Spaces

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1 INTRODUCTION

Following common terminology, a Hilbert space $\mathcal{H}$ consisting of functions defined on some set X with values in a Hilbert space $\mathcal{E}$ is called *reproducing kernel Hilbert space* if all point evaluations

$$\delta(z) : \mathcal{H} \to \mathcal{E} \ , \ f \mapsto f(z) \quad (z \in X)$$

are continuous. Equivalently, there exists a function $K : X \times X \to B(\mathcal{E})$ such that all functions of the form $K(\cdot, z)x : X \to \mathcal{E}$ belong to $\mathcal{H}$ and, moreover, satisfy the equality

$$\langle f, K(\cdot, z)x \rangle = \langle f(z), x \rangle \quad (f \in \mathcal{H} \ , x \in \mathcal{E}, z \in X).$$

The function K is easily seen to be unique with these properties and is usually called the *reproducing kernel* of $\mathcal{H}$.
An operator-valued function $\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)$ is called a *multiplier* between two reproducing kernel Hilbert spaces $\mathcal{H}_1 \subset \mathcal{E}_1^X$ and $\mathcal{H}_2 \subset \mathcal{E}_2^X$ if the pointwise product $\phi \cdot f$ belongs to $\mathcal{H}_2$ for every $f \in \mathcal{H}_1$. The collection of all such multipliers $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ obviously is a linear space. For every multiplier $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ we can define the associated multiplication operator $M_\phi : \mathcal{H}_1 \to \mathcal{H}_2 \ , \ f \mapsto \phi \cdot f$, which is easily seen to be continuous by the closed graph theorem.
It is more than obvious that every multiplication operator M_ϕ has the property that $(M_\phi f)(z) = 0$ holds whenever $f(z) = 0$. It is the main result (Theorem 2.1) of this note that the converse of this statement is surprisingly true - at least if the space $\mathcal{H}_1$ is non-degenerate in an appropriate sense. This result can be regarded as a purely algebraic characterization of multiplication operators. As an application of this result, we prove (Corollary 2.2) that the space $M_{\mathcal{S}}(\mathcal{H}_1, \mathcal{H}_2)$, consisting of all multiplication operators M_ϕ given by $\mathcal{S}$-valued multipliers ϕ, is a reflexive subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$ whenever $\mathcal{S}$ is a reflexive subspace of $B(\mathcal{E}_1, \mathcal{E}_2)$. In particular, the space $M(\mathcal{H}_1, \mathcal{H}_2)$ of

all multiplication operators is always reflexive and therefore weakly closed in $B(\mathcal{H}_1, \mathcal{H}_2)$.

As a second consequence of our main result, we obtain an alternative proof of the following interpolation result proved by Beatrous and Burbea (cf. [3], Theorem 3.5): *Whenever $\mathcal{H}$ is a holomorphic reproducing kernel Hilbert space without common zeroes on a domain $D \subset \mathbb{C}^d$, and $E \subset D$ is a set of uniqueness for $\mathcal{O}(D)$, then every multiplier ψ on the restricted space $\mathcal{H}_{|E}$ can be lifted uniquely to a multiplier ϕ on the whole space without increasing the multiplier norm.* The advantage of the approach presented in this paper is that it works in the vector-valued setting as well.

2 ABSTRACT REPRESENTATION OF MULTIPLICATION OPERATORS

In the sequel, a reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{E}^X$ is called *non-degenerate* if each point evaluation $\delta(z) : \mathcal{H} \to \mathcal{E}$ is either onto or zero. One can easily show that this is fulfilled precisely if $K(z,z) = \delta(z)\delta(z)^*$ is either invertible or zero for all $z \in X$. It is clear that scalar spaces $\mathcal{H}$ (here, scalar means that $\mathcal{E} = \mathbb{C}$) or, more generally, spaces whose reproducing kernel is of the form $K \cdot 1_{\mathcal{E}}$ with a scalar kernel K, are always non-degenerate.

Theorem 2.1. *Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces, and that $\mathcal{H}_1 \subset \mathcal{E}_1^X$ and $\mathcal{H}_2 \subset \mathcal{E}_2^X$ are reproducing kernel Hilbert spaces, $\mathcal{H}_1$ non-degenerate. Then, for $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, the following assertions are equivalent:*

(i) $(Tf)(z) = 0$ holds for all $f \in \mathcal{H}_1$ and $z \in X$ with $f(z) = 0$.

(ii) There exists $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ such that $T = M_\phi$.

Proof. Throughout the proof, the point evaluations on $\mathcal{H}_1$ and $\mathcal{H}_2$ are denoted by $\delta_1(z)$ and $\delta_2(z)$, respectively. The fact that $\mathcal{H}_1$ is non-degenerate implies in particular that the mappings $\delta_1(z)^*$ have closed range for all $z \in X$. We infer that (i) is equivalent to $T^* \operatorname{ran} \delta_2(z)^* \subset \operatorname{ran} \delta_1(z)^*$ for all $z \in X$. This allows us to fix, for every $y \in \mathcal{E}_2$ and $z \in X$, an element $x_{z,y} \in \mathcal{E}_1$ such that $T^*\delta_2(z)^* y = \delta_1(z)^* x_{z,y}$.

Now let us define $X_0 = \{z \in X \ ; \ \delta_1(z) \text{ is onto}\}$. Equivalently, $X \backslash X_0$ is the set of common zeroes of $\mathcal{H}_1$. For all $z \in X_0$, we can choose operators $i(z) \in B(\mathcal{E}_1, \mathcal{H}_1)$ such that $\delta_1(z) i(z) = 1_{\mathcal{E}_1}$. We show that the function

$$\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2) \ , \ \phi(z) = \begin{cases} \delta_2(z) T i(z) & ; \quad z \in X_0 \\ 0 & ; \quad z \notin X_0 \end{cases}$$

has all desired properties. To this end, fix $f \in \mathcal{H}_1$. Then the equality

$$\begin{aligned} \langle \phi(z) f(z), y \rangle &= \langle i(z) f(z), T^* \delta_2(z)^* y \rangle = \langle i(z) f(z), \delta_1(z)^* x_{z,y} \rangle \\ &= \langle f(z), x_{z,y} \rangle = \langle f, \delta_1(z)^* x_{z,y} \rangle = \langle f, T^* \delta_2(z)^* y \rangle \\ &= \langle (Tf)(z), y \rangle \end{aligned}$$

holds for all $y \in \mathcal{E}_2$ and $z \in X_0$ (and trivially for all remaining points $z \in X$). Hence $Tf = \phi \cdot f$, which completes the proof. $\square$

As one of many possible applications, we present the following reflexivity result for multiplier spaces which may be known in special cases. Recall that a linear space $\mathcal{S} \subset B(H_1, H_2)$ of operators is called *reflexive* (following the notions of [7]) if it coincides with its reflexive closure, that is,

$$\mathcal{S} = \{T \in B(H_1, H_2)\ ;\ Tx \in \overline{\mathcal{S}x} \text{ for all } x \in H_1\}.$$

Clearly, this definition generalizes the usual definition of reflexive algebras (cf. [8]).

Corollary 2.2. *Suppose that, in the situation of Theorem 2.1, $\mathcal{S}$ is a reflexive subspace of $B(\mathcal{E}_1, \mathcal{E}_2)$. Then the space*

$$M_{\mathcal{S}}(\mathcal{H}_1, \mathcal{H}_2) = \{M_\phi\ ; \phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \text{ and } \phi(z) \in \mathcal{S} \text{ for all } z \in X\}$$

is a reflexive subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$ and, in particular, weakly closed.

Proof. Let us consider $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $Tf \in \overline{M_{\mathcal{S}}(\mathcal{H}_1, \mathcal{H}_2)f}$ holds for all $f \in \mathcal{H}_1$. Fix $f \in \mathcal{H}_1$ and $z \in X$ with $f(z) = 0$. Then there exists a sequence $(\phi_n)_n$ in $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\lim M_{\phi_n} f = Tf$ holds. The continuity of the point evaluations clearly implies that $(Tf)(z) = \lim_n \phi_n(z) f(z) = 0$, which yields by Theorem 2.1 that $T = M_\psi$ for an appropriate $\psi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$. It remains to show that ψ takes values in $\mathcal{S}$. Without loss of generality, we can assume that $\psi(z) = 0$ holds for all $z \in X \backslash X_0$, where as before $X_0 = \{z \in X\ ;\ \delta_1(z) \text{ is onto}\}$. In particular, $\psi(z) \in \mathcal{S}$ for these z. On the other hand, for $z \in X_0$ and $x \in \mathcal{E}_1$, we can choose a function $f \in \mathcal{H}_1$ such that $f(z) = x$. As above, there exists a sequence $(\phi_n)_n$ of $\mathcal{S}$-valued multipliers such that $Tf = \lim_n M_{\phi_n} f$. From this, we easily obtain $\psi(z)x = (M_\psi f)(z) = (Tf)(z) = \lim_n (M_{\phi_n} f)(z) = \lim_n \phi_n(z)x \in \overline{\mathcal{S}x}$. The reflexivity of $\mathcal{S}$ now shows that $\psi(z) \in \mathcal{S}$ for all $z \in X$, which means that $\psi \in \mathcal{M}_{\mathcal{S}}(\mathcal{H}_1, \mathcal{H}_2)$. $\square$

3 APPLICATION TO HOLOMORPHIC SPACES

In this section, we aim to show that the statement of Theorem 2.1 can be strengthened remarkably if the underlying reproducing kernel Hilbert spaces consist of holomorphic functions. So throughout this section, let D denote an open subset of $\mathbb{C}^d$ and recall that, for every reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{O}(D, \mathcal{E})$, the function $\delta : D \to B(\mathcal{H}, \mathcal{E})\ , \delta(z)f = f(z)$ is weakly holomorphic, and hence holomorphic. The space $\mathcal{H}$ is called *analytically non-degenerate* if there is an analytic function $i : D \to B(\mathcal{E}, \mathcal{H})$ such that $\delta(z)i(z) = 1_{\mathcal{E}}$ for all $z \in D$. The following example shows that most reproducing kernel Hilbert spaces arising in applications are analytically non-degenerate.

Example 3.1. *Suppose that $\mathcal{H} \subset \mathcal{O}(D, \mathcal{E})$ is a reproducing kernel Hilbert space satisfying at least one of the following conditions:*

(1) $\mathcal{H}$ contains the constant functions,

(2) there exists $z_0 \in D$ such that $K(z, z_0)$ is invertible for all $z \in D$,

(3) $\delta(z)$ is onto for all z and D is a domain of holomorphy.

Then $\mathcal{H}$ is analytically non-degenerate. In fact, in the first case, one checks that the analytic function $i : D \to B(\mathcal{E}, \mathcal{H})$, $i(z)x = \underline{x}$ (here, $\underline{x}$ denotes the constant function with value x) is a well defined (by the closed graph theorem) pointwise right inverse for δ. In the second case, one can choose $i(z) = \delta^*_{z_0} K(z, z_0)^{-1}$, and in the third case, the claim follows from a result of Allan [1] and Leiterer [6] (cf. Section 4.6 in [5])

The improved version of Theorem 2.1 now reads as follows.

Theorem 3.2. *Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces, and that $\mathcal{H}_1 \subset \mathcal{O}(D, \mathcal{E}_1)$ and $\mathcal{H}_2 \subset \mathcal{O}(D, \mathcal{E}_2)$ are reproducing kernel Hilbert spaces such that $\mathcal{H}_1$ is analytically non-degenerate. Furthermore, let $E \subset D$ be a set of uniqueness for $\mathcal{O}(D)$. Then, for every $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, the following assertions are equivalent:*

(i) $(Tf)(z) = 0$ holds for all $f \in \mathcal{H}_1$ and $z \in E$ with $f(z) = 0$.

(ii) There exists $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ such that $T = M_\phi$.

Here as usual, E is called a *set of uniqueness* for $\mathcal{O}(D)$ if the restriction mapping $f \mapsto f_{|E}$ is one-to-one on $\mathcal{O}(D)$.

Proof. We choose a holomorphic pointwise right inverse $i : D \to B(\mathcal{E}_1, \mathcal{H}_1)$ for δ_1. As in the proof of Theorem 2.1, we define $\phi : D \to B(\mathcal{E}_1, \mathcal{E}_2)$, $\phi(z) = \delta_2(z)Ti(z)$, which is obviously a holomorphic function. Following the original proof, for $f \in \mathcal{H}_1$, we obtain that $\phi(z)f(z) = (Tf)(z)$ for all $z \in E$. Since E is a set of uniqueness for $\mathcal{O}(D, \mathcal{E}_2)$ as well, we deduce that $\phi \cdot f = Tf$ holds on the whole of D. This clearly completes our proof. □

As an application, we obtain the following lifting theorem for multipliers, which is a vector-valued generalization of results of Szafraniec [9] and of Beatrous and Bourbea [3] (Theorem 3.5). Before we state the result, we briefly recapitulate some well-known facts about restrictions of reproducing kernel Hilbert spaces (see for example [2] for an overview of this topic): Let $\mathcal{H} \subset \mathcal{E}^X$ be a reproducing kernel Hilbert space with kernel K, and let Y be a non-empty subset of X. Then the linear space $\mathcal{H}_{|Y} = \{f_{|Y} \ ; \ f \in \mathcal{H}\}$, endowed with the quotient norm $\|u\| = \inf\{\|f\| \ ; \ f_{|Y} = u\}$, is the reproducing kernel Hilbert space with reproducing kernel $K_{|Y \times Y}$. The restriction mapping $\rho : \mathcal{H} \to \mathcal{H}_{|Y}$ satisfies $(\rho^* u)(z) = u(z)$ for all $z \in Y$, and consequently is a coisometry with $\ker \rho = \{f \in \mathcal{H} \ ; \ f_{|Y} = 0\}$.

Corollary 3.3. *Suppose that, in the situation of Theorem 3.2, ψ is a multiplier in $\mathcal{M}(\mathcal{H}_{1|E}, \mathcal{H}_{2|E})$. Then there exists a multiplier $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\phi_{|E} = \psi$ and $\|M_\phi\| = \|M_\psi\|$. In other words, the restriction mapping*

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{M}(\mathcal{H}_{1|E}, \mathcal{H}_{2|E}) \ , \ \phi \mapsto \phi_{|E}$$

is an isometric isomorphism.

Proof. We start by observing that the restriction operators $\rho_i : \mathcal{H}_i \to \mathcal{H}_{i|E}$ $(i = 1, 2)$ are unitary, since E is a set of uniqueness for $\mathcal{O}(D)$. Now let us define an operator $T = \rho_2^* M_\psi \rho_1 \in B(\mathcal{H}_1, \mathcal{H}_2)$, and consider $f \in \mathcal{H}_1$ and $z \in E$ with $f(z) = 0$. Then

$$(Tf)(z) = (\rho_2^* M_\psi \rho_1)(z) = \psi(z)(\rho_1 f)(z) = \psi(z) f(z) = 0,$$

which shows that $T = M_\phi$ for some $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ by Theorem 3.2. Then clearly $\|M_\phi\| = \|T\| = \|M_\psi\|$. To see that ϕ actually extends ψ, fix $z \in E$ and $x \in \mathcal{E}_1$. Then, using the fact that $\mathcal{H}_1$ is analytically non-degenerate, we choose $f \in \mathcal{H}_1$ with $f(z) = x$ (for example $f = i(z)x$), and observe that

$$\phi(z)x = (M_\phi f)(z) = (\rho_2^* M_\psi \rho_1 f)(z) = \psi(z)(\rho_1 f)(z) = \psi(z) f(z) = \psi(z)x.$$

To complete the proof, we have to show that the restriction mapping $\phi \mapsto \phi_{|E}$ is a well-defined contraction between $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{M}(\mathcal{H}_{1|E}, \mathcal{H}_{2|E})$. To this end recall that a function $\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)$ is a multiplier between arbitrary reproducing kernel Hilbert spaces $\mathcal{H}_1 \subset \mathcal{E}_1^X$ and $\mathcal{H}_2 \subset \mathcal{E}_2^X$ with $\|M_\phi\| \leq 1$ precisely if the function

$$X \times X \to B(\mathcal{E}_2) \ , \ (z, w) \mapsto K_2(z, w) - \phi(z) K_1(z, w) \phi(w)^*$$

is positive definite (where of course, K_1 and K_2 are the reproducing kernels of $\mathcal{H}_1, \mathcal{H}_2$). A proof of this fact can be found in [4]. Since restrictions of positive definite functions remain positive definite, the claim is proved. □

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