

Universität des Saarlandes



Fachrichtung 6.1 – Mathematik

Preprint

**Partial Regularity for Variational Integrals  
with  $(s, \mu, q)$ -Growth**

Michael Bildhauer and Martin Fuchs

Preprint No. 21

Saarbrücken 2000

Universität des Saarlandes



Fachrichtung 6.1 – Mathematik

**Partial Regularity for Variational Integrals  
with  $(s, \mu, q)$ -Growth**

*Michael Bildhauer*

Saarland University  
Department of Mathematics  
Postfach 15 11 50  
D-66041 Saarbrücken  
Germany  
E-Mail: [bibi@math.uni-sb.de](mailto:bibi@math.uni-sb.de)

*Martin Fuchs*

Saarland University  
Department of Mathematics  
Postfach 15 11 50  
D-66041 Saarbrücken  
Germany  
E-Mail: [fuchs@math.uni-sb.de](mailto:fuchs@math.uni-sb.de)

submitted: October 12

Preprint No. 21

Saarbrücken 2000

Edited by  
FR 6.1 – Mathematik  
Im Stadtwald  
D-66041 Saarbrücken  
Germany

Fax: + 49 681 302 4443  
e-mail: [preprint@math.uni-sb.de](mailto:preprint@math.uni-sb.de)  
WWW: <http://www.math.uni-sb.de/>

## Abstract

We introduce integrands  $f: \mathbb{R}^{nN} \rightarrow \mathbb{R}$  of  $(s, \mu, q)$ -type, which are, roughly speaking, of lower (upper) growth rate  $s \geq 1$  ( $q > 1$ ) satisfying in addition  $D^2f(Z) \geq \lambda(1 + |Z|^2)^{-\mu/2}$  for some  $\mu \in \mathbb{R}$ . Then, if  $q < 2 - \mu + s\frac{2}{n}$ , we prove partial  $C^1$ -regularity of local minimizers  $u \in W_{1,loc}^1(\Omega, \mathbb{R}^N)$  by the way including integrands  $f$  being controlled by some  $N$ -function and also integrands of anisotropic power growth. Moreover, we extend the known results up to a certain limit and present examples which are not covered by the standard theory.

*AMS Subject Classification:* 49N60, 49N99, 35J45

*Key words:* regularity, minimizers, convexity, non-standard growth

## 1 Introduction

In this paper we study the problem of partial  $C^1$ -regularity for local minimizers  $u \in W_{1,loc}^1(\Omega, \mathbb{R}^N)$  of strictly convex variational integrals

$$J(u) = \int_{\Omega} f(Du) dx \quad (1.1)$$

under rather general and also non-standard growth conditions. Here  $\Omega$  is some domain in Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ , and we assume that the integrand  $f: \mathbb{R}^{nN} \rightarrow [0, \infty)$  is a function of class  $C^2$  whose second derivative  $D^2f(Z)$  has to satisfy certain coercivity conditions to be specified below. Thus, we do not touch the quasiconvex case (compare e. g. [EV], [FH], [EG1], [AF1], [AF2], [CFM]) and before presenting our results, we briefly summarize the conditions under which partial regularity is available in the framework of strong convexity. Roughly speaking, we can consider three different cases:

### A. power growth

For some number  $m > 1$  and with constants  $\lambda, \Lambda > 0$  the integrand  $f$  satisfies

$$\lambda(|Z|^m - 1) \leq f(Z) \leq \Lambda(|Z|^m + 1) \quad \text{for all } Z \in \mathbb{R}^{nN}, \quad (1.2)$$

in particular,  $f$  has the same growth rate from above and from below. Then, if also  $D^2f(Z) > 0$  holds for any matrix  $Z$ , Anzellotti and Giaquinta proved in [AG] that for any local minimizer  $u \in W_{1,loc}^1(\Omega, \mathbb{R}^N)$  of (1.1) there is an open set  $\Omega_0$  such that  $|\Omega \sim \Omega_0| = 0$ , i. e. the singular set has measure zero, and  $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ . We emphasize that the paper [AG] also includes the case of linear growth ( $m = 1$ ) with corresponding local minimizers in the

space  $BV_{loc}(\Omega, \mathbb{R}^N)$ . Moreover, the reader will find there further comments on earlier results obtained under condition (1.2).

*B. growth conditions involving  $N$ -functions*

The model  $f(Z) = |Z| \ln(1 + |Z|)$  serves as a typical example for integrands  $f$  not satisfying (1.2) for any power  $m \geq 1$ . Generally speaking, the quantity  $|Z|^m$  occurring in (1.2) is now replaced by  $A(|Z|)$  for some arbitrary  $N$ -function  $A: [0, \infty) \rightarrow [0, \infty)$  satisfying a  $\Delta_2$ -condition. If we add an appropriate ellipticity and growth condition on  $D^2 f(Z)$ , then in [FO] partial regularity was shown to hold up to a certain dimension  $n$ . The particular class of integrands  $f$  with logarithmic structure (i. e.  $f$  is  $C^2$ -close to  $|Z| \ln(1 + |Z|)$ ) was studied first in [FS] with the result that minimizers are partially  $C^1$  provided that  $n \leq 4$ . Later on Esposito and Mingione [EM2] removed the restriction on  $n$ , moreover, Mingione and Siepe [MS] proved for  $f(Z) = |Z| \ln(1 + |Z|)$  in fact that the singular set is empty which of course can not be expected in the general case. We would like to remark that some extensions of the results obtained in [MS] can be found in [FM].

*C. anisotropic power growth*

was introduced by Marcellini [M1]–[M4] as a natural extension of (1.2) where now  $f$  is allowed to have different growth rates from above and from below, precisely: with numbers  $1 < p < q$  we have

$$\lambda (|Z|^p - 1) \leq f(Z) \leq \Lambda (|Z|^q + 1) \quad \text{for all } Z \in \mathbb{R}^{nN} \quad (1.3)$$

(plus corresponding conditions involving  $D^2 f(Z)$ , for example  $D^2 f(Z) \geq \lambda(1 + |Z|^2)^{(p-2)/2}$ ). Condition (1.3) is motivated by the integral ( $n = 2$ )

$$J(u) = \int_{\Omega} \left\{ (1 + |\partial_1 u|^2)^{\frac{p}{2}} + (1 + |\partial_2 u|^2)^{\frac{q}{2}} \right\} dx$$

where the derivatives occur with different powers. It should be noted that B. is not a subcase of C. For formal reasons this should be obvious by considering energies of logarithmic type. On the other hand, partial regularity in the anisotropic case has been studied by Acerbi and Fusco [AF4] and later by Passarelli Di Napoli and Siepe [PS] under quite restrictive assumptions: in [PS] they impose the condition

$$2 \leq p < q < \min \left\{ p + 1, \frac{pn}{n-1} \right\} \quad (1.4)$$

thus excluding any subquadratic growth.

The purpose of our paper is twofold: first, we would like to give a unified approach including all the different cases. Secondly, we present certain improvements by extending for example the results of [FO] (see Remark 5.) below) and by constructing integrands to which the results mentioned in A. – C. do not apply but which can be handled with the help of our techniques. We consider integrands  $f$  of  $(s, \mu, q)$ -growth which are defined as follows: let  $F: [0, \infty) \rightarrow [0, \infty)$  denote a continuous function such that for some  $s \geq 1$  we have

$$\lim_{t \rightarrow \infty} \frac{F(t)}{t} = \infty, \quad F(t) \geq c_0 t^s \text{ for large values of } t. \quad (1.5)$$

The integrand  $f$  is a non-negative  $C^2$ -function such that for all  $Z, Y \in \mathbb{R}^{nN}$

$$c_1 F(|Z|) \leq f(Z); \quad (1.6)$$

$$\lambda (1 + |Z|^2)^{-\frac{\mu}{2}} |Y|^2 \leq D^2 f(Z)(Y, Y) \leq \Lambda (1 + |Z|^2)^{\frac{q-2}{2}} |Y|^2. \quad (1.7)$$

Here  $\mu \in \mathbb{R}$ ,  $q > 1$  and  $c_0, c_1, \lambda, \Lambda$  denote positive constants. In addition, we require the  $(s, \mu, q)$ -condition, i. e.

$$q < 2 - \mu + s \frac{2}{n}. \quad (1.8)$$

Note that on account of  $q > 1$  (1.8) gives the upper bound

$$\mu < 1 + \frac{2}{n}. \quad (1.9)$$

In the case that  $f$  is  $C^2$  close to  $|Z| \ln(1 + |Z|)$  we can take  $s = 1$ ,  $\mu = 1$ ,  $q = 1 + \varepsilon$  (for any  $\varepsilon > 0$ ) and  $F(t) = t \ln(1 + t)$ , hence (1.8) holds. Now our main result reads as follows:

**THEOREM 1.1** *Let conditions (1.5)–(1.8) hold and let  $u \in W_{1,loc}^1(\Omega, \mathbb{R}^N)$  denote a local minimizer of (1.1), i. e.  $f(Du) \in L_{loc}^1(\Omega)$  and*

$$\int_{spt(u-v)} f(Du) dx \leq \int_{spt(u-v)} f(Dv) dx$$

*for any  $v \in W_{1,loc}^1(\Omega, \mathbb{R}^N)$  such that  $spt(u - v) \Subset \Omega$ . Then there is an open subset  $\Omega_0$  of  $\Omega$  of full measure, i. e.  $|\Omega \setminus \Omega_0| = 0$ , such that  $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$  for any  $0 < \alpha < 1$ .*

Let us briefly comment on our conditions:

1.) The  $(s, \mu, q)$ -condition was introduced in [BFM] where full regularity (i. e.  $\Omega_0 = \Omega$ ) was established for the scalar case under exactly the same assumptions as stated here. The key ingredient in [BFM] is a local gradient bound in  $L^\infty$  which follows via Moser iteration technique or from DeGiorgi type arguments. In the vectorial setting  $N > 1$  such a bound can not be expected to hold true, thus we could not benefit too much from the arguments in [BFM]. However, as it is shown in [BFM] for the scalar case, it is easy to check that the result of Theorem 1.1 continues to hold if we replace (1.8) by the weaker condition (note that  $s \leq q$  on account of 2.))

$$q < (2 - \mu) \frac{n}{n - 2} \quad (1.8^*)$$

provided we add the balancing condition (introduced in [FO])

$$|D^2 f(Z)| |Z|^2 \leq \text{const} (f(Z) + 1). \quad (\text{B})$$

Note that (1.8\*) makes sense only in the case that  $n \geq 3$  and then (1.8\*) clearly implies (1.9). For the twodimensional case we have to replace (1.8\*) by the requirement that  $\mu < 2$ . We leave the details of the proof of this variant of Theorem 1.1 to the reader.

2.) (1.5) together with the second inequality in (1.7) implies (see [AF3], Lemma 2.1, if  $q < 2$ ) the bound  $s \leq q$ .

3.) In the case  $\mu \geq 1$  we have  $2 - \mu \leq s$ . If  $\mu \leq 0$  we clearly may assume that  $2 - \mu \leq s$  since  $2 - \mu$  is a lower bound for the growth of  $f$ , hence we can replace  $s$  by  $\max\{s, 2 - \mu\}$ . For  $0 < \mu < 1$  this inequality is also reasonable: from [AF3], Lemma 2.1, and the first inequality in (1.7) we get again that  $2 - \mu$  is a lower growth rate for  $f$ . Comparing this to (1.5) we may directly assume that  $2 - \mu \leq s$ . In particular we have by 2.) that  $2 - \mu \leq q$ .

4.) Suppose we are given numbers  $1 < p < q$  such that for all  $Z \in \mathbb{R}^{nN}$

$$\begin{aligned} a(|Z|^p - 1) &\leq f(Z) \leq b(|Z|^q + 1), \\ \lambda(1 + |Z|^2)^{\frac{p-2}{2}} &\leq D^2 f(Z) \leq \Lambda(1 + |Z|^2)^{\frac{q-2}{2}}. \end{aligned}$$

Then we may let  $\mu = 2 - p$ ,  $s = p$ , and we deduce partial regularity if

$$q < p \frac{n + 2}{n}$$

which is much weaker than (1.4). (Note: [PS] do not need an upper bound for  $D^2f(Z)$ .)

5.) In [FO] partial regularity was established under the assumptions (1.5)–(1.7),  $q \leq 2$ ,  $\mu < 4/n$  together with condition (B) (see Remark 1.). Clearly  $q \leq 2$  and  $\mu < 4/n$  imply (1.8\*) so that we have included the result of [FO] on account of the first remark. But, what is even more important, Theorem 1.1 does not need any balancing condition of the form (B), the regularity of local minimizers follows from  $\mu < 2 - q + s2/n$  which for  $q$  close to 1 and large values of  $n$  is a much weaker hypothesis than  $\mu < 4/n$ .

6.) Let us now sketch an example of an integrand of  $(s, \mu, q)$ –growth which is not of type A., B. or C. As shown in [BFM], Section 3, there exists for each  $k \in \mathbb{N}$  and  $t > 1$  a function  $\Phi_t^k: \mathbb{R}^k \rightarrow [0, \infty)$  such that

$$\Phi_t^k(\eta) \geq a|\eta|^t \quad \text{for large values of } \eta \in \mathbb{R}^k, \quad (1.10)$$

$$\text{and } 0 \leq D^2\Phi_t^k(\eta)(\tau, \tau) \leq b(1 + |\eta|^2)^{\frac{t-2}{2}}, \quad \eta, \tau \in \mathbb{R}^k, \quad (1.11)$$

hold with positive constants  $a, b$ . Roughly speaking, the function  $\Phi_t^k$  is constructed by first considering  $(1 + |Z|^2)^{t/2}$ , then redefining equidistant parts to be linear and finally smoothing the result of the first two steps. By definition of  $\Phi_t^k$ , the exponents in (1.10) and (1.11) can not be improved, moreover, due to the degeneracy of  $D^2\Phi_t^k$ , the lower bound of (1.11) is the best possible. Next consider numbers  $s, \mu, q$  such that  $1 < s < q$  and  $2 - \mu < s$ . Again, according to [BFM], Section 3, we can construct a function  $\Phi: \mathbb{R}^{nN} \rightarrow [0, \infty)$  satisfying

$$D^2\Phi(Z)(\tau, \tau) \geq c(1 + |Z|^2)^{-\frac{\mu}{2}} |\tau|^2.$$

For instance, we may choose  $\Phi(Z) = \varphi(|Z|)$  were  $\varphi$  is defined via

$$\varphi(r) = \int_0^r \int_0^s (1 + |t|^2)^{-\frac{\mu}{2}} dt ds, \quad r \in \mathbb{R}_0^+.$$

In the case  $\mu \geq 1$   $\Phi$  is of lower growth than any power  $|Z|^{1+\vartheta}$ ,  $\vartheta > 0$ , for  $\mu < 1$  we get  $\Phi(Z) \leq d(1 + |Z|^2)^{(2-\mu)/2}$ , and it is not possible to improve the exponents. We then define  $(z = (z_1^i, \dots, z_n^i)_{1 \leq i \leq N} \in \mathbb{R}^{nN})$

$$f(Z) = \Phi(Z) + \Phi_s^N(z_1) + \Phi_q^{(n-1)N}(z_2, \dots, z_n).$$

Then (1.5)–(1.7) hold and if we also impose (1.8) then regularity of local minimizers follows which can not be deduced from the results stated in A.–C.



Our paper is organized as follows: in Section 2 we introduce a suitable regularization  $v_\varepsilon$  of our local minimizer  $u$  which converges weakly and in energy to  $u$  on compact subsets. Section 3 investigates higher weak differentiability of  $v_\varepsilon$ . As a consequence we obtain uniform local estimates in  $L^q$  for  $Dv_\varepsilon$  which allow us to give local a priori bounds for  $\|Du\|_{L^q}$ . Moreover, we prove certain Caccioppoli-type inequalities. Finally, Section 4 contains the proof of Theorem 1.1 via blow-up arguments by considering the cases  $q \geq 2$  and  $1 < q < 2$  more or less separately.

## 2 Approximation and some preliminary results

Let  $\varepsilon$  denote a sequence of positive real numbers converging to zero, where we do not care about relabelling if necessary. Then we define  $u_\varepsilon$  as the  $\varepsilon$ -mollification of  $u$  through  $\varphi_\varepsilon$ , where  $\{\varphi_t\}_{t>0}$  is a family of smooth mollifiers. Moreover, let us fix  $R > 0$  and  $x_0 \in \Omega$ . Letting  $B_r := B_r(x_0)$  we assume  $B_{2R} \subset \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$ . For  $\delta \in (0, 1]$  we define

$$f_\delta(Z) := f(Z) + \delta(1 + |Z|^2)^{\frac{q}{2}}$$

and denote by  $v_{\varepsilon, \delta}$  the unique solution of the variational problem

$$J_\delta(w) := \int_{B_{2R}} f_\delta(Dw) dx \rightsquigarrow \min \quad \text{in } u_\varepsilon + \overset{\circ}{W}_q^1(B_{2R}, \mathbb{R}^N).$$

**LEMMA 2.1** *If  $\varepsilon$  and  $\delta$  are connected via*

$$\delta = \delta(\varepsilon) := \frac{1}{1 + \varepsilon^{-1} + \|Du_\varepsilon\|_{L^q(B_{2R})}^{2q}}$$

*and if  $v_\varepsilon = v_{\varepsilon, \delta(\varepsilon)}$ ,  $f_\varepsilon = f_{\delta(\varepsilon)}$ , then we have as  $\varepsilon \rightarrow 0$ :*

- (i.)  $v_\varepsilon \rightharpoonup u$  in  $W_1^1(B_{2R}, \mathbb{R}^N)$ ,
- (ii.)  $\delta(\varepsilon) \int_{B_{2R}} (1 + |Dv_\varepsilon|^2)^{\frac{q}{2}} dx \rightarrow 0$ ,
- (iii.)  $\int_{B_{2R}} f(Dv_\varepsilon) dx \rightarrow \int_{B_{2R}} f(Du) dx$ ,
- (iv.)  $\int_{B_{2R}} f_\varepsilon(Dv_\varepsilon) dx \rightarrow \int_{B_{2R}} f(Du) dx$ .

**Proof of Lemma 2.1.** We argue as in [BFM], conclusion of Theorem 1.1, i. e. we use the minimality of  $v_\varepsilon$  as well as Jensen's inequality to get

$$\begin{aligned}
\int_{B_{2R}} F(|Dv_\varepsilon|) dx &\leq \int_{B_{2R}} f(Dv_\varepsilon) dx \leq \int_{B_{2R}} f_\varepsilon(Du_\varepsilon) dx \\
&\leq \int_{B_{2R}} f(Du) dx + o(\varepsilon),
\end{aligned} \tag{2.1}$$

i. e. we may suppose that

$$v_\varepsilon \rightharpoonup v \quad \text{weakly in } W_1^1(B_{2R}, \mathbb{R}^N).$$

Passing to the limit  $\varepsilon \rightarrow 0$ , lower semicontinuity implies

$$\int_{B_{2R}} f(Dv) dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{B_{2R}} f(Dv_\varepsilon) dx \leq \int_{B_{2R}} f(Du) dx.$$

Finally, the minimality of  $u$  together with strict convexity of  $f$  (see (1.7)) ensure that  $v = u$ , thus, with (2.1) the lemma is proved.  $\blacksquare$

In the following  $\delta$  is always assumed to be chosen according to Lemma 2.1. To finish this section, some well known properties of  $v_\varepsilon$  are summarized. Part a.) of the following lemma is proved in [AF3], Proposition 2.4 and Lemma 2.5, for the second part we refer the reader to [GM], especially formula (3.3), and to [CA] (compare Theorem 1.1).

**LEMMA 2.2** *a.) In the case  $q < 2$  the approximative solution satisfies:*

- (i.)  $v_\varepsilon \in W_{q,loc}^2(B_{2R}, \mathbb{R}^N)$ , (ii.)  $Df_\varepsilon(Dv_\varepsilon) \in W_{2,loc}^1(B_{2R}, \mathbb{R}^{nN})$ ,
- (iii.)  $(1 + |Dv_\varepsilon|^2)^{\frac{q-2}{4}} Dv_\varepsilon \in W_{2,loc}^1(B_{2R}, \mathbb{R}^{nN})$ ,
- (iv.)  $|D^2 v_\varepsilon \mathbf{1}_{|v_\varepsilon| \leq M}| \in L_{loc}^2(B_{2R})$  for all  $M > 0$ .

*b.) In the case  $q \geq 2$  we have*

- (i.)  $v_\varepsilon \in W_{2,loc}^2(B_{2R}, \mathbb{R}^N)$ , (ii.)  $Df_\varepsilon(Dv_\varepsilon) \in W_{q/(q-1),loc}^1(B_{2R}, \mathbb{R}^{nN})$ ,
- (iii.)  $(1 + |Dv_\varepsilon|^2)^{\frac{q}{4}} \in W_{2,loc}^1(B_{2R})$ ,
- (iv.)  $(1 + |Dv_\varepsilon|^2)^{\frac{q-2}{4}} Dv_\varepsilon \in W_{2,loc}^1(B_{2R}, \mathbb{R}^{nN})$ .

### 3 Apriori $L^q$ -estimates and Caccioppoli-type inequalities

In this section we are going to prove the two main ingredients which will enable us to perform the blow-up procedure in Section 4. The starting point is the following Caccioppoli-type inequality for the approximative solutions.

**LEMMA 3.1** *There is a real number  $c > 0$  such that for all  $\eta \in C_0^1(B_{2R})$ ,  $0 \leq \eta \leq 1$ , and for all  $Q \in \mathbb{R}^{nN}$*

$$\begin{aligned} & \int_{B_{2R}} \eta^2 D^2 f_\varepsilon(Dv_\varepsilon) (\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon) dx \\ & \leq c \|D\eta\|_\infty^2 \int_{B_{2R} \cap \text{spt} D\eta} |D^2 f_\varepsilon(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 dx, \end{aligned}$$

where summation with respect to  $s = 1, \dots, n$  is always assumed in the following. In particular, for all  $Q \in \mathbb{R}^{nN}$

$$\begin{aligned} & \int_{B_{2R}} \eta^2 (1 + |Dv_\varepsilon|^2)^{-\frac{\mu}{2}} |D^2 v_\varepsilon|^2 dx \\ & \leq c \|D\eta\|_\infty^2 \int_{B_{2R} \cap \text{spt} D\eta} |D^2 f_\varepsilon(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 dx. \end{aligned}$$

**Proof of Lemma 3.1.** First of all we recall that  $v_\varepsilon$  solves the regularized problem, i. e.

$$\int_{B_{2R}} Df_\varepsilon(Dv_\varepsilon) : D\varphi dx = 0 \quad \text{for all } \varphi \in \mathring{W}_q^1(B_{2R}, \mathbb{R}^N). \quad (3.1)$$

Next, denote by  $e_s \in \mathbb{R}^n$  the unit coordinate vector in  $x_s$ -direction and let for a function  $g$  on  $\Omega$

$$\Delta_h g(x) = \Delta_h^s g(x) = \frac{g(x + h e_s) - g(x)}{h}, \quad h \in \mathbb{R},$$

denote the difference quotient of  $g$  at  $x$  in the direction  $e_s$ . Then, given  $Q \in \mathbb{R}^{nN}$ ,  $\varphi = \Delta_{-h}(\eta^2 \Delta_h(v_\varepsilon - Qx))$ ,  $\eta \in C_0^\infty(B_{2R})$ , is admissible in (3.1) and by a ‘‘partial integration’’ we obtain

$$\begin{aligned} & \int_{B_{2R}} \eta^2 \Delta_h(Df_\varepsilon(Dv_\varepsilon)) : D\Delta_h v_\varepsilon dx \\ & = -2 \int_{B_{2R}} \eta \Delta_h(Df_\varepsilon(Dv_\varepsilon)) : D\eta \otimes \Delta_h(v_\varepsilon - Qx) dx. \end{aligned} \quad (3.2)$$

Consider now the case  $q \geq 2$ : by Lemma 2.2 and by (1.8)  $Dv_\varepsilon$  is known to be of class  $L^r_{loc}$  for some  $r > q$  and if  $F_h$  denotes the integrand on the right-hand side of (3.2), then the existence of a real number  $c(D\eta)$ , independent of  $h$ , follows such that

$$|F_h| \leq c \left\{ |\Delta_h(Df_\varepsilon(Dv_\varepsilon))|^{l_1} + |\Delta_h v_\varepsilon|^{l_2} \right\} \text{ for some } l_1 < \frac{q}{q-1}, \quad q < l_2 < r,$$

thus, equiintegrability of  $F_h$  in the sense of Vitali's convergence theorem is ensured by Lemma 2.2, b.), (ii.), and passing to the limit  $h \rightarrow 0$  the right-hand side of (3.2) tends to

$$-2 \int_{B_{2R}} \eta \partial_s(Df_\varepsilon(Dv_\varepsilon)) : D\eta \otimes (\partial_s v_\varepsilon - Q_s) dx \in (-\infty, +\infty). \quad (3.3)$$

For the left-hand side of (3.2) we observe

$$\Delta_h(Df_\varepsilon(Dv_\varepsilon)) = \int_0^1 D^2 f_\varepsilon(Dv_\varepsilon + th \Delta_h Dv_\varepsilon)(\Delta_h Dv_\varepsilon, \cdot) dt$$

and get using (3.3), Fatou's lemma and Young's inequality

$$\begin{aligned} & \int_{B_{2R}} \eta^2 D^2 f_\varepsilon(Dv_\varepsilon)(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon) dx \\ & \leq \int_{B_{2R}} \eta^2 \liminf_{h \rightarrow 0} \int_0^1 D^2 f_\varepsilon(Dv_\varepsilon + th \Delta_h Dv_\varepsilon)(\Delta_h Dv_\varepsilon, \Delta_h Dv_\varepsilon) dt dx \\ & \leq \frac{1}{2} \int_{B_{2R}} \eta^2 D^2 f_\varepsilon(Dv_\varepsilon)(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon) dx \\ & \quad + c \|D\eta\|_\infty^2 \int_{B_{2R} \cap \text{spt} D\eta} |D^2 f_\varepsilon(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 dx, \end{aligned}$$

i. e. the lemma is proved for  $q \geq 2$ . If  $q < 2$  then we modify the truncation arguments given in [EM1]. To this purpose fix  $M \gg 1$  and let for  $t \geq 0$

$$\psi(t) := \begin{cases} 0, & t \geq M \\ 1, & t \leq M/2 \end{cases}, \quad |\psi'(t)| \leq 4/M.$$

Given  $\eta, Q$  as above, then, by Lemma 2.2, a.), (iv.), and by [EM1], Lemma 1,  $\varphi = \Delta_{-h}(\eta^2 \partial_s(v_\varepsilon - Qx) \psi(|Dv_\varepsilon|))$  is seen to be admissible, hence

$$\begin{aligned} & \int_{B_{2R}} \eta^2 \psi \Delta_h(Df_\varepsilon(Dv_\varepsilon)) : D\partial_s v_\varepsilon dx \\ & = -2 \int_{B_{2R}} \eta \psi \Delta_h(Df_\varepsilon(Dv_\varepsilon)) : D\eta \otimes \partial_s(v_\varepsilon - Qx) dx \\ & \quad - 2 \int_{B_{2R}} \eta^2 \Delta_h(Df_\varepsilon(Dv_\varepsilon)) : D\psi \otimes \partial_s(v_\varepsilon - Qx) dx. \end{aligned} \quad (3.4)$$

By the definition of  $\psi$  and again on account of Lemma 2.2, a.), (iv.), both integrals on the right-hand side of (3.4) can be written as

$$\begin{aligned} & \int_{spt\eta} \Delta_h(Df_\varepsilon(Dv_\varepsilon)) : \xi(x) dx \\ & \leq \int_{spt\eta} |\Delta_h(Df_\varepsilon(Dv_\varepsilon))|^2 dx + \|\xi\|_{L^2(B_{2R}, \mathbb{R}^{nN})}^2, \end{aligned} \quad (3.5)$$

for a suitable function  $\xi$  of class  $L^2$ . Since Lemma 2.2, a.), (ii.), shows  $\partial_s(Df_\varepsilon(Dv_\varepsilon))$  to be of class  $L^2_{loc}$ , strong convergence of difference quotients (see [MO], Theorem 3.6.8 (b)) implies passing to the limit  $h \rightarrow 0$

$$\begin{aligned} & |\Delta_h(Df_\varepsilon(Dv_\varepsilon))|^2 \rightarrow |\partial_s(Df_\varepsilon(Dv_\varepsilon))|^2 \quad \text{almost everywhere,} \\ & \int_{spt\eta} |\Delta_h(Df_\varepsilon(Dv_\varepsilon))|^2 dx \rightarrow \int_{spt\eta} |\partial_s(Df_\varepsilon(Dv_\varepsilon))|^2 dx. \end{aligned} \quad (3.6)$$

With (3.5) and (3.6) the variant of the dominated convergence theorem, given for example in [EG2], Theorem 4, p. 21, is applicable (note that almost everywhere convergence in (3.6) is needed for a proof of this variant). Thus, we may pass to the limit  $h \rightarrow 0$  on the right-hand side of (3.4). The left-hand side is handled as in the case  $q \geq 2$  and summarizing the results we arrive at (again after applying Young's inequality to the bilinear form  $D^2 f_\varepsilon(Dv_\varepsilon)$ )

$$\begin{aligned} & \int_{B_{2R}} \eta^2 \psi D^2 f_\varepsilon(Dv_\varepsilon)(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon) dx \\ & \leq \frac{1}{2} \int_{B_{2R}} \eta^2 \psi D^2 f_\varepsilon(Dv_\varepsilon)(\partial_s Dv_\varepsilon, \partial_s Dv_\varepsilon) dx \\ & \quad + c \|D\eta\|_\infty^2 \int_{spt D\eta} |D^2 f_\varepsilon(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 dx \\ & \quad + c \int_{spt D\eta} |D^2 f_\varepsilon(Dv_\varepsilon)| |D^2 v_\varepsilon|^2 \mathbf{1}_{[M/2 \leq |Dv_\varepsilon| \leq M]} dx. \end{aligned}$$

Here we use the fact that  $|Dv_\varepsilon - Q| < 2M$  on  $[M/2 \leq |Dv_\varepsilon| \leq M]$  for  $M$  sufficiently large and that  $D(\psi(|Dv_\varepsilon|)) \leq c |D^2 v_\varepsilon|/M$ . Before passing to the limit  $M \rightarrow \infty$  we use Proposition 2.4 of [AF3] once again, i. e. we observe the estimate

$$\int_{B_t} (1 + |Dv_\varepsilon|^2)^{\frac{q-2}{2}} |D^2 v_\varepsilon|^2 dx \leq c(t, t') \int_{B_{t'}} (1 + |Dv_\varepsilon|^2)^{\frac{q}{2}} dx$$

being valid for all  $0 < t < t' < 2R$ . Recalling the growth of  $|D^2 f(Dv_\varepsilon)|$  we immediately get that

$$\int_{\text{spt} D\psi} |D^2 f_\varepsilon(Dv_\varepsilon)| |D^2 v_\varepsilon|^2 \mathbf{1}_{[M/2 \leq |Dv_\varepsilon| \leq M]} dx \xrightarrow{M \rightarrow \infty} 0$$

on account of  $\mathbf{1}_{\text{spt} D\psi} \rightarrow 0$  as  $M \rightarrow \infty$  and the claim of the lemma follows. ■

Besides Lemma 3.1 the following technical proposition is needed to prove uniform  $L^q$ -estimates for  $Dv_\varepsilon$ . So let us introduce  $\Theta(t) := (1 + t^2)^{(2-\mu)/4}$ ,  $t \geq 0$ , and let  $h_\varepsilon := \Theta(|Dv_\varepsilon|)$ .

**PROPOSITION 3.2** *With this notation  $h_\varepsilon \in W_{2,loc}^1(B_{2R})$  and*

$$Dh_\varepsilon = \Theta'(|Dv_\varepsilon|) D|Dv_\varepsilon|.$$

**REMARK 3.3** *If we consider for instance the case  $q \geq 2$ , then the fact that  $h_\varepsilon$  is of class  $W_2^1$  follows from Lemma 2.2, b.), (iii.). However, in Lemma 3.4 we need an explicit formula for the derivative.*

**Proof of Proposition 3.2.** In order to reduce the problem to an application of the usual chain rule for Lipschitz functions, let  $L \gg 1$  be some real number and let

$$\Theta_L(t) = \begin{cases} (1 + t^2)^{\frac{2-\mu}{4}}, & 0 \leq t \leq L \\ (1 + L^2)^{\frac{2-\mu}{4}}, & t \geq L \end{cases}, \quad h_\varepsilon^L := \Theta_L(|v_\varepsilon|).$$

As a consequence of Lemma 2.2,  $h_\varepsilon^L$  is immediately seen to be of class  $W_1^1$  satisfying

$$Dh_\varepsilon^L = \Theta'_L(|Dv_\varepsilon|) D|Dv_\varepsilon|. \quad (3.7)$$

In addition, for  $0 < r < 2R$  we have the estimate

$$\begin{aligned} \int_{B_r} |Dh_\varepsilon^L|^2 dx &\leq \int_{B_r \cap \{|Dv_\varepsilon| \leq L\}} |\Theta'|^2 |D^2 v_\varepsilon|^2 dx \\ &\leq c \int_{B_r} (1 + |Dv_\varepsilon|^2)^{-\frac{\mu}{2}} |D^2 v_\varepsilon|^2 dx, \end{aligned}$$

hence, by Lemma 3.1,  $\|Dh_\varepsilon^L\|_{L^2(B_r, \mathbb{R}^n)}$  is uniformly bounded with respect to  $L$  and we may assume

$$Dh_\varepsilon^L \rightharpoonup W_\varepsilon \quad \text{in } L^2(B_r, \mathbb{R}^n) \quad \text{as } L \rightarrow \infty.$$

On the other hand, the obvious convergence  $h_\varepsilon^L \rightarrow h_\varepsilon$  in  $L^2(B_r)$  as  $L \rightarrow \infty$  implies  $W_\varepsilon = Dh_\varepsilon$ , thus  $h_\varepsilon \in W_2^1(B_r)$ . (3.7) also gives

$$Dh_\varepsilon^L \rightarrow \Theta'(|Dv_\varepsilon|)D|Dv_\varepsilon| \quad \text{almost everywhere,}$$

hence we can identify the limit and the proposition is proved.  $\blacksquare$

As mentioned above, we now turn our attention to (uniform) higher integrability of  $Dv_\varepsilon$ . Let us remark, that with uniform growth estimates for  $f$ , but even without any control on the derivatives, integrability of the gradient can be slightly improved (compare, for instance, [CF]). In the situation at hand, Lemma 3.4 can be proved following the lines of [BFM], Lemma 2.4.

**LEMMA 3.4** *Assume again (1.5)–(1.8) and let  $\chi = \frac{n}{n-2}$  if  $n > 2$ , in the case  $n = 2$  let  $\chi > \frac{2s}{s+2-\mu-q}$ . Then there are real numbers  $c, \beta$ , independent of  $\varepsilon$  such that for all  $r < 2R$*

$$\int_{B_r} (1 + |Dv_\varepsilon|^2)^{\frac{(2-\mu)\chi}{2}} dx \leq c(r, R) \left\{ \int_{B_{2R}} (1 + f_\varepsilon(Dv_\varepsilon)) dx \right\}^\beta.$$

In particular, by Lemma 2.1,  $Dv_\varepsilon \in L_{loc}^{(2-\mu)\chi}(B_{2R}, \mathbb{R}^{nN}) \subset L_{loc}^q(B_{2R}, \mathbb{R}^{nN})$  uniformly with respect to  $\varepsilon$ , i. e.

$$Du \in L_{loc}^{(2-\mu)\chi}(B_{2R}, \mathbb{R}^{nN}) \subset L_{loc}^q(B_{2R}, \mathbb{R}^{nN}).$$

**Proof of Lemma 3.4.** We consider the case  $n \geq 3$ , let  $\alpha = \frac{(2-\mu)n}{2(n-2)}$  and assume without loss of generality  $R < r < 3R/2$ . Moreover, fix  $0 < \rho < R/2$  and  $\eta \in C_0^1(B_{r+\rho/2})$ ,  $\eta \equiv 1$  on  $B_r$ ,  $D\eta \leq 4/\rho$ . Since  $h_\varepsilon$  was proved in Proposition 3.2 to be of class  $W_2^1$ , we obtain using Sobolev's inequality

$$\begin{aligned} \int_{B_r} (1 + |Dv_\varepsilon|^2)^\alpha dx &\leq \int_{B_{2R}} \left( \eta [1 + |Dv_\varepsilon|^2]^\alpha \frac{n-2}{2n} \right)^{\frac{2n}{n-2}} dx \\ &= \int_{B_{2R}} (\eta h_\varepsilon)^{\frac{2n}{n-2}} dx \\ &\leq c \left( \int_{B_{2R}} |D(\eta h_\varepsilon)|^2 dx \right)^{\frac{n}{n-2}} \leq c \{T_1 + T_2\}^{\frac{n}{n-2}}, \end{aligned}$$

where we have set

$$T_1 = \int_{B_{2R}} |D\eta|^2 h_\varepsilon^2 dx, \quad T_2 = \int_{B_{2R}} \eta^2 |Dh_\varepsilon|^2 dx.$$

$T_1$  is directly seen to satisfy

$$T_1 \leq \frac{c}{\rho^2} \int_{B_{2R}} (1 + |Dv_\varepsilon|^2)^{\frac{2-\mu}{2}} dx ,$$

whereas  $T_2$  has to be handled via the representation formula for the derivative of  $h_\varepsilon$  given in Proposition 3.2:

$$T_2 \leq c \int_{B_{r+\rho/2}} (1 + |Dv_\varepsilon|^2)^{-\frac{\mu}{2}} |D^2 v_\varepsilon|^2 dx .$$

With Lemma 3.1 (choosing  $Q = 0$ ) and (1.7) we obtain

$$\begin{aligned} & \int_{B_r} (1 + |Dv_\varepsilon|^2)^\alpha dx \\ & \leq \frac{c}{\rho^2} \left\{ \int_{B_{2R}} (1 + |Dv_\varepsilon|^2)^{\frac{2-\mu}{2}} dx + \int_{B_{r+\rho} \sim B_r} (1 + |Dv_\varepsilon|^2)^{\frac{q}{2}} dx \right\}^\chi , \end{aligned}$$

where the arguments used for the right-hand side are the same as in [BFM], i. e.: the interpolation procedure demonstrated in [ELM] (starting with the inequality given after (4.6) in [ELM]) is modified using (1.5):

$$\|Dv_\varepsilon\|_q \leq \|Dv_\varepsilon\|_s^\theta \|Dv_\varepsilon\|_{(2-\mu)\chi}^{1-\theta} .$$

This inequality holds with  $\theta \in (0, 1)$  defined according to  $\frac{1}{q} = \frac{\theta}{s} + \frac{1-\theta}{(2-\mu)\chi}$ . Note that the subsequent arguments of [ELM] require the bound  $(1-\theta)q/(2-\mu) < 1$  which in case  $n \geq 3$  is equivalent to (1.8). Now let  $n = 2$  and define  $\alpha = \chi(2-\mu)/2$ . Then we have

$$\int_{B_r} (1 + |Dv_\varepsilon|^2)^\alpha dx \leq \int_{B_{2R}} (\eta h_\varepsilon)^{2\chi} dx \leq c \left( \int_{B_{2R}} |D(\eta h_\varepsilon)|^t dx \right)^{\frac{2\chi}{t}} ,$$

where  $t \in (1, 2)$  is defined through  $2\chi = 2t/(2-t)$ . Using Hölder's inequality we get

$$\int_{B_r} (1 + |Dv_\varepsilon|^2)^\alpha dx \leq c \left( \int_{B_{2R}} |D(\eta h_\varepsilon)|^2 dx \right)^\chi ,$$

and we can proceed as before with  $n/(n-2)$  replaced by  $\chi$ . Again we have to satisfy the requirement  $(1-\theta)q/(2-\mu) < 1$  which for  $n = 2$  is equivalent to  $\chi > s/(s+2-\mu-q)$ . But the latter inequality follows from our choice of  $\chi$ , thus Lemma 3.4 is established also in case  $n = 2$ .  $\blacksquare$

Having established higher integrability of  $Du$ , the next proposition gives some preparations needed for the limit version of Caccioppoli's-type inequality.



**PROPOSITION 3.5** *Let  $h = (1 + |Du|^2)^{(2-\mu)/4}$ . Then*

- (i.)  $h \in W_{2,loc}^1(B_{2R})$ ,
- (ii.)  $h_\varepsilon \rightharpoonup h$  in  $W_{2,loc}^1(B_{2R})$  as  $\varepsilon \rightarrow 0$ ,
- (iii.)  $Dv_\varepsilon \rightarrow Du$  almost everywhere on  $B_{2R}$  as  $\varepsilon \rightarrow 0$ .

**Proof of Proposition 3.5.** We fix  $0 < r < \hat{r} < 2R$ , combine Lemma 3.1 and Proposition 3.2 to obtain

$$\|Dh_\varepsilon\|_{L^2(B_r, \mathbb{R}^n)}^2 \leq c(1 + \|Dv_\varepsilon\|_{L^q(B_{\hat{r}}, \mathbb{R}^{nN})}^2),$$

hence, by Lemma 3.4,  $h_\varepsilon$  is uniformly bounded in  $W_{2,loc}^1(B_{2R})$  and we may assume as  $\varepsilon \rightarrow 0$

$$h_\varepsilon \rightharpoonup: \hat{h} \quad \text{weakly in } W_{2,loc}^1(B_{2R}) \text{ and almost everywhere.}$$

The proof of  $\hat{h} = h$  together with the pointwise convergences exactly follows the lines of [FO], Lemma 4.1. ■

Now we can formulate the limit version of Lemma 3.1.

**LEMMA 3.6** *There is a real number  $c$  such that for all  $\eta \in C_0^1(B_{2R})$ ,  $0 \leq \eta \leq 1$ , and for all  $Q \in \mathbb{R}^{nN}$*

$$\int_{B_{2R}} \eta^2 |Dh|^2 dx \leq c \|D\eta\|_\infty^2 \int_{B_{2R} \cap \text{spt} D\eta} |D^2 f(Du)| |Du - Q|^2 dx.$$

**Proof of Lemma 3.6.** Given  $Q$ ,  $\eta$  as above, Proposition 3.5, lower semi-continuity, Lemma 3.1 and Proposition 3.2 together imply

$$\begin{aligned} \int_{B_{2R}} \eta^2 |Dh|^2 dx &\leq \liminf_{\varepsilon \rightarrow 0} \int_{B_{2R}} \eta^2 |Dh_\varepsilon|^2 dx \\ &\leq \liminf_{\varepsilon \rightarrow 0} c \|D\eta\|_\infty^2 \int_{B_{2R} \cap \text{spt} D\eta} |D^2 f_\varepsilon(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 dx \quad (3.8) \\ &= \liminf_{\varepsilon \rightarrow 0} c \|D\eta\|_\infty^2 \int_{B_{2R} \cap \text{spt} D\eta} |D^2 f(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 dx. \end{aligned}$$

Here, for the last equality, we made use of Lemma 2.1, (ii). Next, by the pointwise convergence almost everywhere stated in Proposition 3.5, (iii.), we have

$$|D^2 f(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 \rightarrow |D^2 f(Du)| |Du - Q|^2 \quad \text{a. e. as } \varepsilon \rightarrow 0. \quad (3.9)$$

Finally, by Lemma 3.4 we know that  $|D^2 f(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2$  is uniformly bounded in  $L_{loc}^{1+\tau}(B_{2R})$  for some  $\tau > 0$ , hence

$$\begin{aligned} |D^2 f(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 &\rightharpoonup: \vartheta \quad \text{in } L_{loc}^{1+\tau}(B_{2R}), \\ \int_{B_{2R}} |D^2 f(Dv_\varepsilon)| |Dv_\varepsilon - Q|^2 dx &\rightarrow \int_{B_{2R}} \vartheta dx \end{aligned} \quad (3.10)$$

as  $\varepsilon \rightarrow 0$ . From (3.9), (3.10) we clearly get  $\vartheta = |D^2 f(Du)| |Du - Q|^2$ , which together with (3.8) gives the proof of Lemma 3.6.  $\blacksquare$

## 4 Blow-Up

Now we fix a local minimizer  $u$  which by Lemma 3.4 is known to be of class  $W_{q,loc}^1(\Omega, \mathbb{R}^N)$ . The final step is to prove partial regularity of  $u$  via a blow-up procedure. As usual, the main tool is the decay estimate given in Lemma 4.1. The iteration of Lemma 4.1 leading to partial regularity is well known. Depending on the cases  $q \geq 2$  and  $q < 2$  an appropriate excess function has to be introduced: in the case  $q \geq 2$  we let for balls  $B_r(x) \Subset B_R \subset \Omega$

$$E^+(x, r) := \int_{B_r(x)} |Du - (Du)_{x,r}|^2 dy + \int_{B_r(x)} |Du - (Du)_{x,r}|^q dy,$$

where  $(g)_{x,r}$  denotes the mean value of the function  $g$  with respect to the ball  $B_r(x)$ . In the case  $q < 2$  we define for all  $\xi \in \mathbb{R}^k$ ,  $k \in \mathbb{N}$ ,

$$V(\xi) := (1 + |\xi|^2)^{\frac{q-2}{4}} \xi.$$

The properties of  $V$  are studied for example in [CFM], in particular we refer to Lemma 2.1 of [CFM]. With these preliminaries we let for  $q < 2$

$$E^-(x, r) := \int_{B_r(x)} |V(Du(x)) - V((Du)_{x,r})|^2 dy,$$

a definition which makes sense since  $q/2$  is the growth rate of  $V$ . In both cases we have

**LEMMA 4.1** Fix  $L > 0$ . Then there exists a constant  $C_*(L)$  such that for every  $0 < \tau < 1/4$  there is an  $\varepsilon = \varepsilon(L, \tau)$  satisfying: if  $B_r(x) \Subset B_R$  and if we have

$$|(Du)_{x,r}| \leq L, \quad E(x, r) \leq \varepsilon(L, \tau),$$

then

$$E(x, \tau r) \leq C_*(L) \tau^2 E(x, r).$$

Here and in the following  $E$  denotes – depending on  $q$  –  $E^+$  or  $E^-$  respectively.

**Proof of Lemma 4.1.** The proof is organized in four steps, always distinguishing the cases  $q \geq 2$  and  $q < 2$ . If  $q \geq 2$  then we mostly refer to [FO], the case  $q < 2$  follows the lines of [CFM] and [EM2].

Step 1. (Blow-up and limit equation) To argue by contradiction, assume that  $\overline{L} > 0$  is fixed, the corresponding constant  $C_*(L)$  will be chosen later on (see Step 4). If Lemma 4.1 is not true, then for some  $0 < \tau < 1/4$ , there are balls  $B_{r_m}(x_m) \Subset B_R$  such that

$$|(Du)_{x_m, r_m}| \leq L, \quad E(x_m, r_m) =: \lambda_m^2 \xrightarrow{m \rightarrow \infty} 0, \quad (4.1)$$

$$E(x_m, \tau r_m) > C_* \tau^2 \lambda_m^2. \quad (4.2)$$

Now a sequence of rescaled functions is introduced by letting

$$\begin{aligned} a_m &:= (u)_{x_m, r_m}, \quad A_m := (Du)_{x_m, r_m}, \\ u_m(z) &:= \frac{1}{\lambda_m r_m} [u(x_m + r_m z) - a_m - r_m A_m z] \quad \text{if } |z| \leq 1. \end{aligned}$$

Passing to a subsequence, which is not relabeled, (4.1) implies

$$A_m \rightarrow : A \quad \text{in } \mathbb{R}^{nN}. \quad (4.3)$$

We also observe that

$$Du_m(z) = \lambda_m^{-1} [Du(x_m + r_m z) - A_m], \quad (u_m)_{0,1} = 0, \quad (Du_m)_{0,1} = 0,$$

and concentrate for the moment on the case  $q \geq 2$ . Using (4.1) and (4.2) we have

$$\int_{B_1} |Du_m|^2 dz + \lambda_m^{q-2} \int_{B_1} |Du_m|^q dz = \lambda_m^{-2} E^+(x_m, r_m) = 1, \quad (4.4)$$

$$\int_{B_1} |Du_m - (Du_m)_{0,\tau}|^2 dz + \lambda_m^{q-2} \int_{B_1} |Du_m - (Du_m)_{0,\tau}|^q dz > C_* \tau^2. \quad (4.5)$$

With (4.4) we obtain as  $m \rightarrow \infty$

$$u_m \rightharpoonup \hat{u} \quad \text{in } W_2^1(B_1, \mathbb{R}^{nN}), \quad (4.6)$$

$$\lambda_m Du_m \rightarrow 0 \quad \text{in } L^2(B_1, \mathbb{R}^{nN}) \text{ and almost everywhere,} \quad (4.7)$$

$$\lambda_m^{1-\frac{2}{q}} Du_m \rightharpoonup 0 \quad \text{in } L^q(B_1, \mathbb{R}^{nN}) \text{ if } q > 2. \quad (4.8)$$

Considering the case  $q < 2$  we follow [CFM], Proposition 3.4, Step 1, to see

$$\int_{B_1} |V(Du_m(z))|^2 dz \leq c(L), \quad (4.9)$$

hence the “ $q/2$ -growth” of  $V$  (compare [CFM], Lemma 2.1, (i)) implies the existence of a finite constant, independent of  $m$ , such that

$$\|Du_m\|_{L^q(B_1, \mathbb{R}^{nN})} \leq c.$$

Thus, in the subquadratic situation (4.6)–(4.8) have to be replaced by

$$u_m \rightharpoonup \hat{u} \quad \text{in } W_q^1(B_1, \mathbb{R}^{nN}), \quad (4.10)$$

$$\lambda_m Du_m \rightarrow 0 \quad \text{in } L^q(B_1, \mathbb{R}^{nN}) \text{ and almost everywhere.} \quad (4.11)$$

In both cases the limit  $\hat{u}$  satisfies a blow-up equation stated in

**PROPOSITION 4.2** *There is a constant  $C^*$ , only depending on  $L$ , such that for all  $\varphi \in C_0^1(B_1, \mathbb{R}^N)$*

$$\begin{aligned} \int_{B_1} D^2 f(A)(D\hat{u}, D\varphi) dz &= 0, \\ \int_{B_\tau} |D\hat{u} - (D\hat{u})_\tau|^2 dz &\leq C^* \tau^2. \end{aligned} \quad (4.12)$$

**Proof of Proposition 4.2.** The proof of the limit equation for  $q \geq 2$  is well known and can be taken from [EV], p. 236. The subquadratic case again is treated in [CFM], Step 2. Inequality (4.12) of Proposition 4.2 follows from the theory of linear elliptic systems (compare [Gi], Chapter 3) where the subquadratic case also involves Proposition 2.10 of [CFM]. ■

Step 2. Proceeding in the proof of Lemma 4.1 we have to show the following proposition which will imply strong convergence in the third step.

**PROPOSITION 4.3** *Let  $q \geq 2$  and  $0 < \rho < 1$  or consider the case  $q < 2$  together with  $0 < \rho < 1/3$ . Then*

$$\lim_{m \rightarrow \infty} \int_{B_\rho} (1 + |A_m + \lambda_m D\hat{u} + \lambda_m Dw_m|^2)^{-\frac{\mu}{2}} |Dw_m|^2 dz = 0,$$

where we have set  $w_m = u_m - \hat{u}$ .

**REMARK 4.4** *The restriction  $\rho < 1/3$  in the case  $q < 2$  is needed to apply the Sobolev–Poincaré type inequality, Theorem 2.4 of [CFM].*

**Proof of Proposition 4.3.** Again  $q \geq 2$  is the first case to consider, where the basic ideas are given for example in [EG1]. Here we argue exactly as in [FO], pp. 410, i. e. we use the minimality of  $u$  together with the convexity of  $f$ , and conclude for all  $\varphi \in C_0^1(B_1, \mathbb{R}^N)$ ,  $\varphi \geq 0$ ,

$$\begin{aligned} & \int_{B_1} \int_0^1 \varphi D^2 f(A_m + \lambda_m D\hat{u} + s \lambda_m Dw_m)(Dw_m, Dw_m)(1-s) ds dz \\ &= \lambda_m^{-2} \int_{B_1} \varphi \{f(A_m + \lambda_m Du_m) - f(A_m + \lambda_m D\hat{u})\} dz \\ & \quad - \lambda_m^{-1} \int_{B_1} \varphi Df(A_m + \lambda_m D\hat{u}) : Dw_m dz \\ & \leq c \left\{ \int_{B_1} |D\varphi|^2 |w_m|^2 dz + \lambda_m^{q-2} \int_{B_1} |D\varphi|^q |w_m|^q dz \right\} \tag{4.13} \\ & \quad + \lambda_m^{-1} \int_{B_1} Df(A_m + \lambda_m ((1-\varphi)Du_m + \varphi D\hat{u})) : \\ & \quad \quad (D\varphi \otimes (\hat{u} - u_m)) dz \\ & \quad - \lambda_m^{-1} \int_{B_1} \varphi Df(A_m + \lambda_m D\hat{u}) : Dw_m dz. \end{aligned}$$

Clearly, (4.13) is the analogue to inequality (6.6) in [FO]. As demonstrated in [FO] we can discuss the last two integrals on the right-hand side of (4.13) which finally bounds the left-hand side of (4.13) by the quantity  $c \{I_1 + I_2 + I_3\}$  where we have

$$I_1 := \int_{B_1} |D\varphi|^2 |w_m|^2 dz + \lambda_m^{q-2} \int_{B_1} |D\varphi|^q |w_m|^q dz \xrightarrow{m \rightarrow \infty} 0.$$

The limit behaviour follows from the weak convergence of  $u_m$  in  $W_2^1(B_1, \mathbb{R}^N)$  and from

$$\lambda_m^{1-\frac{2}{q}} w_m \rightarrow 0 \text{ in } L^q(B_1, \mathbb{R}^N) \quad \text{as } m \rightarrow \infty. \tag{4.14}$$

In fact, the latter convergence is obtained by (4.8) and by Poincaré's inequality which together with  $(w_m)_{0,1} = 0$  implies (4.14). Further we have

$$\begin{aligned} I_2 &:= \int_{B_1} |Dw_m| |D\varphi| |w_m| dz + \lambda_m^{q-2} \int_{B_1} |Dw_m|^{q-1} |D\varphi| |w_m| dz \\ &\leq \int_{B_1} |Dw_m| |D\varphi| |w_m| dz \\ &\quad + c(D\varphi) \left( \int_{B_1} \lambda_m^{q-2} |Dw_m|^q dz \right)^{\frac{q-1}{q}} \left( \int_{B_1} \lambda_m^{q-2} |w_m|^q dz \right)^{\frac{1}{q}} \end{aligned}$$

and again we use (4.14) to see  $I_2 \rightarrow 0$  as  $m \rightarrow \infty$ . The third part

$$I_3 := \left| \int_{B_1} \int_0^1 D^2 f(A_m + s \lambda_m D\hat{u})(D\hat{u}, D(\varphi w_m)) ds dz \right|$$

is immediately seen to vanish as  $m \rightarrow \infty$  and the proposition is proved if  $q \geq 2$ .

For  $q < 2$  we now benefit from [EM2] (compare [EV]) since the proof of higher integrability given in [CFM], Step 3, is adapted to balanced structure conditions. Thus, let for  $\xi \in \mathbb{R}^{nN}$

$$f_m(\xi) := \frac{f(A_m + \lambda_m \xi) - f(A_m) - \lambda_m Df(A_m) : \xi}{\lambda_m^2}$$

and define for  $0 < \rho < 1/3$ ,  $w \in W_{1,loc}^1(B_{1/3}, \mathbb{R}^N)$

$$I_\rho^m(w) := \int_{B_\rho} f_m(Dw) dz.$$

The first claim to prove is

$$\limsup_{m \rightarrow \infty} \left\{ I_\rho^m(u_m) - I_\rho^m(\hat{u}) \right\} \leq 0 \quad \text{for almost every } \rho \in (0, 1/3). \quad (4.15)$$

To verify (4.15) we fix  $\rho$  as above, choose  $0 < s < \rho$ ,  $\eta \in C_0^\infty(B_\rho)$ ,  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  on  $B_s$ ,  $|\nabla \eta| \leq c/(\rho - s)$  and define  $\varphi_m = (\hat{u} - u_m)\eta$ . Now,  $u_m$  obviously is a local minimizer of  $I_\rho^m$  and together with Lemma 3.3 of [CFM] this yields

$$\begin{aligned} I_\rho^m(u_m) - I_\rho^m(\hat{u}) &\leq I_\rho^m(u_m + \varphi_m) - I_\rho^m(\hat{u}) \\ &= \int_{B_\rho \sim B_s} \left[ f_m(Du_m + D\varphi_m) - f_m(D\hat{u}) \right] dz \leq \end{aligned}$$

$$\begin{aligned}
&\leq \frac{c(q, \Lambda, L)}{\lambda_m^2} \int_{B_\rho \sim B_s} \left[ |V(\lambda_m D\hat{u})|^2 \right. \\
&\quad \left. + |V(\lambda_m(\hat{u} - u_m) \otimes D\eta + \lambda_m \eta D\hat{u} + \lambda_m(1 - \eta) Du_m)|^2 \right] dz \\
&\leq \frac{c}{\lambda_m^2} \int_{B_\rho \sim B_s} \left[ |V(\lambda_m D\hat{u})|^2 + |V(\lambda_m Du_m)|^2 \right. \\
&\quad \left. + (\rho - s)^{-2} |V(\lambda_m(\hat{u} - u_m))|^2 \right] dz.
\end{aligned}$$

Next, a family of positive, uniformly bounded Radon measures  $\mu^m$  on  $B_{1/3}$  is introduced by letting

$$\mu^m(S) := \int_S \frac{1}{\lambda_m^2} \left[ |V(\lambda_m D\hat{u})|^2 + |V(\lambda_m Du_m)|^2 \right] dz.$$

We may assume that  $\mu^m$  converges in measure to a Radon measure  $\mu$  on  $B_{1/3}$ . Exactly as in [EM2], the Sobolev-Poincaré type inequality proved in [CFM], Theorem 2.4, gives for some  $1 < \theta < 2$

$$I_\rho^m(u_m) - I_\rho^m(\hat{u}) \leq c \left[ \mu^m(B_\rho \sim B_s) + (\rho - s)^{-2} \left( \int_{B_1} |u_m - \hat{u}| dz \right)^{2\theta} \right],$$

hence, by taking first the limit  $m \rightarrow \infty$  and then the limit  $s \uparrow \rho$ , we get (4.15) for any  $0 < \rho < 1/3$  such that  $\mu(\partial B_\rho) = 0$  which is true for a. a.  $\rho$ . Once (4.15) is established for some  $0 < \rho < 1/3$ , the following identity is the starting point to derive an estimate for the left-hand side:

$$\begin{aligned}
I_\rho^m(u_m) - I_\rho^m(\hat{u}) &= \lambda_m^{-2} \int_{B_\rho} \left[ f(A_m + \lambda_m Du_m) - f(A_m + \lambda_m D\hat{u}) \right. \\
&\quad \left. - \lambda_m Df(A_m) : Dw_m \right] dz \\
&= \lambda_m^{-1} \int_{B_\rho} \int_0^1 \left[ Df(A_m + \lambda_m D\hat{u} + t \lambda_m Dw_m) \right. \\
&\quad \left. - Df(A_m + \lambda_m D\hat{u}) \right] : Dw_m dt dz \\
&\quad + \lambda_m^{-1} \int_{B_\rho} \left[ Df(A_m + \lambda_m D\hat{u}) - Df(A_m) \right] : Dw_m dz \\
&=: (I)_m + (II)_m.
\end{aligned}$$

Local smoothness of  $\hat{u}$  immediately implies  $\lim_{m \rightarrow \infty} (II)_m = 0$ . On account of

$$\begin{aligned} (I)_m &= \int_{B_\rho} \int_0^1 \int_0^1 t D^2 f(A_m + \lambda_m D\hat{u} + st \lambda_m Dw_m)(Dw_m, Dw_m) ds dt dz \\ &\geq c \int_{B_\rho} (1 + |A_m + \lambda_m D\hat{u} + \lambda_m Dw_m|^2)^{-\frac{\mu}{2}} |Dw_m|^2 dz \end{aligned}$$

and by (4.15) the proposition is proved for almost all, hence for any  $\rho \in (0, 1/3)$ .  $\blacksquare$

Step 3a. (Strong convergence for  $q \geq 2$ )

**PROPOSITION 4.5** *In the case  $q \geq 2$  we have as  $m \rightarrow \infty$*

$$\begin{aligned} (i.) \quad & Du_m \rightarrow D\hat{u} \text{ in } L_{loc}^2(B_1, \mathbb{R}^{nN}); \\ (ii.) \quad & \lambda_m^{1-\frac{2}{q}} Du_m \rightarrow 0 \text{ in } L_{loc}^q(B_1, \mathbb{R}^{nN}) \text{ if } q > 2. \end{aligned} \tag{4.16}$$

**Proof of Proposition 4.5.** Here we have to distinguish two subcases: For  $\mu \leq 0$  the first convergence follows directly from Proposition 4.3. Using this fact, local smoothness of  $\hat{u}$  and again Proposition 4.3, the next conclusion is

$$\int_{B_\rho} \lambda_m^{-\mu} |Dw_m|^{2-\mu} dz \xrightarrow{m \rightarrow \infty} 0 \quad \text{for all } 0 < \rho < 1. \tag{4.17}$$

The proceed further, we introduce the auxiliary functions  $\psi_m$  (see [FO]),

$$\psi_m := \lambda_m^{-1} \left[ (1 + |A_m + \lambda_m Du_m|^2)^{\frac{2-\mu}{4}} - (1 + |A_m|^2)^{\frac{2-\mu}{4}} \right], \tag{4.18}$$

and by Lemma 3.6, (4.6), (4.8), (1.7) we can estimate ( $0 < \rho < 1$ )

$$\int_{B_\rho} |D\psi_m|^2 dz \leq c(\rho) \int_{B_1} |D^2 f(A_m + \lambda_m Du_m)| |Du_m|^2 dz \leq c(\rho).$$

If we now let  $\Theta(Z) := (1 + |Z|^2)^{(2-\mu)/4}$ ,  $Z \in \mathbb{R}^{nN}$ , then

$$\begin{aligned} |\psi_m| &= \lambda_m^{-1} \left| \int_0^1 \frac{d}{dt} \Theta(A_m + t \lambda_m Du_m) dt \right| \\ &\leq c \left| \int_0^1 Du_m : D\Theta(A_m + t \lambda_m Du_m) dt \right| \\ &\leq c \int_0^1 |Du_m| (1 + |A_m + t \lambda_m Du_m|^2)^{-\frac{\mu}{4}} dt \\ &\leq c \left( |Du_m| + \lambda_m^{-\frac{\mu}{2}} |Du_m|^{1-\frac{\mu}{2}} \right). \end{aligned}$$



With this inequality we obtain

$$\int_{B_\rho} |\psi_m|^2 dz \leq c(\rho) \quad \text{for all } 0 < \rho < 1. \quad (4.19)$$

In fact, (4.19) is obvious for  $\mu = 0$ . If  $\mu < 0$ , then (4.19) is just a consequence of (4.17). Thus, we have proved that

$$\sup_m \|\psi_m\|_{W_2^1(B_\rho)} \leq c(\rho) < \infty \quad \text{for all } 0 < \rho < 1 \quad (4.20)$$

and this will imply (4.16), (ii.): to this purpose we fix some real number  $M \gg 1$  and let  $U_m = U_m(M, \rho) := \{z \in B_\rho : \lambda_m |Du_m| \leq M\}$ . On one hand, local  $L^2$ -convergence and  $q > 2$  prove

$$\begin{aligned} \int_{U_m} \lambda_m^{q-2} |Du_m|^q dz &\leq \int_{U_m} \lambda_m^{q-2} |Dw_m|^q dz + \int_{U_m} \lambda_m^{q-2} |D\hat{u}|^q dz \\ &\leq c \int_{U_m} \lambda_m^{q-2} \left( |Du_m|^{q-2} + |D\hat{u}|^{q-2} \right) |Dw_m|^2 \\ &\quad + \int_{U_m} \lambda_m^{q-2} |D\hat{u}|^q dz \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned} \quad (4.21)$$

On the other hand, observe that for  $M$  sufficiently large and for  $z \in B_\rho \sim U_m$

$$\begin{aligned} \psi_m(z) &\geq c \lambda_m^{-1} \lambda_m^{\frac{2-\mu}{2}} |Du_m(z)|^{\frac{2-\mu}{2}}, \quad \text{i. e.} \\ \lambda_m^{q-2+\frac{\mu q}{2-\mu}} \psi_m^{\frac{2q}{2-\mu}}(z) &\geq c \lambda_m^{q-2} |Du_m(z)|^q. \end{aligned}$$

Since (1.8) guarantees  $2q/(2-\mu) < 2n/(n-2)$ , since by (4.20)  $\psi_m$  is uniformly bounded in  $L^{2n/(n-2)}$  and since  $q-2+\mu q/(2-\mu) \geq 0$  follows from  $q \geq 2-\mu$ , we can conclude

$$\int_{B_\rho \sim U_m} \lambda_m^{q-2} |Du_m|^q dz \rightarrow 0 \quad \text{for all } 0 < \rho < 1 \quad (4.22)$$

as  $m \rightarrow \infty$ . Summarizing the results, (4.21) and (4.22) prove Propostion 4.5 in the case  $\mu \leq 0$ .

Now suppose that  $\mu > 0$ . Proposition 4.3 implies in the case at hand for any  $0 < \rho < 1$

$$\int_{B_\rho} (1 + |\lambda_m Dw_m|^2)^{-\frac{\mu}{2}} |Dw_m|^2 dz \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

which immediately gives

$$\int_{U_m} |Dw_m|^2 dz \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (4.23)$$

Here  $U_m$  is defined as above for fixed  $M$  and  $\rho$ . Also as above we introduce  $\psi_m$  and observe that now  $|\psi_m| \leq c|Du_m|$  is obvious, i. e. (4.20) remains to be true in the case  $\mu > 0$ . If  $M$  is chosen sufficiently large, then

$$|\psi_m|^{\frac{4}{2-\mu}} \lambda_m^{\frac{2\mu}{2-\mu}} \geq |Du_m|^2 \quad \text{on } B_\rho \sim U_m,$$

and since  $4/(2-\mu) \leq 2n/(n-2) \Leftrightarrow \mu \leq 4/n$ , the last inequality being true on account of  $q \geq 2$ , we get

$$\int_{B_\rho \sim U_m} |Dw_m|^2 dz \xrightarrow{m \rightarrow \infty} 0 \quad \text{for all } 0 < \rho < 1. \quad (4.24)$$

With (4.23) and (4.24) the first claim of (4.16) also is proved in the case  $\mu > 0$ . (4.16), (ii.), for  $\mu > 0$  follows exactly as for the case  $\mu \leq 0$  and the proof of the proposition is complete.  $\blacksquare$

Step 3b. (Strong convergence for  $q < 2$ )

**PROPOSITION 4.6** *If  $q < 2$ , then for any  $0 < \rho < 1/3$*

$$\lim_{m \rightarrow \infty} \frac{1}{\lambda_m^2} \int_{B_\rho} |V(\lambda_m Dw_m)|^2 dz = 0.$$

**Proof of Proposition 4.6.**

In the subquadratic case, the auxiliary function  $\psi_m$  introduced in (4.18) is handled via Lemma 2.1, (vi.) of [CFM]. We have

$$\begin{aligned} \int_{B_\rho} |D\psi_m|^2 dz &\leq c \int_{B_1} (1 + |\lambda_m Du_m|^2)^{\frac{q-2}{2}} |Du_m|^2 dz \\ &\leq \frac{c}{\lambda_m^2} \int_{B_1} |V(\lambda_m Du_m)|^2 dz \\ &\leq \frac{c}{\lambda_m^2} \int_{B(x_m, R_m)} |V(Du - A_m)|^2 dx \\ &\leq \frac{c(L)}{\lambda_m^2} \int_{B(x_m, R_m)} |V(Du) - V(A_m)|^2 dx \leq \text{const.} \end{aligned}$$

for any  $0 < \rho < 1$ . In addition we have  $|\psi_m| \leq c |Du_m|$ , hence  $\psi_m \in W_{q,loc}^1(B_1)$ , thus  $\psi_m \in L_{loc}^{q_1}(B_1)$  with  $q_1 := nq/(n-q)$ . Iterating this argument we again have verified

$$\sup_m \|\psi_m\|_{W_2^1(B_\rho)} \leq c(\rho) < \infty \quad \text{for all } 0 < \rho < 1. \quad (4.25)$$

Assume now that  $0 < \rho < 1/3$ . With  $M$  and  $U_m$  as before, (4.23) is again a consequence of Proposition 4.3. Let us write ([CFM], Lemma 2.1)

$$\begin{aligned} \frac{1}{\lambda_m^2} \int_{B_\rho} |V(\lambda_m Dw_m)|^2 dz &\leq \frac{c}{\lambda_m^2} \int_{U_m} |V(\lambda_m Dw_m)|^2 dz \\ &\quad + \frac{c}{\lambda_m^2} \int_{B_\rho \sim U_m} |V(\lambda_m Du_m)|^2 dz \\ &\quad + \frac{c}{\lambda_m^2} \int_{B_\rho \sim U_m} |V(\lambda_m D\hat{u})|^2 dz. \end{aligned}$$

Then, by (4.23)

$$\begin{aligned} \frac{1}{\lambda_m^2} \int_{U_m} |V(\lambda_m Dw_m)|^2 dz &\leq \int_{U_m} |Dw_m|^2 (1 + \lambda_m^2 |Dw_m|^2)^{\frac{q-2}{2}} dz \\ &\leq \int_{U_m} |Dw_m|^2 dz \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

The second term vanishes as  $m \rightarrow \infty$  provided that

$$\int_{B_\rho \sim U_m} \lambda_m^{q-2} |Du_m|^q dz \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

To see this we recall the estimates for  $\psi_m$  stated after (4.21) being valid also in the case under consideration and with the same reasoning we obtain (4.22) where now we make use of the apriori bound (4.25). Finally, we use the local boundedness of  $D\hat{u}$  to see

$$\begin{aligned} \frac{1}{\lambda_m^2} \int_{B_\rho \sim U_m} |V(\lambda_m D\hat{u})|^2 dz &\leq \int_{B_\rho \sim U_m} |D\hat{u}|^2 dz \\ &\leq \|D\hat{u}\|_{L^\infty(B_\rho, \mathbb{R}^{nN})}^2 |B_\rho \sim U_m| \xrightarrow{m \rightarrow \infty} 0 \end{aligned}$$

on account of  $\lambda_m Du_m \rightarrow 0$  a. e. on  $B_1$  as  $m \rightarrow \infty$  (see (4.11)). This completes the proof of Proposition 4.6.  $\blacksquare$

Step 4. (Conclusion) Proposition 4.5 together with (4.5) gives in the case  $q \geq 2$

$$\int_{B_\tau} |D\hat{u} - (D\hat{u})_\tau|^2 dz \geq C_* \tau^2,$$

thus we have a contradiction to (4.12) if we choose  $C_* = 2C^*$ .

If  $q < 2$ , then we estimate according to [CFM], p. 24,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{E^-(x_m, \tau R_m)}{\lambda_m^2} &\leq \lim_{m \rightarrow \infty} \frac{c}{\lambda_m^2} \int_{B_\tau} \left\{ |V(\lambda_m Dw_m)|^2 \right. \\ &\quad \left. + |V(\lambda_m (D\hat{u} - (D\hat{u})_\tau))|^2 \right. \\ &\quad \left. + |V(\lambda_m ((D\hat{u})_\tau - (Du_m)_\tau))|^2 \right\} dz, \end{aligned}$$

where the first integral is handled by Proposition 4.6. The last one vanishes when passing to the limit  $m \rightarrow \infty$  since we may first estimate

$$\int_{B_\tau} |V(\lambda_m ((D\hat{u})_\tau - (Du_m)_\tau))|^2 dz \leq \lambda_m^2 \int_{B_\tau} |(D\hat{u})_\tau - (Du_m)_\tau|^2 dz$$

and then use (4.10) for the right-hand side. The second integral again is estimated by (4.12). Thus, choosing  $C_*$  sufficiently large we also get the contradiction in the case  $q < 2$  and the lemma is proved.  $\blacksquare$

## References

- [AF1] Acerbi, E., Fusco, N., A regularity theorem for minimizers of quasiconvex integrals. Arch. Rat. Mech. Anal. 99 (1987), 261–281.
- [AF2] Acerbi, E., Fusco, N., Local regularity for minimizers of non convex integrals. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 16, No. 4 (1989), 603–636.
- [AF3] Acerbi, E., Fusco, N., Regularity for minimizers of non-quadratic functionals. The case  $1 < p < 2$ . J. Math. Anal. Appl. 140 (1989), 115–135.
- [AF4] Acerbi, E., Fusco, N., Partial regularity under anisotropic  $(p, q)$  growth conditions. J. Diff. Equ. 107, No. 1 (1994), 46–67.
- [AG] Anzellotti, G., Giaquinta, M., Convex functionals and partial regularity. Arch. Rat. Mech. Anal. 102 (1988), 243–272.

- [BFM] Bildhauer, M., Fuchs, M., Mingione, G., Apriori gradient bounds and local  $C^{1,\alpha}$ -estimates for (double) obstacle problems under nonstandard growth conditions. to appear.
- [CA] Campanato, S., Hölder continuity of the solutions of some non-linear elliptic systems. *Adv. Math.* 48 (1983), 16–43.
- [CF] Cianchi, A., Fusco, N., Gradient regularity for minimizers under general growth conditions. *J. reine angew. Math.* 507 (1999), 15–36.
- [CFM] Carozza, M., Fusco, N., Mingione, G., Partial regularity of minimizers of quasiconvex integrals with subquadratic growth. *Ann. Mat. Pura e Appl.* (1998).
- [ELM] Esposito, L., Leonetti, F., Mingione, G., Regularity results for minimizers of irregular integrals with  $(p,q)$ -growth. to appear.
- [EM1] Esposito, L., Mingione, G., Some remarks on the regularity of weak solutions of degenerate elliptic systems. *Rev. Mat. Complut.* 11, No.1 (1998), 203–219.
- [EM2] Esposito, L., Mingione, G., Partial regularity for minimizers of convex integrals with  $L \log L$ -growth. *Nonlinear Diff. Equ. Appl.* 7, No. 4 (2000), 157–175.
- [EV] Evans, L. C., Quasiconvexity and partial regularity in the calculus of variations. *Arch. Rat. Mech. Anal.* 95 (1986), 227–252.
- [EG1] Evans, L. C., Gariepy, R., Blowup, compactness and partial regularity in the calculus of variations. *Indiana Univ. Math. J.* 36 (1987), 361–371.
- [EG2] Evans, L. C., Gariepy, R., Measure theory and fine properties of functions. *Studies in Advanced Mathematics*, CRC Press, Boca Raton–Ann Arbor–London 1992.
- [FH] Fusco, N., Hutchinson, J. E.,  $C^{1,\alpha}$  partial regularity of functions minimizing quasiconvex integrals. *Manus. Math.* 54 (1985), 121–143.
- [FM] Fuchs, M., Mingione, G., Full  $C^{1,\alpha}$ -regularity for free and constrained local minimizers of variational integrals with nearly linear growth. *Manus. Math.* 102 (2000), 227–250.
- [FS] Fuchs, M., Seregin, G., A regularity theory for variational integrals with  $L \log L$ -growth. *Calc. Var.* 6 (1998), 171–187.
- [FO] Fuchs, M., Osmolovski, V., Variational integrals on Orlicz–Sobolev spaces. *Z. Anal. Anw.* 17 (1998), 393–415.
- [Gi] Giaquinta, M., Multiple integrals in the calculus of variations and nonlinear elliptic systems. *Ann. Math. Studies* 105, Princeton University Press, Princeton 1983.
- [GM] Giaquinta, M., Modica, G., Remarks on the regularity of the minimizers of certain degenerate functionals. *Manus. Math.* 57 (1986), 55–99.
- [M1] Marcellini, P., Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Arch. Rat. Mech. Anal.* 105 (1989), 267–284.

- [M2] Marcellini, P., Regularity and existence of solutions of elliptic equations with  $(p, q)$ -growth conditions. *J. Diff. Equ.* 90 (1991), 1–30.
- [M3] Marcellini, P., Regularity for elliptic equations with general growth conditions. *J. Diff. Equ.* 105 (1993), 296–333.
- [M4] Marcellini, P., Everywhere regularity for a class of elliptic systems without growth conditions. *Ann. Scuola Norm. Sup. Pisa* 23 (1996), 1–25.
- [MO] Morrey, C. B., Multiple integrals in the calculus of variations. *Grundlehren der math. Wiss. in Einzeldarstellungen* 130, Springer, Berlin–Heidelberg–New York 1966.
- [MS] Mingione, G., Siepe, F., Full  $C^{1,\alpha}$  regularity for minimizers of integral functionals with  $L \log L$  growth. *Z. Anal. Anw.* 18 (1999), 1083–1100.
- [PS] Passarelli Di Napoli, A., Siepe, F., A regularity result for a class of anisotropic systems. *Rend. Ist. Mat. Univ. Trieste* 28, No.1-2 (1996), 13–31.