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# On local generalized minimizers and local stress tensors for variational problems with linear growth

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**Abstract**

Uniqueness and regularity results for local vector-valued generalized minimizers and for local stress tensors associated to variational problems with linear growth conditions are established. Assuming that the energy density $f$ has the structure $f(Z) = h(|Z|)$, only very weak ellipticity assumptions are required. For the proof we combine arguments from measure theory and convex analysis with the regularity results of [ABF].

# 1 Introduction

Let us first consider the **global** minimization problem

$$J[w,\Omega] = \int_\Omega f(\nabla w)\,\mathrm{d}x \to \min \tag{$\mathcal{P}$}$$

w.r.t. prescribed Dirichlet boundary data $u_0 \in W^1_1(\Omega;\mathbb{R}^N)$, $N \geq 1$, where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain and where $f$ is supposed to be a convex function of linear growth, i.e. with constants $a$, $A > 0$ and $b$, $B \in \mathbb{R}$ we have

$$a|Z| + b \leq f(Z) \leq A|Z| + B \quad \text{for all } \; Z \in \mathbb{R}^{nN} . \tag{1.1}$$

Moreover, we assume that $f$ satisfies suitable smoothness and ellipticity conditions to be made precise in Section 2.

The most prominent example is given by the minimal surface integrand $f(Z) = \sqrt{1+|Z|^2}$ which is discussed in numerous contributions. Here we just mention the works of DeGiorgi (see [Gio] for selected papers), of Ladyzhenskaya, Ural'tseva [LU], of Simon [Si], of Giaquinta, Modica, Souček [GMS] and the monograph of Giusti [Gi].

For linear growth problems arising from physical applications we refer to the works of Anzellotti, Giaquinta [AG1], [AG2], of Strang, Temam [ST], of Suquet [Su] and of Seregin on perfect plasticity [Se1]–[Se4]. We also refer to the monographs of Temam [Te] and of Fuchs, Seregin [FS]. Moreover, the theory of perfectly plastic fluids proposed by v. Mises [Mi] has recently been discussed by Naumann and Bildhauer [BN].

These different examples have one essential point in common: since the energy densities are just of linear growth, the natural classes to work in are non-reflexive spaces like $u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$ and in general one cannot expect to find a minimizer within such classes of comparison functions.

There are two known ways to overcome this difficulty.

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a) The first possibility usually is prefered in the minimal surface case or in the setting of related geometric problems: the functional $J$ is relaxed to the space $BV(\Omega;\mathbb{R}^N)$ of functions of bounded variation, which, following the representation formula of Goffman and Serrin ([GS]), means to consider the functional ($\Omega^{ext} \Supset \Omega$)

$$\hat{J}[w,\Omega^{ext}] \ := \ \int_{\Omega^{ext}} f(\nabla^a w)\,\mathrm{d}x + \int_{\Omega^{ext}} f_\infty\big(\nabla^s w/|\nabla^s w|\big)\,\mathrm{d}|\nabla^s w|$$

for comparison functions $w \in BV(\Omega;\mathbb{R}^N)$ which are **extended by the boundary data** $u_0$ on $\Omega^{ext} - \Omega$. Here the absolutely continuous part of $\nabla w$ w.r.t. the Lebesgue measure is denoted by $\nabla^a w$, the singular part by $\nabla^s w$ and $\nabla^s w/|\nabla^s w|$ is the Radon-Nikodym derivative. The symbol $f_\infty$ stands for recession function (w.l.o.g. $f(0) = 0$)

$$f_\infty(Z) := \lim_{t\to\infty} \frac{f(tZ)}{t}\,.$$

The relaxation to the space $BV$ is done in order to apply the lower semicontinuity theorem of Reshetnyak [Re], which ensures the existence of generalized minimizers by the direct method of the calculus of variations.

b) The mechanical point of view is emphasized in the second possibility: the physical quantity of interest is not the strain but the stress tensor, which is a solution of the dual variational problem **w.r.t. the boundary data** $u_0$. In this case methods from convex analysis provide the existence of a dual solution.

Once the existence of a generalized minimizer or of a dual solution is established, we are interested in uniqueness results. For generalized minimizers we just can hope for uniqueness up to a constant, see [Gi], Example 15.12, p.180. The uniqueness of the stress tensor follows by assuming the strict convexity of the conjugate function of $f$, see [ET], Chapter V, Section 3.2, which in general is hard to verify. An approach based on a natural upper ellipticity bound for $D^2 f$ is given in [Bi1].

Of course it remains to study the smoothness properties of generalized minimizers and the dual solution based on suitable structure and ellipticity assumptions. For an overview, some recent results and a list of references we refer to [Bi2].

Let us finally emphasize that both pictures a) and b) are strongly related through a suitable variant of the duality relation

$$\sigma = \nabla f(\nabla u)$$

being valid for the stress tensor $\sigma$ and a generalized solution $u$.

In this note we are interested in a **local theory** not depending on global boundary data and applicable to energy densities $f$ under quite **weak ellipticity assumptions**, but assuming that $f$ is of special structure in the sense that

$$f(Z) = h(|Z|) \quad \text{for a function } \; h : \mathbb{R} \to \mathbb{R}\,. \tag{1.2}$$

There are several essential problems in the study of the local situation under weak ellipticity assumptions.

i) In order to establish a priori bounds following the lines of [ABF] or [MP], we first have to define a local regularization. Doing so, it is necessary to introduce a local Dirichlet problem. Please note that in this step the arguments of [ABF], i.e. to consider mollified boundary data of a given local minimizer, do not work in the space of functions with bounded variation.

ii) Given $\tilde{\Omega} \Subset \Omega$, the interior and the exterior trace on $\partial\tilde{\Omega}$ of a function $w \in BV(\Omega;\mathbb{R}^N)$ in general are different and the relaxed functional is also supported on $\partial\tilde{\Omega}$. Hence, at the first glance it is not clear (and could not be traced in the literature) how to define a suitable local Dirichlet problem. This in particular means that up to now there is no notion of an appropriate local stress tensor $\sigma_{\tilde{\Omega}}$. Both the local Dirichlet problem and the corresponding dual solution $\sigma_{\tilde{\Omega}}$ are introduced in the following.

iii) It is crucial to prove the uniqueness of $\sigma_{\tilde{\Omega}}$.

iv) Even if we have the uniqueness of $\sigma_{\tilde{\Omega}}$, we need the duality relation

$$\sigma_{\tilde{\Omega}} = \nabla f(\nabla u) \quad \text{on } \; \tilde{\Omega}$$

**for any** generalized local minimizer $u$ in order to show the convergence of the regularization introduced in i).

v) Concerning iii) and iv) we emphasize that the arguments given in [Bi2] cannot be carried over since they strongly depend on the $W^1_{2,loc}$-regularity of the (global) stress tensor. In the situation at hand however, the ellipticity assumptions on the energy density are too weak to imply this starting regularity.

Following more or less i)–v), our note is divided into a series of short sections which are based on different kinds of arguments. Nevertheless, each section relies on the previous one and their order is very essential.

# 2 Notation and main result

First we introduce the notion of a local minimizer of the functional $J[\cdot,\Omega]$ in the $BV$-setting. Exactly as in the global situation we let for any bounded Lipschitz domain $\tilde{\Omega} \subset \Omega$ and any function $w \in BV(\Omega;\mathbb{R}^N)$

$$\hat{J}[w,\tilde{\Omega}] := \int_{\tilde{\Omega}} f(\nabla^a w)\,\mathrm{d}x + \int_{\tilde{\Omega}} f_\infty\big(\nabla^s w/|\nabla^s w|\big)\,\mathrm{d}|\nabla^s w|\,,$$

where the recession function is defined as above and where we assume that $f$ satisfies Assumption 2.1 stated below.

**Definition 2.1.** *A function $u \in BV(\Omega;\mathbb{R}^N)$ is called a local $\hat{J}$-minimizer on $\Omega$ if for any $w \in BV(\Omega;\mathbb{R}^N)$ such that* $\mathrm{spt}\,(u-w) \Subset \Omega$

$$\hat{J}[u,\Omega] \le \hat{J}[w,\Omega]\,.$$

**Remark 2.1.** *a) For a rigorous definition of a local minimizer we should suppose $u \in BV_{loc}(\Omega;\mathbb{R}^N)$ and replace $\Omega$ in the following by a domain $\Omega' \Subset \Omega$. This however would just produce an even more technical notation without essential changes in our local arguments and therefore we work with Definition 2.1.*

*b) If a local $\hat{J}$-minimizer is supposed to be of class $W^1_{1,loc}(\Omega;\mathbb{R}^n)$ and if we restrict the admissible comparison functions to this class, then Definition 2.1 reduces to the usual definition of a local $J$-minimizer within this Sobolev class.*

Our general assumption on the energy density $f$ is given in (compare hypothesis (H1) and (H3) of [ABF])

**Assumption 2.1.** *Suppose that $f$ is of class $C^2$ and that we have the structure condition (1.2) with*

$$\begin{gathered}\text{$h$ is strictly increasing and $h''(t) > 0$ for all $t > 0$ together with}\\ \lim_{t\to 0}\frac{h(t)}{t} = 0 \quad\text{and}\quad \lim_{t\to\infty}\frac{h(t)}{t} = \bar{c} \in (0,\infty)\,.\end{gathered} \tag{2.1}$$

*(Note that the first requirement in (2.1) is a consequence of the second and the third one. Note also that the last condition stated in (2.1) implies the linear growth of $f$.)*

*Moreover, suppose that*

$$\begin{gathered}\text{there exist $\bar{\varepsilon}$, $\bar{h} > 0$, $T_0$, $\kappa$, $\mu \ge 0$ such that for all $t \ge T_0$}\\ \bar{\varepsilon}\frac{h'(t)}{t}(1+t^2)^{-\frac{\mu}{2}} \le h''(t) \le \bar{h}(1+t^2)^{-\frac{\kappa}{2}}h(t)\,.\end{gathered} \tag{2.2}$$

**Remark 2.2.** *Condition (2.2) is inspired by the hypotheses (2.9) and (2.13) of [MP], which is outlined in the next remark, and we will use (2.2) for the study of the regularity properties of locally bounded local minima as done in [ABF] for the superlinear case. The boundedness assumption is justified in Remark 2.6, and without this requirement it is possible to adjust our arguments in the spirit of [MP] which will lead us to assumptions like (2.2) but with exponents depending on the dimension $n$.*

**Remark 2.3.** *Since $h$ is convex with $h(0) = 0$, we have $0 \ge h(t) - th'(t)$ and $h(t) \ge h(t/2) + (t/2)h'(t/2)$, i.e.*

$$\frac{h(t)}{t} \le h'(t) \le 2\frac{h(2t)}{2t}$$

*and therefore by the last assumption in (2.1)*

$$0 \le h'(t) \le C \quad \textit{for all } t \ge 0 \tag{2.3}$$

*for a suitable constant $C$. Since $h'$ is increasing, (2.3) clearly gives*

$$0 < h'(1) \le h'(t) \le C \quad \textit{for all } t \ge 1\,.$$

*Due to this observation we can replace (2.2) by the equivalent requirement*

$$\begin{gathered} \textit{there exist } \bar{\varepsilon},\ \bar{h} > 0,\ T_0 \ge 1,\ \kappa,\ \mu \ge 0 \textit{ such that for all } t \ge T_0 \\ \bar{\varepsilon} t^{-1-\mu} \le h''(t) \le \bar{h} t^{1-\kappa}\,. \end{gathered} \tag{2.4}$$

*Note that (2.4) gives $\kappa \le 2 + \mu$. We observe that (2.4) is related to inequality (2.13) of [MP] and reduces to this inequality if we let $\kappa = 2$, $\mu + 1 =: \gamma \in [1, 1 + 2/n)$, i.e. $\mu \in [0, 2/n)$. This choice of the parameters in connection with linear growth problems has also been discussed in the paper [BF3].*

**Remark 2.4.** *The structure condition (1.2) implies for all $Z$, $Y \in \mathbb{R}^{nN}$*

$$\min\Big[\frac{h'(|Z|)}{|Z|}, h''(|Z|)\Big]|Y|^2 \le D^2 f(Z)(Y,Y) \le \max\Big[\frac{h'(|Z|)}{|Z|}, h''(|Z|)\Big]|Y|^2\,,$$

*so that by (2.3) and (2.4)*

$$D^2 f(Z)(Y,Y) \le \Lambda(1 + |Z|^2)^{\frac{q-2}{2}}|Y|^2$$

*for some positive constant $\Lambda$ and for some exponent $q \ge 1$. On account of the weak ellipticity assumptions imposed on $f$ we cannot suppose $q = 2$, which is of particular importance for defining a suitable regularization (a quadratic regularization usually is applied for problems with linear growth).*

Now the main theorem reads as

**Theorem 2.1.** *Suppose that Assumption 2.1 is satisfied and that $u \in BV(\Omega;\mathbb{R}^N)$ is a locally bounded local $\hat{J}$-minimizer in the sense of Definition 2.1. If*

$$2\mu < \kappa\,,$$

*then $|\nabla u|$ is in the space $L^\infty_{loc}(\Omega)$. Moreover, for all $\tilde{\Omega} \Subset \Omega$ there exists a unique stress tensor $\sigma_{\tilde{\Omega}}$ satisfying the duality relation*

$$\sigma_{\tilde{\Omega}} = \nabla f(\nabla u) \quad \textit{a.e. in } \tilde{\Omega}\,.$$

*By definition, this local stress tensor arises as the solution of a suitable dual variational problem (see Section 5 and Section 7).*

**Remark 2.5.** *a) It is remarkable that the minimal surface case, which is discussed several times in [Bi2] as a limit case, again exactly corresponds to the limit case $\mu = 2$, $\kappa = 4$ in the situation at hand although in general we now have weaker ellipiticty assumptions. We mention the good correspondence to the counterexample sketched in Section 4.4 of [Bi2] based on the ideas of [GMS].*

*b) Our requirement $2\mu < \kappa$ (together with $\kappa < 2 + \mu$) gives the restriction $\mu < 2$.*

**Remark 2.6.** *If we return to the global problem ($\mathcal{P}$) and if we assume that $u_0$ is a bounded function, then the maximum principle established in [BF4] shows the boundedness of any generalized solution. Therefore our assumption $u \in L^\infty_{loc}(\Omega; \mathbb{R}^N)$ for the local minimizer $u$ discussed in Theorem 2.1 is quite natural.*

Before we are going to prove our main theorem we like to recall the following measure theoretic preliminaries:

**Preliminaries 2.1.** *a) The exterior trace $tr(w)^{ext}$ on $\partial B$ of a function $w \in BV(\Omega; \mathbb{R}^N)$ and similarly the interior trace $tr(w)^{int}$ on $\partial B$ w.r.t. a ball $B \Subset \Omega$ are defined according to [Gi], Theorem 2.10. For Sobolev functions $v$ both traces coincide and we just use the symbol $tr(v)$.*

*b) Both $tr(w)^{ext}$ and $tr(w)^{int}$ are of class $L^1(\partial B; \mathbb{R}^N)$.*

*c) Following [Gi], Theorem 2.16, we may choose a function $\psi \in W^1_1(B; \mathbb{R}^N)$ such that $tr(\psi) = tr(w)^{ext}$.*

*d) The function $\psi$ can be extended to a function (again denoted by $\psi$) $\psi \in \overset{\circ}{W}{}^1_1(\Omega; \mathbb{R}^N)$.*

*e) In particular we note that the $W^1_1$-norm of $\psi$ is bounded on the whole domain $\Omega$.*

We finally introduce the following notation: suppose that $B := B_r(x_0) \Subset \Omega$ and that $w \in BV(B; \mathbb{R}^N)$, $\varphi \in BV(\Omega; \mathbb{R}^N)$. Then we let

$$w_\varphi := \begin{cases} w & \text{on} \quad B\,, \\ \varphi & \text{on} \quad \Omega - B \end{cases}$$

and

$$BV_\varphi(B; \mathbb{R}^N) := \{w_\varphi : \, w \in BV(B; \mathbb{R}^N)\}\,.$$

**Remark 2.7.** *Although the notation introduced in [Bi2], Appendix A, is motivated by another (a global) point of view, we strictly follow this notation in order to have precise references. We emphasize that the space $BV_\varphi(B; \mathbb{R}^N)$ by definition is a subspace of $BV(\Omega; \mathbb{R}^N)$.*

# 3 Local minimizers solve a Dirichlet problem w.r.t. a $W_1^1$-trace induced from the exterior

In the following we fix a ball $B = B_r(x_0) \Subset \Omega$ and a local $\hat{J}$-minimizer $u$ defined on $\Omega$. For our purposes it is very important to have a rigorous proof that $u$ satisfies on $B$ a Dirichlet problem for boundary data induced by the **exterior trace** of $u$ and that this boundary data are induced by a function of class $\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$.

**Lemma 3.1.** *Suppose that $u$ is a local $\hat{J}$-minimizer on $\Omega$. Associated to $u$ we choose $\varphi \in \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$ following Preliminaries 2.1, c), d), and define for $w \in BV(B;\mathbb{R}^N)$ the relaxed local functional $\hat{J}_\varphi[w,B]$ w.r.t. the generalized notion of Dirichlet boundary data $\varphi$, i.e. we let*

$$\begin{aligned}\hat{J}_\varphi[w,B] \ :=\ & \int_B f(\nabla^a w)\,\mathrm{d}x + \int_B f_\infty\big(\nabla^s w/|\nabla^s w|\big)\,\mathrm{d}|\nabla^s w| \\ & + \int_{\partial B} f_\infty((tr(\varphi) - tr(w)^{int}) \otimes \nu)\,\mathrm{d}\mathcal{H}^{n-1}\,,\end{aligned}$$

*where $\nu$ denotes the outer unit normal to $\partial B$. Then $u$ minimizes $\hat{J}_\varphi[\cdot,B]$ w.r.t. all comparison functions $w \in BV(B;\mathbb{R}^N)$.*

*Proof.* Suppose that the lemma is false. Then there exists a function $v \in BV(B;\mathbb{R}^N)$ such that $\hat{J}_\varphi[v,B] < \hat{J}_\varphi[u,B]$.

Moreover, recalling that on $\partial B$ we have $tr(\varphi) = tr(u)^{ext} = tr(w_u)^{ext}$ for any $w_u \in BV_u(B;\mathbb{R}^N)$, we note that for any $w \in BV(B,\mathbb{R}^N)$ it holds (see, e.g., [AFP], Theorem 3.77, p.171)

$$\begin{aligned}\hat{J}_\varphi[w,B] \ =\ & \int_B f(\nabla^a w_u)\,\mathrm{d}x + \int_B f_\infty\big(\nabla^s w_u/|\nabla^s w_u|\big)\,\mathrm{d}|\nabla^s w_u| \\ & + \int_{\partial B} f_\infty((tr(w_u)^{ext} - tr(w_u)^{int}) \otimes \nu)\,\mathrm{d}\mathcal{H}^{n-1} \\ =\ & \hat{J}[w_u,\Omega] - \hat{J}[w_u,\Omega - \overline{B}]\,,\end{aligned}$$

i.e. with $\varkappa = \varkappa(u,B) := \hat{J}[u,\Omega - \overline{B}]$ independent of $w$ we get

$$\hat{J}_\varphi[w,B] + \varkappa = \hat{J}[w_u,\Omega] \quad \text{for all } \ w \in BV(B;\mathbb{R}^N)\,,\ \text{ i.e. for all } \ w_u \in BV_u(B;\mathbb{R}^N)\,.$$

This means

$$\hat{J}[v_u,\Omega] = \hat{J}_\varphi[v,B] + \varkappa < \hat{J}_\varphi[u,B] + \varkappa = \hat{J}[u,\Omega]$$

and we immediately obtain a contradiction to the local minimality of $u$ w.r.t. the functional $\hat{J}[w,\Omega]$. □

**Corollary 3.1.** *Consider $B$, $u$ and $\varphi$ as in Lemma 3.1. Then $u_\varphi$ minimizes $\hat{J}[\cdot,\Omega]$ w.r.t. all comparison functions $w_\varphi \in BV_\varphi(B;\mathbb{R}^N)$.*

*Proof.* As above we have using $tr(\varphi) = tr(u)^{ext} = tr(w_\varphi)^{ext}$ for any $w \in BV(B;\mathbb{R}^N)$ on $\partial B$

$$\hat{J}_\varphi[w,B] + \tilde{\varkappa} = \hat{J}[w_\varphi,\Omega] \quad \text{for all } w \in BV(B;\mathbb{R}^N)\,, \ \text{ i.e. for all } w_\varphi \in BV_\varphi(B;\mathbb{R}^N)\,,$$

where $\tilde{\varkappa} = \tilde{\varkappa}(\varphi, B) := \hat{J}[\varphi, \Omega - \overline{B}]$. Lemma 3.1 shows for all $w \in BV(B;\mathbb{R}^N)$

$$\hat{J}[u_\varphi,\Omega] = \hat{J}_\varphi[u,B] + \tilde{\varkappa} \le \hat{J}_\varphi[w,B] + \tilde{\varkappa} = \hat{J}[w_\varphi,\Omega]$$

and the corollary is proved. □

**Remark 3.1.** *With Corollary 3.1 we are exactly in the situation studied in Appendix A of [Bi2] (see also [BF2]), i.e. on B u is a solution of a generalized Dirichlet problem with boundary data $\varphi$ of class $W^1_1(B;\mathbb{R}^N)$. Then the boundary data as well as the relaxed functional are extended to the exterior.*

Finally we consider the Dirichlet problem

$$J[w,B] \to \min \quad \text{in } \varphi + \overset{\circ}{W}{}^1_1(B;\mathbb{R}^N) \qquad (\mathcal{P}_\varphi)$$

and let

$$\mathcal{M} \ := \ \{v \in BV(B;\mathbb{R}^N) : v \text{ is the } L^1\text{-limit of a } J[\cdot,B]\text{-minimizing sequence from } \varphi + \overset{\circ}{W}{}^1_1(B;\mathbb{R}^N)\}\,.$$

Observing that the arguments of [BF2] and of Appendix A.1, [Bi2], work under the present hypotheses, the proofs of [BF2], Theorem 1.2 and of [Bi2], Theorem A.3, give without changes

**Lemma 3.2.** *a) We have*

$$\inf_{w \in \varphi + \overset{\circ}{W}{}^1_1(B;\mathbb{R}^N)} J[w,B] = \inf_{w \in BV(B;\mathbb{R}^N)} \hat{J}_\varphi[w,B]\,.$$

*b) Moreover it holds: $u^* \in \mathcal{M} \Leftrightarrow u^*$ is $\hat{J}_\varphi[\cdot,B]$-minimizing in the class $BV(B;\mathbb{R}^N)$.*

# 4 Regularization

As in the previous section we fix a ball $B \Subset \Omega$ and consider a local $\hat{J}$-minimizer $u$. W.r.t. these data we define the function $\varphi$ associated to $u$ as formulated in Preliminaries 2.1. We recall that variational problems of a regularized type have been investigated in [Bi2] assuming for technical simplicity the smoothness of the boundary data (see [Bi2], p. 17, Remark 2.5). Boundary values of the natural class $W^1_1$ have been treated in [Bi3] but both references are based on a quadratic regularization which according to Remark (2.4) cannot be used in the present setting. Since in addition we definitively do not have more smoothness information on the data than being of class $W^1_1$ , a careful look on the regularization is necessary and will be presented in this section.

We consider a sequence $\{\varphi^m\}$, $\varphi^m \in C^\infty_0(\Omega)$ for all $m \in \mathbb{N}$, such that

$$\varphi^m \to \varphi \quad \text{in } W^1_1(\Omega;\mathbb{R}^N) \text{ as } m \to \infty . \tag{4.1}$$

Let $\tilde{q} > \max\{2, q\}$ with $q$ taken from Remark 2.4, $f_\delta := \delta(1 + |\cdot|^2)^{\tilde{q}/2} + f$ and denote by $u^m_\delta$ the unique solution of the minimzing problem

$$J^m_\delta[w, B] := \int_B f_\delta(\nabla w)\,\mathrm{d}x \to \min \quad \text{in } \varphi^m + \overset{\circ}{W}{}^1_{\tilde{q}}(B;\mathbb{R}^N) , \tag{$\mathcal{P}_{\varphi^m,\delta}$}$$

where $0 < \delta < 1$ and where we choose in the following $\delta = \delta(m)$ sufficiently small.

The next lemma summarizes the first elementary properties of the approximation which will be used to follow the regularity arguments of [ABF] and to prove by passing to the limit that the approximation really produces a generalized minimizer $u^* \in \mathcal{M}$.

**Lemma 4.1.** *a) There exists a positive number $c$ which does not depend on $\delta$, $m$ such that*

$$\delta(m) \int_B (1 + |\nabla u^m_{\delta(m)}|^2)^{\frac{\tilde{q}}{2}}\,\mathrm{d}x \le c , \qquad \int_B |\nabla u^m_{\delta(m)}|\,\mathrm{d}x \le c ;$$

*b) $\|u^m_{\delta(m)}\|_{L^\infty(B;\mathbb{R}^N)}$ is bounded independent of $\delta$ and $m$;*

*c) $u^m_{\delta(m)}$ is of class $W^2_{2,loc} \cap W^1_{\infty,loc}(B;\mathbb{R}^N)$;*

*d) if we let $w^m_{\delta(m)} = u^m_{\delta(m)} + \varphi - \varphi^m$, then the $L^1$-cluster points of $\{w^m_{\delta(m)}\}$ and $\{u^m_{\delta(m)}\}$ coincide and*

$$J[u^m_{\delta(m)}, B] \to a \;\; \text{as } m \to \infty \quad \Leftrightarrow \quad J[w^m_{\delta(m)}, B] \to a \;\; \text{as } m \to \infty .$$

*Proof.* Ad $a$). By the minimality of $u^m_{\delta(m)}$ we have choosing $\delta(m)$ sufficiently small

$$J^m_{\delta(m)}[u^m_{\delta(m)}, B] \le J^m_{\delta(m)}[\varphi^m, B] \le \frac{1}{m} + \int_B f(\nabla\varphi^m)\,\mathrm{d}x . \tag{4.2}$$

Moreover, (2.3) gives the existence of a constant $c$ such that $|f(Z) - f(\tilde{Z})| \le c|Z - \tilde{Z}|$ for all $Z$, $\tilde{Z} \in \mathbb{R}^{nN}$, hence by (4.1)

$$\left| \int_B (f(\nabla\varphi^m) - f(\nabla\varphi))\,\mathrm{d}x \right| \le c \int_B |\nabla\varphi^m - \nabla\varphi|\,\mathrm{d}x \to 0 \tag{4.3}$$

as $m \to \infty$ and together with (4.2) the claim $a$) follows.

Ad $b$). The maximum-principle of [DLM] gives $b$).

Ad $c$). We refer to [GM] and [Ca].

Ad $d$). The first claim is immediate by (4.1), for the second claim we just observe that similar to (4.3)

$$\left| \int_B (f(\nabla w^m_{\delta(m)}) - f(\nabla u^m_{\delta(m)}))\,\mathrm{d}x \right| \le c \int_B |\nabla w^m_{\delta(m)} - \nabla u^m_{\delta(m)}|\,\mathrm{d}x \to 0$$

as $m \to \infty$ and the proof of the lemma is complete. □

**Remark 4.1.** *The arguments leading to assertion* $d$) *of Lemma 4.1 immediately give*

$$\inf_{w \in \varphi^m + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w,B] \to \inf_{w \in \varphi + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w,B] \quad \text{as } m \to \infty\,.$$

# 5 The local stress tensor

Based on the principles of convex analysis presented in the book [ET] of Ekeland and Temam, Section 2.1.1 of [Bi2] summarizes the main results needed in our context. In particular we recall:

*i*) the conjugate function $f^*\colon \mathbb{R}^{nN} \to \mathbb{R}$ is defined as

$$f^*(Z^*) := \sup_{Z \in \mathbb{R}^{nN}} \{Z : Z^* - f(Z)\}\,;$$

*ii*) the duality relation

$$f(Z) + f^*(\nabla f(Z)) = Z : \nabla f(Z)$$

holds for all $Z \in \mathbb{R}^{nN}$;

*iii*) we have the representation formula ($w \in \varphi + \overset{\circ}{W}{}^1_1(B;\mathbb{R}^N)$)

$$\begin{aligned} J[w,B] &:= \int_B f(\nabla w)\,\mathrm{d}x \\ &= \sup_{\varkappa \in L^\infty(B;\mathbb{R}^{nN})} \left[ \int_B \varkappa : \nabla w\,\mathrm{d}x - \int_B f^*(\varkappa)\,\mathrm{d}x \right] =: \sup_{\varkappa \in L^\infty(B;\mathbb{R}^{nN})} l(w,\varkappa)\,; \end{aligned}$$

*iv*) the dual functional $R[\cdot,B]\colon L^\infty(B;\mathbb{R}^{nN}) \to \overline{\mathbb{R}}$ is given by

$$R[\varkappa,B] := \inf_{w \in \varphi + \overset{\circ}{W}{}^1_1(B;\mathbb{R}^N)} l(w,\varkappa) = \begin{cases} -\infty\,, & \text{if } \operatorname{div}\varkappa \neq 0\,, \\ l(\varphi,\varkappa)\,, & \text{if } \operatorname{div}\varkappa = 0\,; \end{cases}$$

*v)* the dual variational problem reads as

$$R[\varkappa, B] \to \max \quad \text{in } L^\infty(B;\mathbb{R}^{nN}); \tag{$\mathcal{P}^*$}$$

*vi)* when discussing Problem ($\mathcal{P}^*$) we may assume w.l.o.g. that $\varkappa(x) \in \operatorname{dom} f^* := \{Z^* \in \mathbb{R}^{nN} : f^*(Z^*) < \infty\}$ a.e.;

*vii)* for any $w \in \varphi + \overset{\circ}{W}{}^1_1(B;\mathbb{R}^N)$ we have

$$J[w,B] \geq \sup_{\varkappa \in L^\infty(B;\mathbb{R}^{nN})} R[\varkappa, B].$$

Referring to the notation of Section 4 we let

$$\begin{aligned}
\tau^m_{\delta(m)} &:= \nabla f(\nabla u^m_{\delta(m)}),\\
\sigma^m_{\delta(m)} &:= \delta(m) X^m_{\delta(m)} + \tau^m_{\delta(m)} \;=\; \nabla f_{\delta(m)}(\nabla u^m_{\delta(m)}),\\
X^m_{\delta(m)} &:= \tilde{q}(1+|\nabla u^m_{\delta(m)}|^2)^{\frac{\tilde{q}-2}{2}} \nabla u^m_{\delta(m)}.
\end{aligned}$$

Lemma 4.1, *a*), implies

$$\|\delta(m)^{\frac{\tilde{q}-1}{\tilde{q}}} X^m_{\delta(m)}\|_{L^{\frac{\tilde{q}}{\tilde{q}-1}}(B;\mathbb{R}^{nN})} \leq c,$$

hence

$$\delta(m) X^m_{\delta(m)} \to 0 \quad \text{in } L^{\frac{\tilde{q}}{\tilde{q}-1}}(B;\mathbb{R}^{nN}) \text{ as } m \to \infty.$$

Recalling once more Lemma 4.1, *a*), and the boundedness of $\nabla f$ we find $\sigma \in L^{\frac{\tilde{q}}{\tilde{q}-1}}(B;\mathbb{R}^{nN})$ such that (without relabelling subsequences)

$$\tau^m_{\delta(m)}, \; \sigma^m_{\delta(m)} \rightharpoonup: \sigma \quad \text{in } L^{\frac{\tilde{q}}{\tilde{q}-1}}(B;\mathbb{R}^{nN}) \text{ as } m \to \infty. \tag{5.1}$$

**Lemma 5.1.** *With the notation introduced above we have:*

*a) the limit $\sigma$ satisfies* $\operatorname{div}\sigma = 0$*;*

*b) $\sigma$ is a solution of the dual variational problem ($\mathcal{P}^*$);*

*c) the* "$\inf - \sup$" *relation holds in the sense of*

$$\inf_{w \in \varphi + \overset{\circ}{W}{}^1_1(B;\mathbb{R}^N)} J[w,B] = \sup_{\varkappa \in L^\infty(B;\mathbb{R}^{nN})} R[\varkappa, B];$$

*d)* $\delta(m) \int_B (1+|\nabla u^m_{\delta(m)}|^2)^{\frac{\tilde{q}}{2}} \, \mathrm{d}x \to 0$ *as* $m \to \infty$*;*

*e) we have*

$$J[u^m_{\delta(m)}, B] \to \inf_{w \in \varphi + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w,B] \quad \text{as } m \to \infty,$$

*and on account of Lemma 4.1, d), any $L^1$-cluster point of the sequence $\{u^m_{\delta(m)}\}$ is a generalized minimizer of $J[\cdot, B]$, i.e. belongs to the class $\mathcal{M}$.*

*Proof.* Ad *a*). Since $u^m_{\delta(m)}$ is defined as the solution of $(\mathcal{P}_{\varphi^m,\delta})$ we have the Euler equation $\operatorname{div}\sigma^m_{\delta(m)} = 0$ and passing to the limit $m \to \infty$ this claim follows.

Ad *b*). As in [BF1], proof of Lemma 3.1, or in [Bi2], Section 4.1.2, formula (10), p.102, (compare also Section 2.1.1 of [Bi2]) we now obtain with the help of *vii*) mentioned above and with the help of Remark 4.1 for any $\varepsilon > 0$ and $m$ sufficiently large

$$\begin{aligned} R[\varkappa, B] &\leq \inf_{w\in\varphi+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w,B] \leq \inf_{w\in\varphi^m+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w,B] + \varepsilon \\ &\leq J_\delta[u^m_{\delta(m)}, B] + \varepsilon \\ &\leq \int_B (\tau^m_{\delta(m)} : \nabla\varphi^m - f^*(\tau^m_{\delta(m)}))\,\mathrm{d}x + \delta(m)\int_B X^m_{\delta(m)} : \nabla\varphi^m\,\mathrm{d}x \\ &\quad +(1-\tilde{q})\delta(m)\int_B (1+|\nabla u^m_{\delta(m)}|^2)^{\frac{\tilde{q}}{2}}\,\mathrm{d}x \\ &\quad +\delta(m)\tilde{q}\int_B (1+|\nabla u^m_{\delta(m)}|^2)^{\frac{\tilde{q}-2}{2}}\,\mathrm{d}x + \varepsilon\,, \end{aligned}$$

where we have to observe that in contrast to the case of fixed boundary data we still have $\varphi^m$ in the first and the second integral on the r.h.s. However, since $\tau^m_{\delta(m)}$ by definition is uniformly bounded in $L^\infty(B;\mathbb{R}^{nN})$ and since we have

$$\left|\int_B \tau^m_{\delta(m)} : (\nabla\varphi^m - \nabla\varphi)\,\mathrm{d}x\right| \leq c\int_B |\nabla\varphi^m - \nabla\varphi|\,\mathrm{d}x$$

as well as

$$\left|\delta(m)\int_B X^m_{\delta(m)} : \nabla\varphi^m\,\mathrm{d}x\right| \leq \delta(m)^{\frac{1}{\tilde{q}}}\|\delta(m)^{\frac{\tilde{q}-1}{\tilde{q}}}X^m_{\delta(m)}\|_{L^{\frac{\tilde{q}}{\tilde{q}-1}}(B;\mathbb{R}^{nN})}\|\nabla\varphi^m\|_{L^{\tilde{q}}(B;\mathbb{R}^{nN})}\,,$$

we may choose $\delta(m)$ sufficiently small and argue exactly as in the proof of Lemma 4.8, [Bi2]. This gives the lemma. □

In order to prove smoothness results **for any** local generalized minimizer of the original problem $(\mathcal{P})$ (and not just for cluster points of the regularizing sequence), we have to establish the uniqueness of the local stress tensor which of course also is a result of general interest. The following lemma is a consequence of the structure condition (1.2) and of some elementary properties of the function $f^*$.

**Lemma 5.2.** *The solution of the dual variational problem $(\mathcal{P}^*)$ is unique. On account of Lemma 5.1, b), this solution is given by the limit $\sigma$ defined in (5.1).*

*Proof.* The uniqueness of the dual solution will follow as soon as we can show the strict convexity of $f^*$ on $\operatorname{dom} f^*$. From Assumption 2.1 and from (2.3) we first deduce that

$$K := \lim_{t\to\infty} h'(t)$$

exists in $(0,\infty)$. Note that $K$ determines the set $\mathrm{im}\,(\nabla f)$: since

$$\nabla f(Z) = \frac{h'(|Z|)}{|Z|} Z\,, \qquad Z \neq 0\,, \qquad \nabla f(0) = 0\,,$$

it is elementary to show that $\nabla f$ is a one-to-one mapping from $\mathbb{R}^{nN}$ onto the open ball $\mathcal{B}_K(0) \subset \mathbb{R}^{nN}$. Observing that $f^*(Z) = h^*(|Z|)$ implies $\mathrm{dom}\, f^* = \{Z \in \mathbb{R}^{nN} : |Z| \in \mathrm{dom}\, h^*\}$, we have to discuss $\mathrm{dom}\, h^*$.

If $s > K$, then

$$h^*(s) = \sup_{t\geq 0} \left[ ts - \int_0^t h'(r)\,\mathrm{d}r \right] \geq \sup_{t \geq 0}[ts - Kt] = +\infty\,,$$

and if $s < K$, then there exists $t \in \mathbb{R}$ such that $s = h'(t)$, hence

$$h^*(s) = h^*(h'(t)) = th'(t) - h(t) < \infty\,,$$

so that either $\mathrm{dom}\, h^* = [0,K)$ or $\mathrm{dom}\, h^* = [0,K]$ and in conclusion:

$$\textbf{either} \quad \mathrm{dom}\, f^* = \mathcal{B}_K(0) \quad \textbf{or} \quad \mathrm{dom}\, f^* = \overline{\mathcal{B}_K(0)}\,.$$

In the first case the lemma follows from $\mathrm{dom}\, f^* = \mathrm{im}\,(\nabla f)$ and from the strict convexity of $f^*$ on $\mathrm{im}\,(\nabla f)$.

In the second case we note that $\lim_{s\uparrow K} h^*(s) < h^*(K)$ immediately would give the lemma and that $h^*(K) < \lim_{s\uparrow K} h^*(s)$ would contradict the strict monotonicty and the convexity of $h^*$ on $[0,K]$. Thus we may suppose that $h^*$ is a continuous function on $[0,K]$. Moreover, by elementary calculations it is easy to see that a continuous function $\phi\colon [a,b] \to \mathbb{R}$ satisfying $\phi'' > 0$ on $(a,b)$ is strictly convex on $[a,b]$. Applying this to $h^*$ the lemma is proved. □

# 6 A priori bounds

**Theorem 6.1.** *Suppose that the assumptions of Theorem 2.1 are satisfied. Passing to subsequences (again not relabelled) we have:*

*a) the sequence $\{u^m_{\delta(m)}\}$ is uniformly bounded in the space $W^1_{\infty,loc}(B;\mathbb{R}^N)$;*

*b) the sequence $\{|\nabla\sigma^m_{\delta(m)}|\}$ is uniformly bounded in the space $L^2_{loc}(B)$, hence (recall $\tilde{q} \geq 2$) $\{\sigma^m_{\delta(m)}\}$ is uniformly bounded in $W^1_{\tilde{q}/(\tilde{q}-1),loc}(B;\mathbb{R}^{nN})$, so that in particular*

$$\sigma^m_{\delta(m)} \to \sigma \;\; a.e. \quad as \;\; m \to \infty\,.$$

*Moreover, we have*

$$\nabla u^m_{\delta(m)} \to \nabla u^* \;\; a.e. \quad as \;\; m \to \infty\,,$$

*where $u^*$ denotes a $L^1$-cluster point of the sequence $\{u^m_{\delta(m)}\}$;*

*c) the duality relation*

$$\sigma = \nabla f(\nabla u^*)$$

*holds a.e. In particular $\sigma$ takes its values in the open set* $\mathrm{im}\,(\nabla f) = \mathcal{B}_K(0)$.

**Remark 6.1.** *a) Once more we point out that local $W^1_2$-regularity of the local stress tensor does not follow along the lines of [Bi2] since we do not assume the condition*

$$D^2 f(Z)(Y,Y) \leq c \frac{1}{\sqrt{1+|Z|^2}} |Y|^2 \, .$$

*b) We also cannot refer to arguments as given in Corollary 6.10 of [Bi2], since these arguments rely on a uniform local $W^2_2$ bound for $\{u^m_{\delta(m)}\}$ which in general cannot be expected since $D^2 f$ may be degenerated in the origin.*

*Proof.* Ad *a*). For notational simplicity we drop the index $m$ and proceed similar to the proof of Theorem 1.1 in [ABF]. Let us also assume the validity of (2.2) and its reformulation (2.4) for all $t \geq 0$, i.e. we have $T_0 = 0$. The necessary adjustments – being of pure technical nature – which are needed for the treatment of the case $T_0 > 0$, can be found in [ABF]. With $\eta \in C^\infty_0(B)$, $0 \leq \eta \leq 1$, $\Gamma := 1 + |\nabla u|^2$ and $s \geq 0$ we obtain with the help of Lemma 4.1, b), c)

$$\begin{aligned}
\int_B \eta^2 h(|\nabla u|)\Gamma^{\frac{s+2}{2}} \,\mathrm{d}x &= \int_B \eta^2 h(|\nabla u|)\Gamma^{\frac{s}{2}} \,\mathrm{d}x + \int_B \partial_\alpha u \cdot \partial_\alpha u \eta^2 h(|\nabla u|)\Gamma^{\frac{s}{2}} \,\mathrm{d}x \\
&= \int_B \eta^2 h(|\nabla u|)\Gamma^{\frac{s}{2}} \,\mathrm{d}x - \int_B u \cdot \partial_\alpha \Big[\partial_\alpha u \eta^2 h(|\nabla u|)\Gamma^{\frac{s}{2}}\Big] \,\mathrm{d}x \\
&\leq \int_B \eta^2 h(|\nabla u|)\Gamma^{\frac{s}{2}} \,\mathrm{d}x + c\Bigg[\int_B \eta|\nabla\eta| h(|\nabla u|)|\nabla u|\Gamma^{\frac{s}{2}} \,\mathrm{d}x \\
&\quad + \int_B \eta^2 |\nabla^2 u| h(|\nabla u|)\Gamma^{\frac{s}{2}} \,\mathrm{d}x + \int_B \eta^2 |\nabla u| h'(|\nabla u|)|\nabla^2 u|\Gamma^{\frac{s}{2}} \,\mathrm{d}x\Bigg] .
\end{aligned}$$

Applying Young's inequality to the second integral on the r.h.s and using $h'(t)t \leq ch(t)$ we get

$$\begin{aligned}
\int_B \eta^2 h(|\nabla u|)\Gamma^{\frac{s+2}{2}} \,\mathrm{d}x &\leq c\Bigg[\int_B (\eta^2 + |\nabla\eta|^2) h(|\nabla u|)\Gamma^{\frac{s}{2}} \,\mathrm{d}x + \int_B \eta^2 |\nabla^2 u| h(|\nabla u|)\Gamma^{\frac{s}{2}} \,\mathrm{d}x\Bigg] \\
&=: c[T_1 + T_2] \qquad (6.1)
\end{aligned}$$

with $c$ depending on $s$, but being independent of the approximation parameter. Another application of Young's inequality yields

$$T_2 \leq \tau \int_B \eta^2 h(|\nabla u|)\Gamma^{\frac{s+2}{2}} \,\mathrm{d}x + c(\tau) \int_B \eta^2 |\nabla^2 u|^2 h(|\nabla u|)\Gamma^{\frac{s-2}{2}} \,\mathrm{d}x$$

and for $\tau$ small enough we deduce from (6.1)

$$\begin{aligned}\int_B \eta^2 h(|\nabla u|)\Gamma^{\frac{s+2}{2}}\,\mathrm{d}x &\le c\left[T_1 + \int_B \eta^2|\nabla^2 u|^2 h(|\nabla u|)\Gamma^{\frac{s-2}{2}}\,\mathrm{d}x\right]\\ &=: c[T_1+T_3]\,, \end{aligned}\tag{6.2}$$

hence it remains to discuss $T_3$. Recalling $h(t)\le th'(t)$ and also the first inequality stated in Remark 2.4 it is easy to see that (2.2) implies

$$T_3 \le \int_B \eta^2 D^2 f(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\Gamma^{\frac{s+\mu}{2}}\,\mathrm{d}x\,. \tag{6.3}$$

Moreover, referring to the Caccioppoli-type inequality Lemma 2.2 in [ABF], we can bound the r.h.s. of (6.3) by the quantity

$$c\int_B |D^2 f(\nabla u)|\Gamma^{\frac{s+2+\mu}{2}}|\nabla\eta|^2\,\mathrm{d}x\,.$$

Again from (2.2) it follows that

$$|D^2 f(\nabla u)| \le c h(|\nabla u|)\Gamma^{\frac{\mu-\kappa}{2}}\,,$$

thus (6.2) and (6.3) show the validity of

$$\int_B \eta^2 h(|\nabla u|)\Gamma^{\frac{s+2}{2}} \le c\left[\int_B (\eta^2+|\nabla\eta|^2)h(|\nabla u|)\Gamma^{\frac{s}{2}}\,\mathrm{d}x + \int_B |\nabla\eta|^2 h(|\nabla u|)\Gamma^{\frac{s+2+2\mu-\kappa}{2}}\,\mathrm{d}x\right]. \tag{6.4}$$

Finally we replace $\eta$ by $\eta^l$ for $l\in\mathbb{N}$ large and apply Young's inequality to the second integral on the r.h.s. of (6.4), which is possible on account of

$$2\mu < \kappa\,,$$

to deduce from (6.4)

$$\int_{B''} h(|\nabla u|)\Gamma^{\frac{s+2}{2}}\,\mathrm{d}x \le c(s,B'',B')\int_{B'} h(|\nabla u|)\Gamma^{\frac{s}{2}}\,\mathrm{d}x$$

valid for balls $B''\Subset B'\Subset B$. Starting with $s=0$, it is now obvious that

$$|\nabla u| \in L^s_{loc}(B) \tag{6.5}$$

for all finite $s$ (uniformly w.r.t. the approximation). Proceeding similar as in [Bi2], proof of Theorem 5.22, (see [ABF], Section 4, for the necessary adjustments) we obtain from (6.5) the local boundedness of $\nabla u$ (uniformly w.r.t. $m$).

Ad $b$). With $a$), in particular with the condition $2\mu < \kappa$ and our structure assumptions, we know that $\{\nabla u^m_{\delta(m)}\}$ is uniformly locally bounded and (11), Section 2.1.3 of [Bi2], gives together with Young's inequality

$$\int_{B'} D^2 f(\nabla u^m_{\delta(m)})(\partial_\gamma \nabla u^m_{\delta(m)}, \partial_\gamma \nabla u^m_{\delta(m)})\,\mathrm{d}x \le c$$

with a local constant $c$ depending on $B' \Subset B$. Thus, (13), Section 2.1.3 of [Bi2], implies the claims for $\{\sigma^m_{\delta(m)}\}$ stated in $b$).
Since $\nabla f$ is a one-to-one mapping $\mathbb{R}^{nN} \to \mathcal{B}_K(0)$ with inverse given by

$$Y \mapsto \frac{1}{|Y|}(h')^{-1}(|Y|)Y\,,$$

we have the a.e. convergence of $\{\nabla u^m_{\delta(m)}\}$ by the a.e. convergence of $\{\sigma^m_{\delta(m)}\}$, by the definition of $\sigma^m_{\delta(m)}$ and by Lemma 5.1, $d$).

Ad $c$). This is an immediate consequence of $b$). □

# 7 A uniqueness result for local generalized minimizers

In Lemma 5.1, $e$), it is established that each $L^1$-cluster point $\bar{u}$ of the regularizing sequence $\{u^m_{\delta(m)}\}$ is a local generalized minimizer of Problem $(\mathcal{P})$, i.e. $\bar{u} \in \mathcal{M}$.

Moreover, we have the duality relation $c$) of Theorem 6.1, which gives information on the unique stress tensor by knowing one particular generalized minimizer $u^* \in \mathcal{M}$. Since $\mathcal{B}_K(0) = \mathrm{im}\,(\nabla f)$ is an open set and since $\nabla f$ is continuous, we have for any $M > 0$ and for any ball $\mathcal{B}_M(0) \subset \mathbb{R}^{nN}$ with radius $M$ and center $0 \in \mathbb{R}^{nN}$

$$\nabla f(\overline{\mathcal{B}_M(0)}) \Subset \mathcal{B}_K(0)\,. \tag{7.1}$$

Now we refer to the local uniform bounds for $\{\nabla u^m_{\delta(m)}\}$ established in Theorem 6.1, $a$), which together with the duality relation imply using (7.1): for any $B' \Subset B$ there is a ball $\mathcal{B}_M \subset \mathbb{R}^{nN}$ such that $\{\sigma(x) : x \in B'\} \subset \mathcal{B}_M$. This means that on any ball $B' \Subset B$ the stress tensor takes values in a compact set $S(B') \Subset \mathcal{B}_K(0)$. Hence, given $\lambda \in C^\infty_0(B;\mathbb{R}^{nN})$ and $t$ sufficiently small, the function $\sigma_t := \sigma + t\lambda$ is an admissible variation of $\sigma$ as first observed by Seregin [Se4] in a different setting and later used in [BF3], Section 5, and in the proof of [Bi2], Theorem A.9, p.182. Following this proof we additionally just need the $\inf - \sup$ relation and the maximality of the local stress tensor and finally arrive at

**Theorem 7.1.** *Any generalized minimizer $u^* \in \mathcal{M}$ satisfies a.e. in $B$ the duality relation $\sigma = \nabla f(\nabla u^*)$, hence on account of Lemma 5.2 and of Lemma 3.2, $b$), local generalized minimizers in the sense of Definition 2.1 are unique up to a constant.*

Once more recalling Theorem 6.1, $a$), the proof of Theorem 2.1 is complete. □

**Remark 7.1.** *If a global Dirichlet problem is considered, then the above results show that there is only one way to define a suitable local stress tensor associated to the global solution of the original problem.*

# Appendix A Examples

*Example 1.* We fix $r \in (0,1)$ and let (compare [MP])

$$h(t) := t - t^r + 1 \quad \text{for all } t \geq 1 .$$

For $t < 1$ the function $h$ is extended as a smooth function satisfying our general hypotheses. With $\mu := 1 - r$ and $\kappa := 3 - r$ we then have Assumption 2.1 and the example is admissible for our theory. However, we note that the arguments of Section 4.2, [Bi2], also apply to this example.

*Example 2.* For any $k \in \mathbb{N}$ we fix $\varepsilon_k \in (0, 1/2)$ and $a_k > 0$. With $I_k := [k - \varepsilon_k, k + \varepsilon_k]$ we then define $\theta$: $[0,\infty) \to [0,\infty)$ via the following properties:

- $\theta \in C^0([0,\infty))$;
- $\theta \equiv 0$ on $[0,\infty) - \bigcup_{k=1}^{\infty} I_k$;
- $\theta \leq a_k$ on $I_k$ and $\theta(k) = a_k$.

With

$$g(t) := \int_0^t \left[ \int_0^s \theta(\xi)\, \mathrm{d}\xi \right] \mathrm{d}s , \qquad t \geq 0 ,$$

we claim that for a suitable choice of the parameters

$$\lim_{t\to\infty} \frac{g(t)}{t} \ \text{ exists in } (0,\infty) . \tag{A.1}$$

In fact, if for any $k \in \mathbb{N}$

$$\varepsilon_k \leq \frac{1}{k^\alpha a_k} \qquad \text{for some exponent } \alpha > 1 , \tag{A.2}$$

then it follows that

$$0 \leq g'(t) = \int_0^t \theta(\xi)\, \mathrm{d}\xi \leq 2 \sum_{k=1}^{\infty} \varepsilon_k a_k < \infty . \tag{A.3}$$

Since $g$ is a convex function, we get

$$0 = g(0) \geq g(t) - t g'(t) ,$$

and (A.3) gives

$$\frac{g(t)}{t} \le g'(t) \le const\,. \tag{A.4}$$

Once more by the convexity of $g$ we obtain for any $0 < \gamma < 1$

$$\frac{g(\gamma t) - g(0)}{\gamma t} \le \frac{g(t) - g(\gamma t)}{t - \gamma t}\,,$$

which is equivalent to

$$\frac{g(\gamma t)}{\gamma t} \le \frac{g(t)}{t}\,.$$

This means that the function $s \mapsto g(s)/s$ is increasing and together with (A.4) the claim (A.1) is established.

Adding the density from the first example, $\tilde{h}(t) := h(t) + g(t)$, Assumption (2.1) is satisfied for $\tilde{h}$ and we have ($\mu = 1 - r$, $\kappa = 3 - r$) for constants $\bar{\varepsilon}$, $\bar{h} > 0$

$$\bar{\varepsilon} t^{-1-\mu} \le \tilde{h}''(t) \le \bar{h} t^{1-\kappa} + \theta(t)\,.$$

If $r > 1/2$, $\theta(k) = k^m$ for $0 < m < 2r - 1$, then (2.4) is satisfied with upper exponent $\tilde{\kappa} := 1 - m$ and the assumptions of Theorem 2.1 hold. However, the results of Section 4.2, [Bi2], do not apply.

**Remark A.1.** *We have the duality relation*

$$\tilde{h}^*(\tilde{h}'(t)) = t\tilde{h}'(t) - \tilde{h}(t)$$

*which implies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\tilde{h}^*(\tilde{h}'(t))\big] = t\tilde{h}''(t)\,,$$

*i.e.*

$$\tilde{h}^*(\tilde{h}'(t)) = \int_0^t t\tilde{h}''(t)\,\mathrm{d}t\,.$$

*Hence, still keeping (A.2), it is evident that in general one cannot decide whether* $\mathrm{dom}\,\tilde{h}^*$ *is an open or a closed interval.*

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