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**Regularity Results for Local Minimizers of Energies
with General Densities Having Superquadratic
Growth**

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Abstract

We consider variational integrals whose energy densities are represented by N -functions h of at least quadratic growth. Under rather general conditions on h almost everywhere regularity of vector-valued local minimizers is established, and it is possible to include the case of higher order variational problems without essential changes in the arguments.

1 Introduction

In recent years increasing attention has been paid to the study of the regularity properties of local minimizers $u : \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}^M$ of variational integrals of the form

$$(1.1) \quad I[w, \Omega] = \int_{\Omega} H(\nabla w) dx$$

with density $H : \mathbb{R}^{nM} \rightarrow [0, \infty)$ being a strictly convex function of “nonstandard growth”, which means in rough words that we do not have an ellipticity estimate of the form $\lambda(|Z|)|Y|^2 \leq D^2H(Z)(Y, Y) \leq \Lambda(|Z|)|Y|^2$ for all $Y, Z \in \mathbb{R}^{nM}$ with functions λ, Λ such that

$$c_1 \leq \Lambda(t)/\lambda(t) \leq c_2$$

holds for all $t \geq 0$ and with constants $c_1, c_2 > 0$. One major class showing such a behaviour is generated by so-called integrands of anisotropic power growth, e.g. $H(\nabla u) = (1 + |\nabla u|^2)^{p/2} + (1 + |\partial_n u|^2)^{q/2}$ with exponents $p < q$, for which we have $\lambda(t) = a(1 + t^2)^{\frac{p-2}{2}}$, $\Lambda(t) = A(1 + t^2)^{\frac{q-2}{2}}$ as functions characterizing the growth of D^2H .

In this setting and if in addition the scalar case (i.e. $M = 1$) is considered Marcellini proved in [Ma1] and [Ma4] the interior $C^{1,\alpha}$ -regularity of local minimizers under conditions of the form $q < c(n)p$ relating the exponents p and q , where $c(n)$ is rather large for low dimensions n , but $c(n) \searrow 1$ as $n \rightarrow \infty$. As a matter of fact - without further hypotheses on H - one can only hope for almost everywhere regularity in the general vector case $M \geq 2$. Here we mention the important contributions of Passarelli Di Napoli and Siepe [PS], of Acerbi and Fusco [AF] and of Esposito, Leonetti and Mingione [ELM1,2] as well as the references mentioned in these papers. In addition, also the publications [BF1,2] are devoted to the partial regularity theory for variational

problems with anisotropic (p, q) -growth, and it should be emphasized that in all the above mentioned papers again an inequality like $q < c(n)p$ is needed.

A second class of integrands with nonstandard behaviour arises if H satisfies

$$(1.2) \quad H(Z) = h(|Z|), \quad Z \in \mathbb{R}^{nM},$$

for a given N -function h . In this case we have (compare v) below)

$$\lambda(t) = \min \left\{ \frac{h'(t)}{t}, h''(t) \right\}, \Lambda = \max \{ \dots \},$$

so that the condition of uniform ellipticity in general is violated. Concentrating on the vector case interior $C^{1,\alpha}$ -regularity of local minimizers of integrals depending on the modulus of the gradient has been proved by many authors: starting with the work of Uhlenbeck [Uh] on the p -growth case, i.e. $h(t) = t^p$ for some $p > 1$, which was later on extended by Giaquinta and Modica [GM], Marcellini and Marcellini and Papi studied very general N - functions h in [Ma1–3] and [MP], whereas the case of nearly linear growth, i.e. $h(t) = t \ln(1 + t)$, is due to Mingione and Siepe [MS]. Further contributions concerning the regularity problem under condition (1.2) are given in the recent paper [ABF].

The purpose of the present note is threefold:

1. In Theorem 1.1 below we will establish almost everywhere regularity of vector-valued local minimizers requiring (1.2) but under (in a sense to be made precise) more general hypotheses as for example used in [MP].
2. The methods apply to variational problems of higher order (see Theorem 1.2) providing some extensions of the results obtained in [ApF].
3. We think that with some additional work our results can be transferred to local minima $u : \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}^n$ of integrals like $\int_{\Omega} h(|\varepsilon(u)|) dx$ subject to the constraint $\operatorname{div} u = 0$. Here $\varepsilon(u)$ is the symmetric gradient of u , and for $n = 2$ this variant of the stationary Stokes problem has been the subject of the recent paper [Fu2]. To our knowledge the hypothesis (1.2) seems to be a natural assumption for the study of fluids: it occurs for example in the setting of electrorheological fluids for which partial regularity has been proved by Acerbi and Mingione [AM].

With respect to 1. our result is not optimal: we expect that the singular set is empty but we could not rule out the occurrence of singular points. For variational problems of higher order or in the framework of fluids it seems to be even harder to develop methods which use the structure condition (1.2) in order to exclude singularities.

Let us give a precise formulation of our assumptions: let $n \geq 3$, $M \geq 1$ and consider an open set $\Omega \subset \mathbb{R}^n$. Let the density $H : \mathbb{R}^{nM} \rightarrow [0, \infty)$ satisfy (1.2), where $h : [0, \infty) \rightarrow [0, \infty)$ is of class C^2 . We will impose the following hypotheses on h :

$$(A1) \quad h \text{ is strictly increasing and convex together with } h''(0) > 0 \text{ and } \lim_{t \downarrow 0} \frac{h(t)}{t} = 0;$$

$$(A2) \quad \text{there exists a constant } \bar{k} > 0 \text{ such that } h(2t) \leq \bar{k} h(t) \text{ for all } t \geq 0;$$

$$(A3) \quad \begin{cases} \text{for an exponent } \omega \geq 0 \text{ and a constant } a \geq 0 \text{ we have} \\ \frac{h'(t)}{t} \leq h''(t) \leq a(1+t^2)^{\frac{\omega}{2}} \frac{h'(t)}{t} \quad \text{for all } t \geq 0. \end{cases}$$

Let us give some comments on (A1-3):

- i) We have $h(0) = h'(0)$, and by convexity h' is an increasing function with $h'(t) > 0$ for all $t > 0$: otherwise it would follow that $h' = 0$ on some interval $[0, t_0]$, $t_0 > 0$, contradicting the first part of (A1).
- ii) The inequality $\frac{h'(t)}{t} \leq h''(t)$ implies that the function $t \mapsto \frac{h'(t)}{t}$ is increasing, moreover we deduce the lower bound

$$(1.3) \quad h(t) \geq \frac{1}{2} h''(0) t^2, \quad t \geq 0.$$

(A1) combined with (1.3) shows that h is a N -function in the sense of [Ad, Section 8.2].

- iii) (A2) states that h satisfies a global $(\Delta 2)$ -condition, and it is easy to see that

$$h(t) \leq c(t^m + 1)$$

for a suitable exponent $m \geq 2$ and a constant c . The convexity of h then implies that $h'(t)$ can be bounded in terms of t^{m-1} .

- iv) From (A2) and from the convexity of h we deduce the inequality

$$(1.4) \quad \bar{k}^{-1} h'(t)t \leq h(t) \leq th'(t), \quad t \geq 0.$$

- v) From (1.2) it follows for all $Y, Z \in \mathbb{R}^{nM}$

$$\begin{aligned} \min \quad & \left\{ \frac{h'(|Z|)}{|Z|}, h''(|Z|) \right\} |Y|^2 \leq D^2 H(Z)(Y, Y) \leq \\ \max \quad & \left\{ \frac{h'(|Z|)}{|Z|}, h''(|Z|) \right\} |Y|^2, \end{aligned}$$

hence by (A3)

$$(1.5) \quad \frac{h'(|Z|)}{|Z|} |Y|^2 \leq D^2 H(Z)(Y, Y) \leq a(1 + |Z|^2)^{\frac{\varepsilon}{2}} \frac{h'(|Z|)}{|Z|} |Y|^2.$$

Recalling iii) and using (see ii)) $\frac{h'(|Z|)}{|Z|} \geq h''(0)$, we get from (1.5) with exponent $q := m + \omega$

$$(1.6) \quad h''(0) |Y|^2 \leq D^2 H(Z)(Y, Y) \leq C(1 + |Z|^2)^{\frac{q-2}{2}} |Y|^2,$$

and (1.6) means that H is of anisotropic $(2, q)$ -growth.

Definition 1.1. A function $u \in W_{1,\text{loc}}^1(\Omega; \mathbb{R}^M)$ is a local minimizer of the energy I from (1.1) if for any subdomain Ω' with compact closure in Ω it holds $I[u, \Omega'] < \infty$ and $I[u, \Omega'] \leq I[v, \Omega']$, where v is an arbitrary function from $W_{1,\text{loc}}^1(\Omega; \mathbb{R}^M)$ such that $\text{spt}(u - v) \subset \Omega'$.

For a definition of the Sobolev spaces $W_{p,\text{loc}}^k(\Omega; \mathbb{R}^M)$ and related classes we refer the reader to the monograph [Ad]. Note that a local minimizer actually belongs to the Orlicz-Sobolev class $W_{h,\text{loc}}^1(\Omega; \mathbb{R}^M)$, in particular (see (1.3)) local minimizers are in the space $W_{2,\text{loc}}^1(\Omega; \mathbb{R}^M)$.

Let us now state our first result

Theorem 1.1. Let (A1) - (A3) hold and consider a local minimizer $u \in W_{1,\text{loc}}^1(\Omega; \mathbb{R}^M)$ of the functional I from (1.1) with integrand H defined according to (1.2). Suppose further that there is a finite constant c such that

$$(1.7) \quad t^\omega \leq c \left[h(t)^{\frac{2}{n-2}} + 1 \right]$$

for all sufficiently large t . Then there is an open subset Ω_0 of Ω with full Lebesgue measure such that $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^M)$ for any $0 < \alpha < 1$.

Remark 1.1. According to (1.3) we have the validity of (1.7) if we require $\omega \leq 4/(n-2)$.

Remark 1.2. Let us compare Theorem 1.1 with the recent result obtained by Marcellini and Papi [MP]. Roughly speaking, they replace the second inequality from (A3) by the requirement (compare (2.9) of [MP])

$$(1.8) \quad h''(t) \leq \text{const} \left(\frac{h'(t)}{t} \right)^{\frac{n}{n-1}}.$$

Then, full interior regularity is established, i.e. it holds $\Omega_0 = \Omega$, whereas we could not exclude the occurrence of singular points despite of our structure condition (1.2). But if we consider functions h such that

$$h'' = 1 \text{ on } [0, \infty) - \bigcup_{i=1}^{\infty} I_i,$$

where $I_i := [a_i - \varepsilon, a_i + \varepsilon]$ for an appropriate sequence $a_i \rightarrow \infty$ with corresponding very small ε_i , and if we let $h''(a_i) = a_i^\kappa$ for a suitable power κ , then (1.8) is violated, whereas the right-hand side of (A3) together with (1.7) can be guaranteed.

Remark 1.3. *The regularity problem under the hypotheses (A1-3) has been addressed from a different point of view in the paper [ABF]: assuming that the local minimizer is also a locally bounded function we could prove $\Omega_0 = \Omega$ replacing (1.7) by the dimensionless condition $\omega < 2$.*

Remark 1.4. *As stated after (1.6) our variational integral also falls into the category of anisotropic energies and it is possible to deduce Theorem 1.1 from the papers on this subject provided (compare (1.6) and choose $p = 2$) we know that $q < p \frac{n}{n-2}$ (or a similar inequality) is true. But if we recall the definition of q , it is immediate that such a bound for q is too restrictive.*

Let us now consider the case of higher order variational problems, i.e. we discuss local minima of variational integrals like

$$(1.9) \quad J[w, \Omega] = \int_{\Omega} h(|\nabla^k w|) dx,$$

where $k \geq 2$ denotes a given integer. The symbol $\nabla^k w$ stands for the tensor of all k^{th} order (weak) partial derivatives of the function w , and w is assumed to be an element of the space $W_{1,\text{loc}}^k(\Omega; \mathbb{R}^M)$. The appropriate modification of Definition 1.1 for the situation at hand is immediate, and we have

Theorem 1.2. *Let (A1-3) and (1.7) hold and let $u \in W_{1,\text{loc}}^k(\Omega; \mathbb{R}^M)$, $M \geq 1$, $k \geq 2$, denote a local minimizer of the functional J from (1.9). Then there is an open set $\Omega_0 \subset \Omega$ of full Lebesgue measure such that $u \in C^{k,\nu}(\Omega_0; \mathbb{R}^M)$ for any $0 < \nu < 1$.*

Remark 1.5. *In [ApF] we proved partial regularity for local minimizers of anisotropic variational problems of higher order but as outlined in Remark 1.4 the result of Theorem 1.2 follows from Theorem 1.1 in [ApF] only under very restrictive assumptions on h .*

Remark 1.6. *We note that with obvious simplifications our results apply to the case $\Omega \subset \mathbb{R}^2$, which means that then hypothesis (1.7) becomes superfluous.*

Our paper is organized as follows: in Section 2 we collect some auxiliary results among which Lemma 2.2 is of separate interest since it contains a statement on the local higher integrability of ∇u without using the hypothesis (1.7).

Section 3 is devoted to the proof of Theorem 1.1, which is based on the blow-up technique. In Section 4 we sketch how to adjust these arguments to the higher order case described in Theorem 1.2.

2 Preliminary results

Throughout this section we consider a local minimizer u as defined in Definition 1.1 and assume the validity of (A1-3), whereas (1.7) is not required. The following calculations can be made precise by replacing u , H , I by a suitable local regularization u_δ , H_δ , I_δ , $\delta > 0$, with exponent q as outlined for example in [BF1].

Lemma 2.1. *Let $\Psi := \int_0^{|\nabla u|} \sqrt{\frac{h'(t)}{t}} dt$. Then it holds for balls $B_R(x_0) \subset \Omega$ and $0 < t < 1$*

$$(2.1) \quad \int_{B_{tR}(x_0)} |\nabla \Psi|^2 dx \leq cR^{-2}(1-t)^{-2} \int_{B_R(x_0)} h(|\nabla u|) dx.$$

Proof: Let $\eta \in C_0^\infty(\Omega)$. We have (summation w.r.t. $\alpha = 1, \dots, n$)

$$\begin{aligned} 0 &= \int_{\Omega} \partial_\alpha (DH(\nabla u)) : \nabla (\eta^2 \partial_\alpha u) dx \\ &= \int_{\Omega} \partial_\alpha (DH(\nabla u)) : \partial_\alpha \nabla u \eta^2 dx \\ &\quad + \int_{\Omega} \partial_\alpha (DH(\nabla u)) : (\nabla \eta^2 \otimes \partial_\alpha u) dx, \end{aligned}$$

where “:” is the scalar product of matrices and where “ \otimes ” represents the tensor product. We obtain

$$\int_{\Omega} \partial_\alpha (DH(\nabla u)) : \partial_\alpha \nabla u \eta^2 dx = \int_{\Omega} DH(\nabla u) : \partial_\alpha (\nabla \eta^2 \otimes \partial_\alpha u) dx.$$

Using the first inequality in (1.5) and observing $|DH(Z)| \leq h'(|Z|)$ we get

$$\begin{aligned} \int_{\Omega} \eta^2 \frac{h'(|\nabla u|)}{|\nabla u|} |\nabla^2 u|^2 dx &\leq \int_{\Omega} h'(|\nabla u|) |\nabla u| |\nabla^2 \eta^2| dx \\ &\quad + 2 \int_{\Omega} h'(|\nabla u|) \eta |\nabla \eta| |\nabla^2 u| dx \\ &\stackrel{1.4)}{\leq} c \int_{\Omega} |\nabla^2 \eta^2| h(|\nabla u|) dx \\ &\quad + 2 \int_{\Omega} \left(\frac{h'(|\nabla u|)}{|\nabla u|} \right)^{1/2} \eta |\nabla^2 u| (h'(|\nabla u|) |\nabla u|)^{1/2} |\nabla \eta| dx, \end{aligned}$$

and Young’s inequality together with another application of (1.4) yields

$$(2.2) \quad \int_{\Omega} \eta^2 \frac{h'(|\nabla u|)}{|\nabla u|} |\nabla^2 u|^2 dx \leq c \int_{\Omega} [|\nabla \eta|^2 + |\nabla^2 \eta^2|] h(|\nabla u|) dx.$$

Since $|\nabla \Psi|^2 \leq \frac{h'(|\nabla u|)}{|\nabla u|} |\nabla^2 u|^2$, (2.1) follows from (2.2) by specifying η .

□

Let us give upper and lower bounds for the function Ψ : we have

$$\begin{aligned}\Psi &\geq \int_{|\nabla u|/2}^{|\nabla u|} \sqrt{\frac{h'(t)}{t}} dt \geq \frac{|\nabla u|}{2} \sqrt{h'(|\nabla u|/2)/(|\nabla u|/2)} \\ &= \sqrt{\frac{1}{2}|\nabla u|h'(|\nabla u|/2)} \stackrel{(1.4)}{\geq} \sqrt{h(|\nabla u|/2)}\end{aligned}$$

by the monotonicity of $t \mapsto \frac{h'(t)}{t}$, hence by (A2)

$$(2.3) \quad h(|\nabla u|) \leq c\Psi^2.$$

On the other hand we clearly have $\Psi \leq |\nabla u| \sqrt{\frac{h'(|\nabla u|)}{|\nabla u|}}$, hence

$$(2.4) \quad \Psi^2 \leq ch(|\nabla u|).$$

The r.h.s. of (2.4) is in $L^1_{\text{loc}}(\Omega)$, which together with (2.1) implies $\Psi \in W^1_{2,\text{loc}}(\Omega)$. This gives by Sobolev's theorem in combination with (2.3):

Lemma 2.2. *We have $h(|\nabla u|)^{\frac{n}{n-2}} \in L^1_{\text{loc}}(\Omega)$.*

Next we recall the Caccioppoli-type inequality (see, e.g., [BF1])

$$(2.5) \quad \int_{\Omega} \eta^2 D^2 H(\nabla u)(\partial_{\alpha} \nabla u, \partial_{\alpha} \nabla u) dx \leq c \int_{\Omega} |D^2 H(\nabla u)| |\nabla \eta|^2 |\nabla u - Q|^2 dx$$

valid for $\eta \in C_0^{\infty}(\Omega)$ and $Q \in \mathbb{R}^{nM}$. From (2.5) it follows

$$(2.6) \quad \int_{\Omega} \eta^2 \frac{h'(|\nabla u|)}{|\nabla u|} |\nabla^2 u|^2 dx \leq c \int_{\Omega} |\nabla \eta|^2 (1 + |\nabla u|^2)^{\frac{\omega}{2}} \frac{h'(|\nabla u|)}{|\nabla u|} |\nabla u - Q|^2 dx.$$

Note that the integrands on the right-hand sides of (2.5) and (2.6) behave like $|\nabla u|^{\omega} h(|\nabla u|)$, and by Lemma 2.2 their local integrability will follow if condition (1.7) is valid.

3 Proof of Theorem 1.1 via blow-up

From now on let the assumptions of Theorem 1.1 hold. Let u denote a local I -minimizer and suppose that $\omega > 0$ in (A3). (Otherwise the claim of Theorem 1.1 follows with $\Omega_0 = \Omega$ from [MP].) We further let

$$\tilde{h}(t) := t^{\omega} h(t), \quad t \geq 0,$$

and observe that \tilde{h} is a N -function. From (1.7) and Lemma 2.2 it follows that $u \in W_{h,\text{loc}}^1(\Omega; \mathbb{R}^M)$, hence the excess-function

$$E(x, r) := \int_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2 dy + \int_{B_r(x)} \tilde{h}(|\nabla u - (\nabla u)_{x,r}|) dy$$

for balls $B_r(x) \Subset \Omega$ is well-defined. Here and in what follows $\int \dots f, (f) \dots$ denote the mean value of a function f .

Lemma 3.1. *Fix $L > 0$ and a subdomain $\Omega' \Subset \Omega$. Then there is a constant $C_*(L)$ such that for every $\tau \in (0, 1)$ one can find a number $\varepsilon = \varepsilon(L, \tau)$ with the following property: if $B_r(x) \subset \Omega'$ and if*

$$(3.1) \quad |(\nabla u)_{x,r}| \leq L, \quad E(x, r) \leq \varepsilon,$$

then it holds

$$(3.2) \quad E(x, \tau r) \leq C_*(L) \tau^2 E(x, r).$$

Once having established Lemma 3.1, it is standard (see, e.g. Giaquinta's textbook [Gi]) to prove the desired partial regularity result. It turns out that the regular set Ω_0 is given by

$$\Omega_0 = \left\{ x \in \Omega : \sup_{r>0} |(\nabla u)_{x,r}| < \infty \text{ and } \liminf_{r \downarrow 0} E(x, r) = 0 \right\},$$

i.e. Lemma 3.1 shows that the set on the r.h.s. is open and $\nabla u \in C^{0,\alpha}$ there for any $0 < \alpha < 1$. Obviously it is a set of full Lebesgue measure.

We divide the proof of Lemma 3.1 into several steps.

Step 1. Scaling

We argue by contradiction. Let $L > 0$ and choose $C_* = C_*(L)$ as outlined in Step 2. Then, for some $\tau \in (0, 1)$, there is a sequence of balls $B_{r_m}(x_m) \Subset \Omega'$ such that

$$(3.3) \quad |(\nabla u)_{x_m, r_m}| \leq L, \quad E(x_m, r_m) =: \lambda_m^2 \rightarrow 0, \quad \text{as } m \rightarrow \infty,$$

$$(3.4) \quad E(x_m, \tau r_m) > C_* \tau^2 \lambda_m^2.$$

Letting $a_m := (u)_{x_m, r_m}$, $A_m := (\nabla u)_{x_m, r_m}$ we define for $z \in B_1 := B_1(0)$

$$u_m(z) := \frac{1}{\lambda_m r_m} \left[u(x_m + r_m z) - a_m - r_m A_m z \right]$$

and get from (3.3)

$$(3.5) \quad |A_m| \leq L, \quad \int_{B_1} |\nabla u_m|^2 dz + \lambda_m^{-2} \int_{B_1} \tilde{h}(\lambda_m |\nabla u_m|) dz = 1.$$

On the other hand, (3.4) reads after scaling

$$(3.6) \quad \int_{B_\tau} |\nabla u_m - (\nabla u_m)_{0,\tau}|^2 dz + \lambda_m^{-2} \int_{B_\tau} \tilde{h}(\lambda_m |\nabla u_m - (\nabla u_m)_{0,\tau}|) dz > C_* \tau^2.$$

After passing to suitable subsequences we obtain from (3.5)

$$(3.7) \quad \begin{aligned} A_m &\rightharpoonup: A, u_m \rightharpoonup: \bar{u} \quad \text{in } W_2^1(B_1; \mathbb{R}^M), \\ \lambda_m \nabla u_m &\rightarrow 0 \quad \text{in } L^2(B_1; \mathbb{R}^{nM}) \text{ and a.e.}, \end{aligned}$$

where obviously $(\bar{u})_{0,1} = 0$, $(\nabla \bar{u})_{0,1} = 0$.

Step 2. Limit equation

For any $\varphi \in C_0^\infty(B_1; \mathbb{R}^M)$ the Euler equation satisfied by u implies after scaling

$$(3.8) \quad \begin{aligned} &\int_{B_1} D^2 H(A_m)(\nabla u_m, \nabla \varphi) dz \\ &= - \int_{B_1} \int_0^1 \left[D^2 H(Z_m) - D^2 H(A_m) \right] (\nabla u_m, \nabla \varphi) ds dz, \end{aligned}$$

where we have abbreviated

$$Z_m := Z_m(s, z) := A_m + s \lambda_m \nabla u_m(z).$$

The first line of (3.7) yields

$$(3.9) \quad \text{l.h.s. of (3.8)} \xrightarrow{m \rightarrow \infty} \int_{B_1} D^2 H(A)(\nabla \bar{u}, \nabla \varphi) dz.$$

For discussing the r.h.s of (3.8) we let $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ such that

$$(3.10) \quad \int_A |\nabla \varphi|^2 dz \leq \varepsilon$$

for any measurable subset A of B_1 such that $\mathcal{L}^n(A) \leq \delta$. The second line of (3.7) gives the existence of $S \subset B_1$ such that $\mathcal{L}^n(B_1 - S) \leq \delta$ and

$$(3.11) \quad \lambda_m \nabla u_m \rightharpoonup 0 \text{ on } S.$$

From (3.11) we immediatly infer (using also (3.5))

$$(3.12) \quad \begin{aligned} &\left| \int_S \int_0^1 \left[D^2 H(Z_m) - D^2 H(A_m) \right] (\nabla u_m, \nabla \varphi) ds dz \right| \\ &\leq \sup_{S \times [0,1]} \left| [\dots] \right| \left(\int_{B_1} |\nabla u_m|^2 dz \right)^{1/2} \left(\int_{B_1} |\nabla \varphi|^2 dz \right)^{1/2} \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. At the same time we use (1.5) to see

$$\begin{aligned}
T &:= \left| \int_{B_1-S} \int_0^1 [\dots](\nabla u_m, \nabla \varphi) ds dz \right| \\
&\leq c \int_{B_1-S} \int_0^1 \{1 + |D^2 H(Z_m)|\} |\nabla u_m| |\nabla \varphi| ds dz \\
&\leq c \int_{B_1-S} \int_0^1 \left\{ 1 + (1 + |Z_m|^2)^{\frac{\omega}{2}} \frac{h'(|Z_m|)}{|Z_m|} \right\} |\nabla u_m| |\nabla \varphi| ds dz \\
&= c \left\{ \int_{(B_1-S) \cap M_1} \int_0^1 \dots ds dz + \int_{(B_1-S) \cap M_2} \int_0^1 \dots ds dz \right\} \\
&=: c\{T_1 + T_2\},
\end{aligned}$$

where we have abbreviated

$$M_1 := [\lambda_m |\nabla u_m| \leq K], M_2 := [\lambda_m |\nabla u_m| > K]$$

for a sufficiently large number K . On M_1 we have

$$|Z_m| \leq \sup_m |A_m| + K \leq L + K =: k$$

so that

$$\begin{aligned}
(1 + |Z_m|^2)^{\frac{\omega}{2}} \frac{h'(|Z_m|)}{|Z_m|} &\leq c(K) \frac{h'(k)}{k} \\
&\stackrel{(A3)}{\leq} c(K) h''(k)
\end{aligned}$$

and in conclusion

$$\begin{aligned}
T_1 &\leq c(K) \int_{B_1-S} |\nabla u_m| |\nabla \varphi| dz \\
&\leq c(K) \left(\int_{B_1-S} |\nabla u_m|^2 dz \right)^{1/2} \left(\int_{B_1-S} |\nabla \varphi|^2 dz \right)^{1/2} \\
(3.13) \quad &\stackrel{(3.10)}{\leq} c(K) \sqrt{\varepsilon}.
\end{aligned}$$

For T_2 we obtain (assuming $|Z_m| \leq 2\lambda_m |\nabla u_m|$ on M_2)

$$\begin{aligned}
T_2 &\leq c \int_{(B_1-S) \cap M_2} |\nabla u_m| |\nabla \varphi| dz \\
&\quad + c \int_{(B_1-S) \cap M_2} (\lambda_m |\nabla u_m|)^{\omega} \frac{h'(2\lambda_m |\nabla u_m|)}{2\lambda_m |\nabla u_m|} |\nabla u_m| |\nabla \varphi| dz,
\end{aligned}$$

and as before the first integral on the r.h.s. is bounded by $c\sqrt{\varepsilon}$. Using (1.4) and (A2) we get

$$T_2 \leq c\sqrt{\varepsilon} + c\lambda_m^{-1} \int_{(B_1-S) \cap M_2} (\lambda_m |\nabla u_m|)^{\omega-1} h(\lambda_m |\nabla u_m|) |\nabla \varphi| dz.$$

Recalling the definition of \tilde{h} , we see

$$\omega t^{\omega-1} h(t) \leq \tilde{h}'(t),$$

hence

$$(3.14) \quad T_2 \leq c\sqrt{\varepsilon} + c\lambda_m^{-1} \int_{(B_1-S) \cap M_2} \tilde{h}'(\lambda_m |\nabla u_m|) |\nabla \varphi| dz,$$

and it remains to discuss the integral on the r.h.s. of (3.14). To this purpose we let for $\tau > 0$ $g_\tau(t) := \tau \tilde{h}^*(t)$, \tilde{h}^* being the conjugate function of \tilde{h} , and use Young's inequality in the form

$$\alpha\beta \leq g_\tau(\alpha) + g_\tau^*(\beta)$$

for numbers $\alpha, \beta \geq 0$. With $\alpha := \tilde{h}'(\lambda_m |\nabla u_m|)$, $\beta := |\nabla \varphi|$ we obtain

$$\begin{aligned} \tilde{h}'(\lambda_m |\nabla u_m|) |\nabla \varphi| &\leq \tau \tilde{h}^* \left(\tilde{h}'(\lambda_m |\nabla u_m|) \right) + g_\tau^*(|\nabla \varphi|) \\ &= \tau \left[\lambda_m |\nabla u_m| \tilde{h}'(\lambda_m |\nabla u_m|) - \tilde{h}(\lambda_m |\nabla u_m|) \right] + g_\tau^*(|\nabla \varphi|) \\ &\leq c\tau \tilde{h}(\lambda_m |\nabla u_m|) + g_\tau^*(|\nabla \varphi|), \end{aligned}$$

$$g_\tau^*(|\nabla \varphi|) = \sup_{\gamma \geq 0} \left[\beta \gamma - \tau \tilde{h}^*(\gamma) \right] = \tau \sup_{\gamma \geq 0} \left[\frac{\beta}{\tau} \gamma - \tilde{h}^*(\gamma) \right] = \tau \tilde{h}(\beta/\tau) = \tau \tilde{h}(|\nabla \varphi|/\tau).$$

This gives

$$\begin{aligned} \xi_m &:= \lambda_m^{-1} \int_{(B_1-S) \cap M_2} \tilde{h}'(\lambda_m |\nabla u_m|) |\nabla \varphi| dz \\ &\leq c\lambda_m^{-1} \tau \int_{B_1} \tilde{h}(\lambda_m |\nabla u_m|) dz + c\lambda_m^{-1} \tau \int_{B_1} \tilde{h} \left(\frac{1}{\tau} |\nabla \varphi| \right) dz. \end{aligned}$$

Now let $\tau := \lambda_m^{\delta-1}$ for a small $\delta > 0$. Then it holds

$$\xi_m \leq c \left[\lambda_m^{\delta-2} \int_{B_1} \tilde{h}(\lambda_m |\nabla u_m|) dz + \lambda_m^{\delta-2} \int_{B_1} \tilde{h}(\lambda_m^{1-\delta} |\nabla \varphi|) dz \right].$$

The definition of \tilde{h} gives for any $\ell > 0$

$$\tilde{h}(t) \leq \text{const}(\ell) t^{2+\omega} \quad \forall 0 \leq t \leq \ell,$$

where $\text{const}(\ell)$ is depending on the value of ℓ . Here we choose $\ell := \|\nabla \varphi\|_\infty$, hence

$$\lambda_m^{\delta-2} \int_{B_1} \tilde{h}(\lambda_m^{1-\delta} |\nabla \varphi|) dz \leq c(\|\nabla \varphi\|_\infty) \lambda_m^{\delta-2+(1-\delta)(2+\omega)},$$

and since ω is positive, it is possible to choose $\delta > 0$ such that

$$\lambda_m^{\delta-2} \int_{B_1} \tilde{h}(\lambda_m^{1-\delta} |\nabla \varphi|) dz \rightarrow 0, \quad m \rightarrow \infty.$$

Since according to (3.5)

$$\lambda_m^{\delta-2} \int_{B_1} \tilde{h}(\lambda_m |\nabla u_m|) dz \rightarrow 0, \quad m \rightarrow \infty,$$

we arrive at $\lim_{m \rightarrow \infty} \xi_m = 0$. Combining this fact with the estimates (3.12) - (3.14) it is shown that

$$(3.15) \quad \limsup_{m \rightarrow \infty} |\text{r.h.s. of (3.8)}| \leq c\sqrt{\varepsilon}.$$

Now ε can be chosen as small as we want, hence (3.8) and (3.9) together with (3.15) lead to the elliptic system with constant coefficients

$$\int_{B_1} D^2 H(A)(\nabla \bar{u}, \nabla \varphi) dz = 0$$

satisfied by \bar{u} . According to [Gi] \bar{u} is of class $C^\infty(B_1; \mathbb{R}^M)$, and we have the Campanato estimate

$$\int_{B_\tau} |\nabla \bar{u} - (\nabla \bar{u})_{0,\tau}|^2 dz \leq C^* \tau^2 \int_{B_1} |\nabla \bar{u} - (\nabla \bar{u})_{0,1}|^2 dz$$

for a constant $C^* = C^*(L)$. Observing (see (3.5)) $\int_{B_1} |\nabla \bar{u}|^2 dz \leq 1$ and recalling $(\nabla \bar{u})_{0,1} = 0$, we get

$$(3.16) \quad \int_{B_\tau} |\nabla \bar{u} - (\nabla \bar{u})_{0,\tau}|^2 dz \leq C^* \tau^2.$$

Letting $C_* := 2C^*$ we see that (3.16) contradicts (3.6) as soon as we can show the validity of

$$(3.17) \quad \nabla u_m \rightarrow \nabla \bar{u} \text{ in } L_{\text{loc}}^2(B_1; \mathbb{R}^{nM}),$$

$$(3.18) \quad \lambda_m^{-2} \int_{B_r} \tilde{h}(\lambda_m |\nabla u_m|) dz \rightarrow 0, \quad r < 1.$$

In fact, (3.17) implies the convergence of the first integral on the l.h.s. of (3.6) towards the integral on the l.h.s. of (3.16). We further have

$$\begin{aligned} & \lambda_m^{-2} \int_{B_\tau} \tilde{h}(\lambda_m |\nabla u_m - (\nabla u_m)_{0,\tau}|) dz \\ & \leq c \left[\lambda_m^{-2} \int_{B_\tau} \tilde{h}(\lambda_m |\nabla u_m|) dz + \lambda_m^{-2} \tilde{h}(\lambda_m |(\nabla u_m)_{0,\tau}|) \right] \end{aligned}$$

which follows from the convexity and the $(\Delta 2)$ -property of \tilde{h} . By (3.18) the first term on the r.h.s. of the above inequality vanishes as $m \rightarrow \infty$, and (3.17) yields $|(\nabla u_m)_{0,\tau}| \rightarrow |(\nabla \bar{u})_{0,\tau}|$ as $m \rightarrow \infty$, hence $\lambda_m^{-2} \tilde{h}(\lambda_m |(\nabla u_m)_{0,\tau}|)$ behaves like $\lambda_m^{-2} \lambda_m^{2+\omega} \xrightarrow{m \rightarrow \infty} 0$, and therefore the second term on the l.h.s. of (3.6) vanishes as $m \rightarrow \infty$.

Step 3. Proof of (3.17) and (3.18)

We return to (2.6), observe $\frac{h'(t)}{t} \geq h''(0)$, choose $Q = A_m$ and obtain after scaling (for a suitable choice of η)

$$(3.19) \quad \int_{B_t} |\nabla^2 u_m| dz \leq C(1-t)^{-2} \int_{B_1} |D^2 H(\lambda_m \nabla u_m + A_m)| |\nabla u_m|^2 dz$$

valid for $0 < t < 1$. On $[\lambda_m |\nabla u_m| \leq K]$ we have

$$|D^2 H(A_m + \lambda_m \nabla u_m)| |\nabla u_m|^2 \leq c(K) |\nabla u_m|^2,$$

whereas on $[\lambda_m |\nabla u_m| \geq K]$ it holds (K large enough)

$$\begin{aligned} & |D^2 H(\lambda_m \nabla u_m + A_m)| |\nabla u_m|^2 \\ & \leq c(K) \left[1 + (\lambda_m |\nabla u_m|)^\omega \frac{h'(\lambda_m |\nabla u_m|)}{\lambda_m |\nabla u_m|} \right] |\nabla u_m|^2 \\ & \leq c(K) \left[|\nabla u_m|^2 + \lambda_m^{-2} \tilde{h}(\lambda_m |\nabla u_m|) \right]. \end{aligned}$$

(3.19) therefore implies

$$(3.20) \quad \int_{B_t} |\nabla^2 u_m|^2 dz \leq c(1-t)^{-2} \left[\int_{B_1} |\nabla u_m|^2 dz + \lambda_m^{-2} \int_{B_1} \tilde{h}(\lambda_m |\nabla u_m|) dz \right],$$

and by (3.5) the r.h.s. of (3.20) is bounded. This together with (3.7) proves (3.17). Let us denote by $c(t)$ a bound for the r.h.s. of (3.20). Then another application of (2.6) yields for $t \in (0, 1)$

$$(3.21) \quad \int_{B_t} \frac{h'(|\lambda_m \nabla u_m + A_m|)}{|\lambda_m \nabla u_m + A_m|} |\nabla^2 u_m|^2 dz \leq c(t).$$

We introduce the auxiliary functions

$$\Psi_m := \frac{1}{\lambda_m} \left\{ \int_0^{|\lambda_m \nabla u_m + A_m|} \sqrt{\frac{h'(t)}{t}} dt - \int_0^{|A_m|} \sqrt{\frac{h'(t)}{t}} dt \right\}$$

and observe that by (3.21)

$$(3.22) \quad \int_{B_t} |\nabla \Psi_m|^2 dz \leq c(t).$$

On $[\lambda_m |\nabla u_m| \leq K]$ it holds

$$\begin{aligned} |\Psi_m| &= \frac{1}{\lambda_m} \left| \int_{|A_m|}^{|A_m + \lambda_m \nabla u_m|} \sqrt{\frac{h'(t)}{t}} dt \right| \\ &\leq \frac{1}{\lambda_m} \left| |A_m + \lambda_m \nabla u_m| - |A_m| \right| \sqrt{\frac{h'(k)}{k}}, \end{aligned}$$

where $k := K + \sup_m |A_m|$. We therefore get

$$|\Psi_m| \leq \sqrt{\frac{h'(k)}{k}} |\nabla u_m|$$

provided $\lambda_m |\nabla u_m| \leq K$. For K large enough we can assume that

$$|A_m + \lambda_m \nabla u_m| \leq 2\lambda_m |\nabla u_m|$$

on $[\lambda_m |\nabla u_m| \geq K]$, hence

$$\begin{aligned} |\Psi_m| &\leq c \frac{1}{\lambda_m} \lambda_m |\nabla u_m| \sqrt{\frac{h'(2\lambda_m |\nabla u_m|)}{2\lambda_m |\nabla u_m|}} \stackrel{(1.4), (A2)}{\leq} c \lambda_m^{-1} h(\lambda_m |\nabla u_m|)^{1/2} \\ &\leq c \lambda_m^{-1} \tilde{h}(\lambda_m |\nabla u_m|)^{1/2}, \end{aligned}$$

if $\lambda_m |\nabla u_m| \geq K$. Recalling the bounds from (3.5) it is shown that

$$\int_{B_1} \Psi_m^2 dz \leq c < \infty$$

which together with (3.22) implies ($0 < t < 1$)

$$(3.23) \quad \|\Psi_m\|_{W_2^1(B_t)} \leq c(t) < \infty.$$

Let $A_K(t) := B_t \cap [\lambda_m |\nabla u_m| \leq K]$. We use $\tilde{h}(t) \leq c_K t^{2+\omega} =: c_K t^s$ for $t \leq K$ to get

$$\begin{aligned} &\int_{A_K(t)} \lambda_m^{-2} \tilde{h}(\lambda_m |\nabla u_m|) dz \\ &\leq c \left[\lambda_m^{s-2} \int_{A_K(t)} |\nabla u_m - \nabla \bar{u}|^s dz + \lambda_m^{s-2} \int_{A_K(t)} |\nabla \bar{u}|^s dz \right] \\ &\leq c \left[\lambda_m^{s-2} \int_{A_K(t)} \{ |\nabla u_m|^{s-2} + |\nabla \bar{u}|^{s-2} \} |\nabla u_m - \nabla \bar{u}|^2 dz + \lambda_m^{s-2} \int_{A_K(t)} |\nabla \bar{u}|^s dz \right]. \end{aligned}$$

Using the local boundedness of $\nabla \bar{u}$ as well as (3.17) we deduce

$$(3.24) \quad \lambda_m^{-2} \int_{A_K(t)} \tilde{h}(\lambda_m |\nabla u_m|) dz \longrightarrow 0, \quad m \rightarrow \infty.$$

Now consider the set $\tilde{A}_K(t) := B_t \cap [\lambda_m |\nabla u_m| \geq K]$, on which the following estimates are valid:

$$\begin{aligned}
\Psi_m &= \frac{1}{\lambda_m} \int_{|A_m|}^{|\lambda_m \nabla u_m + A_m|} \sqrt{\frac{h'(t)}{t}} dt \\
&\geq \frac{1}{\lambda_m} \int_{\frac{1}{2}\lambda_m |\nabla u_m|}^{\frac{2}{3}\lambda_m |\nabla u_m|} \sqrt{\frac{h'(t)}{t}} dt \\
&\geq c \frac{1}{\lambda_m} \lambda_m |\nabla u_m| \sqrt{\frac{h'(\lambda_m |\nabla u_m|/2)}{\frac{1}{2}\lambda_m |\nabla u_m|}} \geq \frac{c}{\lambda_m} \sqrt{h(\lambda_m |\nabla u_m|)}.
\end{aligned}$$

This gives

$$(3.25) \quad \Psi_m^2 \geq c \lambda_m^{-2} h(\lambda_m |\nabla u_m|) \text{ on } \tilde{A}_K(t),$$

and therefore

$$\begin{aligned}
&\lambda_m^{-2} \int_{\tilde{A}_K(t)} \tilde{h}(\lambda_m |\nabla u_m|) dz \\
&= \lambda_m^{-2} \int_{\tilde{A}_K(t)} (\lambda_m |\nabla u_m|)^\omega h(\lambda_m |\nabla u_m|) dz \\
&\stackrel{(3.25)}{\leq} c \int_{\tilde{A}_K(t)} (\lambda_m |\nabla u_m|)^\omega \Psi_m^2 dz \\
&\leq c \left(\int_{B_t} \Psi_m^{\frac{2n}{n-2}} dz \right)^{1-2/n} \left(\int_{\tilde{A}_K(t)} (\lambda_m |\nabla u_m|)^{\omega \frac{n}{2}} dz \right)^{2/n} \\
&\stackrel{(3.23)}{\leq} c(t) \left(\int_{\tilde{A}_K(t)} (\lambda_m |\nabla u_m|)^{\omega \frac{n}{2}} dz \right)^{2/n} \\
&\stackrel{(1.7)}{\leq} c(t) \left[\mathcal{L}^n(\tilde{A}_K(t)) + \int_{\tilde{A}_K(t)} h(\lambda_m |\nabla u_m|)^{\frac{n}{n-2}} dz \right].
\end{aligned}$$

The second line of (3.7) shows $\mathcal{L}^n(\tilde{A}_K(t)) \rightarrow 0$ as $m \rightarrow \infty$, moreover we have

$$\begin{aligned}
\int_{\tilde{A}_K(t)} h(\lambda_m |\nabla u_m|)^{\frac{n}{n-2}} dz &\stackrel{(3.25)}{\leq} c \lambda_m^{\frac{2n}{n-2}} \int_{B_t} \Psi_m^{\frac{2n}{n-2}} dz \\
&\stackrel{(3.23)}{\leq} c(t) \lambda_m^{\frac{2n}{n-2}} \rightarrow 0, \quad m \rightarrow \infty,
\end{aligned}$$

hence $\int_{\tilde{A}_K(t)} \lambda_m^{-2} \tilde{h}(\lambda_m |\nabla u_m|) dz \rightarrow 0$, $m \rightarrow \infty$. This together with (3.24) proves (3.18) which completes the proof of Lemma 3.1 and thereby of Theorem 1.1. \square

4 Sketch of the proof of Theorem 1.2

Clearly we can restrict ourselves to the case $k = 2$ together with $M = 1$. Then we introduce a local regularization as done in [ApF], Section 2, and prove - dropping the approximation parameter:

Lemma 4.1. *The function*

$$\Psi := \int_0^{|\nabla^2 u|} \sqrt{\frac{h'(t)}{t}} dt$$

is of class $W_{2,\text{loc}}^1(\Omega)$ (uniformly w.r.t. the approximation).

A complete proof of Lemma 4.1 is presented in [Fu1], Step 2 of the proof of Theorem 1. Lemma 2.2 has to be replaced by

Lemma 4.2. *The function $h(|\nabla^2 u|)^{\frac{n}{n-2}}$ is in the space $L_{\text{loc}}^1(\Omega)$ (uniformly w.r.t. the regularization).*

A version of the Caccioppoli inequality (2.5) valid for the higher order case has been established in (2.11) of [ApF]. During the blow-up procedure we need the following adjustments: now we let

$$E(x, r) := \int_{B_r(x)} |\nabla^2 u - (\nabla^2 u)_{x,r}|^2 dy + \int_{B_r(x)} \tilde{h}(|\nabla^2 u - (\nabla^2 u)_{x,r}|) dy$$

and require $|(\nabla^2 u)_{x,r}| \leq L$ in (3.1). We define a_m, A_m as after (3.4) and let $\Theta_m := (\nabla^2 u)_{x_m, r_m}$ as well as

$$\begin{aligned} \hat{u}_m(z) := & \frac{1}{\lambda_m r_m^2} \left[u_m(x_m + r_m z) - a_m - r_m A_m z \right. \\ & \left. - \frac{1}{2} r_m^2 \Theta_m(z, z) + \frac{1}{2} r_m^2 \int_{B_1} \Theta_m(\tilde{z}, \tilde{z}) d\tilde{z} \right], z \in B_1. \end{aligned}$$

(3.7) has to be replaced by

$$(4.1) \quad \begin{cases} \Theta_m \rightarrow: \Theta, \hat{u}_m \rightarrow: \hat{u} \text{ in } W_2^2(B_1), \\ \lambda_m \nabla^2 \hat{u}_m \rightarrow 0 \text{ in } L^2(B_1; \mathbb{R}^{n \times n}) \text{ and a.e.} \end{cases}$$

Let $H(Z) := h(|Z|)$. From (4.1) we obtain along the lines of Step 2 of Section 3 the limit equation

$$(4.2) \quad \int_{B_1} D^2 H(\Theta)(\nabla^2 \hat{u}, \nabla^2 \varphi) dz = 0$$

valid for all $\varphi \in C_0^\infty(B_1)$, and to (4.2) we can apply the Campanato-type estimate (3.11) of [ApF] by the way fixing the value of C_* . Finally we prove (see (3.17) and (3.18))

$$(4.3) \quad \begin{cases} \nabla^2 \hat{u}_m \rightarrow \nabla^2 \hat{u} \text{ in } L_{\text{loc}}^2(B_1; \mathbb{R}^{n \times n}), \\ \lambda_m^{-2} \int_{B_r} \tilde{h}(\lambda_m |\nabla^2 \hat{u}_m|) dz \rightarrow 0, r < 1, \end{cases}$$

which is done exactly as in Step 3 of Section 3. But as before (4.3) will lead to the desired contradiction by the way proving Theorem 1.2.

□

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