Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 239

Generalizations Of Korn's Inequality Based On Gradient Estimates In Orlicz Spaces And Applications To Variational Problems In 2D Involving The Trace Free Part Of The Symmetric Gradient

Martin Fuchs

Saarbrücken 2009

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 239
submitted: 17^{th} of August, 2009

Generalizations Of Korn's Inequality Based On Gradient Estimates In Orlicz Spaces And Applications To Variational Problems In 2D Involving The Trace Free Part Of The Symmetric Gradient

Martin Fuchs

Saarland University
Dep. of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

AMS Classification: 49 N 60, 83 C

Keywords: generalized Korn's inequality, Orlicz–Sobolev spaces, regularity of minimizers, 2D problems

Abstract

We prove variants of Korn's inequality involving the deviatoric part of the symmetric gradient of fields $u : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ belonging to Orlicz-Sobolev classes. These inequalities are derived with the help of gradient estimates for the Poisson equation in Orlicz spaces. We apply these Korn type inequalities to variational integrals of the form $\int_\Omega h\left(|\varepsilon^D(u)|\right)\, dx$ occurring in General Relativity and prove $C^{1,\alpha}$–regularity results for minimizers under rather general conditions on the N–function h. A further useful tool for this analysis is an appropriate version of the (Sobolev-) Poincaré inequality with $\varepsilon^D(u)$ measuring the distance of u to the holomorphic functions.

1 Introduction

Korn's inequality is a classical tool in the analysis of problems arising in (linear) elasticity or in fluid mechanics (see, e.g., [Ze]). Recently Dain [Da] discussed the following variant: let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain. For fields $u : \Omega \to \mathbb{R}^n$ we consider the symmetric gradient

$$\varepsilon(u) := \frac{1}{2}\left(\partial_i u^j + \partial_j u^i\right)_{1\le i,j\le n}$$

and its deviatoric part

$$\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}(\operatorname{div} u)\,\mathbf{1},$$

$\mathbf{1}$ denoting the unit matrix. Then we have the inequality

$$\int_\Omega |\nabla u|^2\, dx \le C\left[\int_\Omega |u|^2\, dx + \int_\Omega \left|\varepsilon^D(u)\right|^2\, dx\right] \tag{1.1}$$

being valid for functions u from the Sobolev class $W^1_2(\Omega;\mathbb{R}^n)$ (see [Ad] for a definition) and for dimensions n at least 3. This is the result of Theorem 1.1 in [Da], and as remarked later inequality (1.1) is false if the case $n = 2$ is considered: the validity of (1.1) for domains $\Omega \subset \mathbb{R}^2$ would imply that the kernel of ε^D is of finite dimension but $\varepsilon^D(u) = 0$

Acknowledgement. The author likes to thank Professor Shulin Zhou for his valuable comments and suggestions.

on Ω if and only if u is holomorphic. The interest for estimates of the form (1.1) becomes clear, when we look at variational integrals like

$$I[u,\Omega] := \int_\Omega H\left(\varepsilon^D(u)\right)\, dx \tag{1.2}$$

and ask for their coercivity if for example Dirichlet boundary data are prescribed. Here we note that energies involving $\varepsilon^D(u)$ naturally occur in General Relativity as outlined in the paper of Bartnik and Isenberg [BI]. Before passing to the twodimensional case we refer the reader to the thesis [Sc] in which for $n \geq 3$ several variants of (1.1) valid in the spaces $W^1_p(\Omega;\mathbb{R}^n)$ and $\overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n)$ with arbitrary exponent $p \in (1,\infty)$ are presented together with applications concerning the question of smoothness of (local) minima of the energy from (1.2) at least for densities of quadratic growth. From now on we assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^2$. Then we have

THEOREM 1.1. *For any $p \in (1,\infty)$ there is a constant $C = C(p,\Omega)$ such that*

$$\|\nabla u\|_{L^p(\Omega)} \leq C \left\|\varepsilon^D(u)\right\|_{L^p(\Omega)} \tag{1.3}$$

is true for any function $u \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^2)$.

COROLLARY 1.1. *For $p \in (1,\infty)$ let $X_p := \left\{u \in L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^2) : \varepsilon^D(u) \in L^p_{\mathrm{loc}}(\Omega;\mathbb{S}^2)\right\}$, where $\varepsilon^D(u)$ is defined in the sense of distributions and where $\mathbb{S}^2$ is the space of symmetric (2×2)–matrices. Then we have $X_p = W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^2)$.*

Both results have been established in the recent paper [FS], where it is also shown how to get $C^{1,\alpha}$–regularity of local I–minimizers with the help of (1.3) at least when H is uniformly elliptic, i.e. it holds

$$\lambda\,|\tau|^2 \leq D^2H(\sigma)(\tau,\tau) \leq \Lambda\,|\tau|^2 \tag{1.4}$$

for all σ, $\tau \in \mathbb{S}^2$ with constants λ, $\Lambda > 0$.

The main purpose of the present note is to establish an interior regularity result for (local) minima of the functional defined in (1.2) in the case that

$$H(\sigma) = h\left(|\sigma|\right),\ \sigma \in \mathbb{S}^2\,, \tag{1.5}$$

for a suitable N-function h, which means that we consider densities depending on the modulus of $\varepsilon^D(u)$, and from the foregoing explanations it should be clear that this requires a variant of Theorem 1.1 for Orlicz spaces. To be precise, let $h : [0,\infty) \to [0,\infty)$ denote a function of class C^2 satisfying the following hypotheses:

(A1) h is strictly increasing and convex together with $h''(0) > 0$ and $\displaystyle\lim_{t\searrow 0}\frac{h(t)}{t} = 0$;

(A2) there is a constant $\overline{k} > 0$ such that $h(2t) \le \overline{k}\; h(t)$ for all $t \ge 0$;

(A3) we have $\dfrac{h'(t)}{t} \le h''(t)$ for all $t \ge 0$.

Let us add some comments: from (A1) it follows that $h(0) = 0 = h'(0)$ and $h'(t) > 0$ for all $t > 0$. (A3) implies that $t \mapsto h'(t)/t$ is increasing, moreover we get

$$h(t) \ge \frac{1}{2} h''(0) t^2,\ t \ge 0\,, \tag{1.6}$$

so that h is of at least quadratic growth. (A2) is the $(\Delta 2)$–condition, hence with $m \ge 2$ and $c > 0$ we find (see, e.g., [RR])

$$h(t) \le c(t^m + 1)\,, \tag{1.7}$$

and by convexity $h'(t)$ grows at most like t^{m-1}. Note also that (A1) together with (1.6) gives that h is a N-function in the sense of [Ad, Section 8.2], for which

$$\frac{1}{\overline{k}} h'(t) t \le h(t) \le t h'(t),\ t \ge 0\,, \tag{1.8}$$

holds. (1.8) is a simple consequence of of the above assumptions. According to (1.5) we have ($\sigma \in \mathbb{S}^2$)

$$\min \quad \left\{ h''(|\sigma|),\ \frac{h'(|\sigma|)}{|\sigma|} \right\} \le D^2 H(\sigma) \le \max\{\ldots\}$$

(in the sense of bilinear forms), hence by (A3)

$$h''(0) \le \frac{h'(|\sigma|)}{|\sigma|} \le D^2 H(\sigma),\ \sigma \in \mathbb{S}^2\ , \tag{1.9}$$

which means that

$$h''(0) |\tau|^2 \le \frac{h'(|\sigma|)}{|\sigma|} |\tau|^2 \le D^2 H(\sigma)(\tau, \tau)$$

is valid for all τ, $\sigma \in \mathbb{S}^2$.

Observing

$$t h'(t) = \int_0^t \frac{d}{ds} \left[s h'(s) \right] ds = h(t) + \int_0^t s h''(s)\, ds$$

we see that (A3) implies the validity of

$$a(h) := \inf_{t > 0} \frac{t h'(t)}{h(t)} \ge 2\,.$$

Therefore h is a N–function of (global) type $(\nabla 2)$, which follows from Corollary 4 on p. 26 in [RR].

After these preparations we can state the following variants of Theorem 1.1 and Corollary 1.1

THEOREM 1.2. *Suppose that h shares the properties (A1–3) stated above. Then there is a constant $C = C(h, \Omega)$ such that*

$$\int_\Omega h\left(|\nabla u|\right) dx \le C \int_\Omega h\left(\left|\varepsilon^D(u)\right|\right) dx \tag{1.10}$$

holds for any function from the Orlicz-Sobolev class $\overset{\circ}{W}{}^1_h(\Omega;\mathbb{R}^2)$, and this is true for N-functions h just satisfying $(\Delta 2) \cap (\nabla 2)$.

COROLLARY 1.2. *With h as in Theorem 1.2 we define the local space*

$$X_h := \left\{ u \in L_{h,\mathrm{loc}}(\Omega;\mathbb{R}^2) : \ \varepsilon^D(u) \in L_{h,\mathrm{loc}}(\Omega;\mathbb{S}^2) \right\} .$$

Then it holds $X_h = W^1_{h,\mathrm{loc}}(\Omega;\mathbb{R}^2)$.

REMARK 1.1. *For a definition of the Orlicz classes $L_{h,\mathrm{loc}}(\ldots)$ and the related Orlicz-Sobolev spaces $\overset{\circ}{W}{}^1_h(\ldots)$, $W^1_{h,\mathrm{loc}}(\ldots)$ we refer the reader to the textbooks [Ad] or [RR].*

REMARK 1.2. *The proof of Theorem 1.2 is based on Theorem 3.1 established by Jia, Li and Wang in their work [JLW], and there is also a strong connection to the recent papers [YSZ], [BYZ] of Byun, Sun, Yao and Zhou. It will become clear that we do not need the full strength of our assumptions (A1–3) for the function h in order to establish Theorem 1.2. This will be the case for the investigation of the regularity of I–minimizers.*

Suppose next that we are given a function

$$u_0 \in W^1_h(\Omega;\mathbb{R}^2) \tag{1.11}$$

and define the class

$$\mathbb{K} := u_0 + \overset{\circ}{W}{}^1_h(\Omega;\mathbb{R}^2) . \tag{1.12}$$

Then we have

THEOREM 1.3. *Suppose that h satisfies (A1–3) and let (1.5) hold. Then, with I, u_0 and $\mathbb{K}$ being defined in (1.2), (1.11) and (1.12) respectively, the variational problem*

$$I[\,\cdot\,,\Omega] \to \min \ \ \text{in } \mathbb{K} \tag{1.13}$$

admits a unique solution u such that $u \in W^2_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2)$ and therefore

$$|\nabla u| \in L^s_{\mathrm{loc}}(\Omega) \quad \text{for any } s < \infty . \tag{1.14}$$

If in addition to (A3) it holds

$$h''(t) \le a(1+t^2)^{\frac{\omega}{2}} \frac{h'(t)}{t}, \ t \ge 0 , \tag{1.15}$$

for some $a > 0$ and with exponent $\omega \in [0, 2)$, then u is of class $C^{1,\alpha}(\Omega;\mathbb{R}^2)$ for any $\alpha > 0$. These regularity results extend to local minima $v \in X_h$ of the functional $I[\,\cdot\,,\Omega]$.

Our paper is organized as follows: Section 2 contains the proofs of Theorem 1.2 and Corollary 1.2. Moreover, as an application, we sketch how to get the existence part of Theorem 1.3. Section 3 is devoted to the proof of the higher integrability result (1.14) following ideas used in [Fu] in the setting fluids. The interior differentiability of the minimizer is discussed in Section 4 based on arguments developed with Bildhauer and Zhong in the paper [BFZ]. Here we will also make use of a particular Sobolev–Poincaré type inequality, whose proof is presented in the Appendix.

2 Proof of Theorem 1.2 and application to problem (1.13)

Let ϕ denote a N–function of type $(\Delta 2) \cap (\nabla 2)$. If $w \in \overset{\circ}{W}{}^1_\phi(\Omega)$ is a weak solution of $\Delta w = \operatorname{div} f$ with $f \in L_\phi(\Omega; \mathbb{R}^2)$, then it was kindly pointed out to us by S. Zhou, that Jia, Li and Wang [JLW] have shown the validity of the basic estimate

$$\int_\Omega \phi\left(|\nabla w|\right)\, dx \le C \int_\Omega \phi\left(|f|\right)\, dx \tag{2.1}$$

for a constant C being independent of w and f by the way extending earlier work of Meyers [Me] concerning the case $\phi(t) = t^p$. Now, if v is in $C_0^\infty(\Omega; \mathbb{R}^2)$, we obtain from formula (26) in Dain's paper [Da] the equation

$$\Delta v^j = 2\partial_i \varepsilon^D(v)_{ij},\ j = 1, 2\ , \tag{2.2}$$

where here and it what follows the sum is taken w.r.t. indices repeated twice. Letting $w = v^j, f = (2\varepsilon^D(v)_{ij})_{1\le i\le 2}$, we have by (2.2) that $\Delta w = \operatorname{div} f$ and by applying (2.1) to each component v^j we deduce the validity of (1.10) for test functions v and for N–functions h from the class $(\Delta 2) \cap (\nabla 2)$. Since $\overset{\circ}{W}{}^1_h(\Omega; \mathbb{R}^2)$ is the closure of $C_0^\infty(\Omega; \mathbb{R}^2)$ in $W^1_h(\Omega; \mathbb{R}^2)$, we finally arrive at inequality (1.10) for any function in $\overset{\circ}{W}{}^1_h(\Omega; \mathbb{R}^2)$. Of course, inequality (2.1) is of central importance for proving Theorem 1.2. We therefore add the following comment: if we require that we have $a(h) > 2$, then $\phi(t) := h(\sqrt{t}), t \ge 0$, is a N–function in $(\Delta 2) \cap (\nabla 2)$, and Theorem 1.12 in [BYZ] gives the estimate

$$\int_\Omega \phi\left(|\nabla w|^2\right)\, dx \le C \int_\Omega \phi\left(|f|^2\right)\, dx$$

for the unique weak solution w of $\Delta w = \operatorname{div} f$ with zero trace, provided we know $|f|^2 \in L_\phi(\Omega)$. By the definition of ϕ this implies (2.1) now for h, which means that under the additional hypothesis $a(h) > 2$ Theorem 1.2 can be derived from [BYZ]. The statement of Corollary 1.2 follows from standard arguments: fix a subdomain $\Omega' \Subset \Omega$ and a function $\eta \in C_0^\infty(\Omega), 0 \le \eta \le 1$, such that $\eta \equiv 1$ on Ω'. Moreover, let $w^{(\nu)}$ be a sequence of mollifications of a given function $w \in X_h$. From

$$\left|\varepsilon^D\left(\eta w^{(\nu)}\right)\right| \le c\left[\eta\left|\varepsilon^D(w^\nu)\right| + |\nabla\eta||w^{(\nu)}|\right]$$

in combination with standard properties of the mollification operator we deduce

$$\sup_{\nu} \int_{\Omega} h\left(|\varepsilon^D(\eta w^{(\nu)})|\right)\, dx < \infty\,,$$

and this together with (1.10) shows that $\eta w^{(\nu)}$ is a bounded sequence in the reflexive space $\overset{\circ}{W}{}^1_h(\Omega;\mathbb{R}^2)$, hence $\eta w^{(\nu)} \rightharpoondown: v$ for some v from this space. At the same time we have $\eta w^{(\nu)} \to \eta w$ in e.g. $L^2(\Omega;\mathbb{R}^2)$, thus $\eta w = v$, and we end up with $w = v$ on Ω, which shows that $w \in W^1_{h,\mathrm{loc}}(\Omega;\mathbb{R}^2)$ by arbitrariness of Ω'.

Next we establish the existence of a unique solution to problem (1.13): since $\mathbb{K} \neq \emptyset$, we can consider a minimizing sequence $\{u_k\} \subset \mathbb{K}$. Clearly it holds

$$\sup_{k} \int_{\Omega} h\left(|\varepsilon^D(u_k)|\right)\, dx < \infty\,,$$

hence

$$\sup_{k} \int_{\Omega} h\left(|\varepsilon^D(u_k - u_0)|\right)\, dx < \infty\,,$$

and (1.10) gives boundedness of $\{u_k - u_0\}$ in $\overset{\circ}{W}{}^1_h(\Omega;\mathbb{R}^2)$. By the reflexivity of the space $\overset{\circ}{W}{}^1_h(\Omega;\mathbb{R}^2)$ (see [Ad], 8.28 Theorem) we find a weak limit $\overline{u}$ in this class at least for a suitable subsequence of $\{u_k - u_0\}$, and obviously we have that $u := u_0 + \overline{u}$ is in $\mathbb{K}$ together with

$$I[u,\Omega] = \inf_{\mathbb{K}} I[\,\cdot\,,\Omega]\,, \tag{2.3}$$

since by lower semicontinuity we have

$$\int_{\Omega} H\left(\varepsilon^D(u)\right)\, dx \le \liminf_{k\to\infty} \int_{\Omega} H\left(\varepsilon^D(u_k)\right)\, dx\,.$$

Let us assume that (2.3) holds for a second function $\widetilde{u} \in \mathbb{K}$. If $\varepsilon^D(u) \neq \varepsilon^D(\widetilde{u})$ on a set with positive measure, then the strict convexity of H implies on this set

$$H\left(\varepsilon^D\left(\frac{u+\widetilde{u}}{2}\right)\right) < \frac{1}{2}H\left(\varepsilon^D(u)\right) + \frac{1}{2}H\left(\varepsilon^D(\widetilde{u})\right)\,,$$

which together with (2.3) leads to the contradiction

$$I\left[\frac{u+\widetilde{u}}{2},\Omega\right] < \inf_{\mathbb{K}} I[\,\cdot\,,\Omega]\,.$$

Therefore we must have $\varepsilon^D(u - \widetilde{u}) = 0$ and by quoting (1.10) one more time we get $\nabla u = \nabla\widetilde{u}$, hence $u = \widetilde{u}$. $\square$

3 Proof of the higher integrability result (1.14)

Let all the assumptions of Theorem 1.3 hold and consider the unique solution u of problem (1.13) constructed in the previous Section. Since we do not know, if u is smooth enough in order to carry out the subsequent calculations, we have to replace (1.13) by a suitable regularization admitting regular solutions. We recall the following technical lemma being established in [BF]:

LEMMA 3.1. *For $\ell \in \mathbb{N}$ consider functions $\eta_\ell \in C^1([0,\infty))$ such that $0 \le \eta_\ell < 1$, $\eta_\ell' \le 0$, $|\eta'| \le c/\ell$, $\eta \equiv 1$ on $[0, \frac{3}{2}\ell]$ and $\eta \equiv 0$ on $[2\ell, \infty)$. Let*

$$h_\ell(t) := \int_0^t s\, g_\ell(s)\, ds, \ t \ge 0 \, ,$$

where

$$g_\ell(t) := g(0) + \int_0^t \eta_\ell(s) g'(s)\, ds$$

with $g(t) := h'(t)/t, \ t \ge 0$.

Then it holds:

a) h_ℓ satisfies (A1–3) with constants being independent of ℓ. If h satisfies (1.15) then the same is true for h_ℓ, again with an uniform constant.

b) $H_\ell(\sigma) := h_\ell(|\sigma|) \le h(|\sigma|) = H(\sigma)$, $\lim_{\ell\to\infty} H_\ell(\sigma) = H(\sigma)$, $\sigma \in \mathbb{S}^2$.

c) *H_ℓ is of quadratic growth which follows from*

$$c\,|\tau|^2 \le D^2 H_\ell(\sigma)(\tau,\tau) \le \Lambda(\ell)\,|\tau|^2 \, , \quad \sigma, \tau \in \mathbb{S}^2 \, , \tag{3.1}$$

with $c > 0$ independent of ℓ and $\Lambda(\ell)$ not necessarily bounded as $\ell \to \infty$.

On account of (3.1) the variational problem

$$I_\ell[v,\Omega] := \int_\Omega H_\ell\left(\varepsilon^D(v)\right)\, dx \to \min \ \text{in } u_0 + \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^2) \tag{3.2_ℓ}$$

is well-posed (recall (1.11) and (1.6)) with unique solution u_ℓ. This follows along the same lines as in Section 2 using (1.3) with $p = 2$. The regularity properties of the functions u_ℓ have been investigated in [FS] with the result:

LEMMA 3.2. *The solutions u_ℓ of $(3.2)_\ell$ belong to the space $C^{1,\alpha}(\Omega;\mathbb{R}^2) \cap W^2_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2)$.*

After these preparations we observe that by the minimality of u_ℓ and by Lemma 3.2 we have

$$0 = \int_\Omega \partial_\mu \left[DH_\ell\left(\varepsilon^D(u_\ell)\right)\right] : \varepsilon^D(\Psi)\, dx \ , \ \ \mu = 1,2 \ ,$$

valid for $\Psi \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^2)$, and if we choose $\Psi = \eta^2 \partial_\mu u_\ell$, $\eta \in C_0^\infty(\Omega)$, the above equation turns into (summation only w.r.t. $\mu = 1,2$!)

$$\begin{aligned}
(3.2) \qquad & \int_\Omega D^2 H_\ell\left(\varepsilon^D(u_\ell)\right)\left(\varepsilon^D(\partial_\mu u_\ell), \varepsilon^D(\partial_\mu u_\ell)\right) \eta^2\, dx \\
&\qquad = - \int_\Omega \partial_\mu \left[DH_\ell\left(\varepsilon^D(u_\ell)\right)\right] : W^\mu \, dx \ , \\
&\quad W^\mu \ := (W^\mu_{ij})_{1\le i,j\le 2} = \eta \left[\left(\partial_i \eta \partial_\mu u^j_\ell + \partial_j \eta \partial_\mu u^i_\ell\right)_{1\le i,j\le 2} - (\partial_\mu u \cdot \nabla \eta)\, \mathbf{1}\right] \ .
\end{aligned}$$

In order to simplify our exposition we just write v in place of u_ℓ. After an integration by parts we deduce from (3.2)

$$\begin{aligned}
(3.3) \qquad & \int_\Omega D^2 H_\ell\left(\varepsilon^D(v)\right)\left(\varepsilon^D(\partial_\mu v), \varepsilon^D(\partial_\mu v)\right) \eta^2\, dx \\
&\qquad = \int_\Omega DH_\ell\left(\varepsilon^D(v)\right) : \partial_\mu W^\mu \, dx \ .
\end{aligned}$$

Note that the inequality stated before (1.9) holds for H_ℓ and h_ℓ too, moreover, according to Lemma 3.1 a) we have (A3) for h_ℓ, hence

$$(3.4) \qquad \text{l.h.s. of (3.3)} \ge \int_\Omega \frac{h'_\ell(|\varepsilon^D(v)|)}{|\varepsilon^D(v)|} \left|\nabla \varepsilon^D(v)\right|^2 \eta^2 \, dx \ .$$

At the same time we obtain (in what follows c always denotes a finite constant independent of ℓ)

$$(3.5) \qquad \text{r.h.s. of (3.3)} \le c \int_\Omega h'_\ell\left(|\varepsilon^D(v)|\right)\left[\left(|\nabla\eta|^2 + |\nabla^2\eta|\right)|\nabla v| + \eta|\nabla\eta||\nabla^2 v|\right] \, dx \ .$$

Let $\xi(\eta) := \|\nabla\eta\|^2_{L^\infty(\Omega)} + \|\nabla^2\eta\|_{L^\infty(\Omega)}$. Then it holds

$$\int_\Omega h'_\ell\left(|\varepsilon^D(v)|\right)|\nabla v| \left[|\nabla\eta|^2 + |\nabla^2\eta|\right] \, dx \le c\,\xi(\eta) \left\{ \int_{\operatorname{spt}\eta} h'_\ell\left(|\varepsilon^D(v)|\right)^2 \, dx + \int_\Omega |\nabla v|^2 \, dx \right\} ,$$

and from Theorem 1.1 with $p = 2$ we get

$$\begin{aligned}
\int_\Omega |\nabla v|^2\,dx &\le c\left\{\int_\Omega |\nabla u_0|^2 + \int_\Omega \left|\varepsilon^D(v-u_0)\right|^2\,dx\right\}\\
&\le c\left\{\int_\Omega |\nabla u_0|^2\,dx + \int_\Omega \left|\varepsilon^D(v)\right|^2\,dx\right\}\\
&\overset{(3.1)}{\le} c\left\{\int_\Omega |\nabla u_0|^2\,dx + \int_\Omega H_\ell\left(\varepsilon^D(v)\right)\,dx\right\}\\
&\le c\left\{\int_\Omega |\nabla u_0|^2\,dx + \int_\Omega H_\ell\left(\varepsilon^D(u_0)\right)\,dx\right\}\\
&\le c\left\{\int_\Omega |\nabla u_0|^2\,dx + \int_\Omega H\left(\varepsilon^D(u_0)\right)\,dx\right\}\,,
\end{aligned}$$

where we have used the $I_\ell[\,\cdot\,,\Omega]$– minimality of $v = u_\ell$ and the fact that $H_\ell \le H$. This yields

$$\int_\Omega h'_\ell\left(|\varepsilon^D(v)|\right)|\nabla v|\left[|\nabla\eta|^2 + |\nabla^2\eta|\right]\,dx \le c\,\xi(\eta)\left[1 + \int_{\operatorname{spt}\eta} h'_\ell\left(|\varepsilon^D(v)|\right)^2\,dx\right] \tag{3.6}$$

for a constant c depending on the boundary datum u_0. The remaining term on the right-hand side of (3.5) is estimated as follows

$$\begin{aligned}
&\int_\Omega h'_\ell\left(|\varepsilon^D(v)|\right)|\nabla\eta|\eta|\nabla^2 v|\,dx\\
&\quad\le \delta\int_\Omega \eta^2\left|\nabla^2 v\right|^2\,dx + c(\delta)\xi(\eta)\int_{\operatorname{spt}\eta} h'_\ell\left(|\varepsilon^D(v)|\right)^2\,dx
\end{aligned} \tag{3.7}$$

with arbitrary parameter $\delta \in (0,1)$. Putting together (3.4) - (3.7) we find

$$\begin{aligned}
&\int_\Omega \eta^2\frac{h'_\ell(|\varepsilon^D(v)|)}{|\varepsilon^D(v)|}\left|\nabla\varepsilon^D(v)\right|^2\,dx - \delta\int_\Omega \eta^2\left|\nabla^2 v\right|^2\,dx\\
&\quad\le c(\delta)\xi(\eta)\left[1 + \int_{\operatorname{spt}\eta} h'_\ell\left(|\varepsilon^D(v)|\right)^2\,dx\right]\,.
\end{aligned} \tag{3.8}$$

Here δ and η are still under our disposal. We discuss the δ–term: it holds by Theorem 1.1 ($p = 2$)

$$\begin{aligned}
\int_\Omega \eta^2\left|\nabla^2 v\right|^2\,dx &\le c\left[\int_\Omega |\nabla(\eta\nabla v)|^2\,dx + \int_\Omega |\nabla\eta|^2|\nabla v|^2\,dx\right]\\
&= c\left[\int_\Omega \nabla(\eta\partial_\mu v) : \nabla(\eta\partial_\mu v)\,dx + \int_\Omega |\nabla\eta|^2|\nabla v|^2\,dx\right]\\
&\le c\left[\int_\Omega \varepsilon^D(\eta\partial_\mu v) : \varepsilon^D(\eta\partial_\mu v)\,dx + \int_\Omega |\nabla\eta|^2|\nabla v|^2\,dx\right]\\
&\le c\left[\int_\Omega \eta^2\varepsilon^D(\partial_\mu v) : \varepsilon^D(\partial_\mu v)\,dx + \int_\Omega |\nabla\eta|^2|\nabla v|^2\,dx\right]\\
&= c\left[\int_\Omega \eta^2\left|\nabla\varepsilon^D(v)\right|^2\,dx + \int_\Omega |\nabla\eta|^2|\nabla v|^2\,dx\right]\,,
\end{aligned}$$

and the integral involving $|\nabla v|$ has been estimated before (3.6). Recalling that $h'_\ell(t)/t$ is bounded from below by the positive constant occurring on the left-hand side of (3.1) and choosing δ in an appropriate way we deduce from (3.8)

$$\begin{aligned}(3.9)\qquad &\int_\Omega \eta^2 \frac{h'_\ell(|\varepsilon^D(v)|)}{|\varepsilon^D(v)|}\left|\nabla\varepsilon^D(v)\right|^2 dx\\ &\leq c\,\xi(\eta)\left[1+\int_{\operatorname{spt}\eta} h'_\ell\left(|\varepsilon^D(v)|\right)^2 dx\right] .\end{aligned}$$

The remaining integral on the right-hand side of (3.9) is now discussed similar to the pressure - term in [Fu]: let us fix a disc $B_R(z)$ and a number $L>0$. Using (1.8) with h_ℓ in place of h we get

$$\begin{aligned}\int_{B_R(z)} h'_\ell\left(|\varepsilon^D(v)|\right)^2 dx &= \int_{B_R(z)\cap[|\varepsilon^D(v)|\leq L]} \dots\, dx + \int_{B_R(z)\cap[|\varepsilon^D(v)|>L]} \dots\, dx\\ &\leq\ h'_\ell(L)^2\pi R^2 + cL^{-2}\int_{B_R(z)\cap[|\varepsilon^D(v)|>L]} h_\ell\left(|\varepsilon^D(v)|\right)^2 dx\\ &\leq\ \pi R^2 h'_\ell(L)^2 + cL^{-2}\int_{B_R(z)} h_\ell\left(|\varepsilon^D(v)|\right)^2 dx\\ &\leq\ \pi R^2 h'(L)^2 + cL^{-2}\int_{B_R(z)} h_\ell\left(|\varepsilon^D(v)|\right)^2 dx\ ,\end{aligned}$$

where in the last inequality we have used (see Lemma 3.1)

$$h'_\ell(t) = tg_\ell(t) \leq tg(t) = h'(t)\ .$$

Let $r<R$ and specify η (in (3.9)) such that $\eta=1$ on $B_r(z)$, $\operatorname{spt}\eta\subset B_R(z)$, $0\leq\eta\leq 1$ and $|\nabla^\nu\eta|\leq c(R-r)^{-\nu}$, $\nu=1,2$. We further let $L=\lambda^{-1}(R-r)^{-1}$ for some $\lambda\in(0,1)$ and recall that $h'(L)^2\leq cL^{2m-2}$. Then (3.9) implies

$$\begin{aligned}(3.10)\qquad &\int_{B_r(z)} \frac{h'_\ell(|\varepsilon^D(v)|)}{|\varepsilon^D(v)|}\left|\nabla\varepsilon^D(v)\right|^2 dx\\ &\leq c(\lambda)(R-r)^{-\beta} + c\lambda^2\int_{B_R(z)} h_\ell\left(|\varepsilon^D(v)|\right)^2 dx\ .\end{aligned}$$

In (3.10) β denotes a suitable positive exponent and for the derivation of (3.10) we have used that $\xi(\eta)\leq c(R-r)^{-2}$ according to the choice of η. Note further that (3.10) is valid for all $\lambda\in(0,1)$ and any radii $0<r<R\leq 1$ such that $B_R(z)\subset\Omega$.

Now we select $\rho\in(0,R)$ and define $r:=(\rho+R)/2$. With $\eta\in C_0^\infty(B_r(z))$, $0\leq\eta\leq 1$,

$\eta \equiv 1$ on $B_\rho(z)$ and $|\nabla\eta| \le c/(r-\rho)(= 2c/(R-\rho))$ we get with Sobolev's inequality

$$\begin{aligned}
\int_{B_\rho(z)} h_\ell\left(|\varepsilon^D(v)|\right)^2 dx &\le \int_{B_r(z)} \left(\eta h_\ell\left(|\varepsilon^D(v)|\right)\right)^2 dx \\
&\le c\left[\int_{B_r(z)} |\nabla\eta| h_\ell\left(|\varepsilon^D(v)|\right) dx + \int_{B_r(z)} h'_\ell\left(|\varepsilon^D(v)|\right)|\nabla\varepsilon^D(v)|\, dx\right]^2 \\
&\le c(R-\rho)^{-2}\left[\int_{B_R(z)} h_\ell\left(|\varepsilon^D(v)|\right) dx\right]^2 + c\left[\int_{B_r(z)} h'_\ell\left(|\varepsilon^D(v)|\right)|\nabla\varepsilon^D(v)|\, dx\right]^2 .
\end{aligned}$$

The I_ℓ–minimality of v gives

$$\int_{B_R(z)} h_\ell\left(|\varepsilon^D(v)|\right) dx \le \int_\Omega h_\ell\left(|\varepsilon^D(v)|\right) dx \le I_\ell[u_0,\Omega] \le I[u_0,\Omega] ,$$

hence

$$\begin{aligned}
\int_{B_\rho(z)} & h_\ell\left(|\varepsilon^D(v)|\right)^2 dx \\
&\le c(R-\rho)^{-2} + c\left[\int_{B_r(z)} h'_\ell\left(|\varepsilon^D(v)|\right)|\nabla\varepsilon^D(v)|\, dx\right]^2 .
\end{aligned} \tag{3.11}$$

Hölder's inequality implies

$$\begin{aligned}
\Big[\ldots\Big]^2 &= \left[\int_{B_r(z)} \left(\frac{h'_\ell\left(|\varepsilon^D(v)|\right)}{|\varepsilon^D(v)|}\right)^{1/2} |\nabla\varepsilon^D(v)| \cdot \left(h'_\ell\left(|\varepsilon^D(v)|\right)|\varepsilon^D(v)|\right)^{1/2} dx\right]^2 \\
&\le \int_{B_r(z)} h'_\ell\left(|\varepsilon^D(v)|\right)|\varepsilon^D(v)|\, dx \int_{B_r(z)} \frac{h'_\ell\left(|\varepsilon^D(v)|\right)}{|\varepsilon^D(v)|}|\nabla\varepsilon^D(v)|^2\, dx ,
\end{aligned}$$

and according to (1.8) and the minimality of v we have as usual

$$\int_{B_r(z)} h'_\ell\left(|\varepsilon^D(v)|\right)|\varepsilon^D(v)|\, dx \le cI[u_0,\Omega] .$$

Therefore (3.11) implies the bound

$$\begin{aligned}
\int_{B_\rho(z)} & h_\ell\left(|\varepsilon^D(v)|\right)^2 dx \\
&\le c(R-\rho)^{-2} + c\int_{B_r(z)} \frac{h'_\ell\left(|\varepsilon^D(v)|\right)}{|\varepsilon^D(v)|}\left|\nabla\varepsilon^D(v)\right|^2 dx ,
\end{aligned} \tag{3.12}$$

and if we use (3.10) on the right-hand side of (3.12), we end up with (recall the choice of r)

$$\begin{aligned}
\int_{B_\rho(z)} & h_\ell\left(|\varepsilon^D(v)|\right)^2 dx \\
&\le c(R-\rho)^{-2} + c(\lambda)(R-\rho)^{-\beta} + c\lambda^2\int_{B_R(z)} h_\ell\left(|\varepsilon^D(v)|\right)^2 dx .
\end{aligned}$$

Since clearly $\beta \geq 2$ this inequality yields after suitable choice of λ

$$\int_{B_\rho(z)} h_\ell \left(|\varepsilon^D(v)|\right)^2 \, dx \leq c(R-\rho)^{-\beta} + \frac{1}{2} \int_{B_R(z)} h_\ell \left(|\varepsilon^D(v)|\right)^2 \, dx \, .$$

Here $\rho < R \leq 1$ are arbitrary with $B_R(z) \subset \Omega$. Lemma 3.1, p.161, of [Gi] then gives

$$\int_{B_\rho(z)} h_\ell \left(|\varepsilon^D(v)|\right)^2 \, dx \leq c(R-\rho)^{-\beta} \, ,$$

hence it is shown that

$$\sup_\ell \int_{\widetilde{\Omega}} h_\ell \left(|\varepsilon^D(u_\ell)|\right)^2 \, dx \quad < \infty \tag{3.13}$$

is true for any subdomain $\widetilde{\Omega} \Subset \Omega$. If we apply (3.13) on the right-hand side of (3.10), we see

$$\sup_\ell \int_{\widetilde{\Omega}} \frac{h'_\ell\left(|\varepsilon^D(u_\ell)|\right)}{|\varepsilon^D(u_\ell)|} \left|\nabla \varepsilon^D(u_\ell)\right|^2 \, dx < \infty \, ,$$

and since $h'_\ell(t)/t$ is increasing, we get

$$h'_\ell(t)/t \geq h''_\ell(0) = h''(0) \, ,$$

and therefore it is shown that

$$\sup_\ell \int_{\widetilde{\Omega}} |\nabla \varepsilon^D(u_\ell)|^2 \, dx < \infty \, . \tag{3.14}$$

Combining (3.14) with the argument used after (3.8) the uniform bound for the local L^2–norm of $\nabla \varepsilon^D(u_\ell)$ implies

$$\sup_\ell \int_{\widetilde{\Omega}} \left|\nabla^2 u_\ell\right|^2 \, dx < \infty \, . \tag{3.15}$$

As stated before (3.6) we have

$$\sup_\ell \int_\Omega |\nabla u_\ell|^2 \, dx < \infty$$

which in combination with (3.15) proves

$$\sup_\ell \|u_\ell\|_{W^2_2(\widetilde{\Omega})} \leq c(\widetilde{\Omega}) < \infty \tag{3.16}$$

for any subdomain $\widetilde{\Omega} \Subset \Omega$. Due to (3.16) and the global W^1_2 - bound for the sequence $\{u_\ell\}$ we find a function $\overline{u} \in W^1_2(\Omega; \mathbb{R}^2) \cap W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^2)$ such that $u_\ell \rightharpoondown \overline{u}$ weakly in $W^1_2(\Omega; \mathbb{R}^2) \cap W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^2)$. Now, proceeding exactly as done at the end of Section 3 in [BF], it can be shown that $\overline{u} = u$, thus $u \in W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^2)$, and our claim (1.14) follows by Sobolev's embedding theorem. □

4 Proof of the interior differentiability of the minimizer

Suppose that all the hypotheses of Theorem 1.3 are satisfied. We use the same notation as in the previous Section and study the approximations h_ℓ, H_ℓ, $v = u_\ell$ introduced in Lemma 3.1 and Lemma 3.2. Again we start from the equation

$$0 = \int_\Omega \partial_\mu \left[DH_\ell(\varepsilon^D(v)) \right] : \varepsilon^D(\Psi)\, dx, \quad \mu = 1,2\,,$$

and choose $\Psi = \eta^2(\partial_\mu v - \kappa_\mu)$, where $\eta \in C_0^\infty(B_{2r}(z))$ for a disc $B_{2r}(z) \Subset \Omega$ is such that $0 \le \eta \le 1$, $\eta = 1$ on $B_r(z)$ and $|\nabla \eta| \le c/r$. Moreover, κ_μ denotes a holomorphic function $B_{2r}(z) \to \mathbb{C}$ being specified below. This gives

$$\begin{aligned}
(4.1) \qquad & \int_{B_{2r}(z)} D^2 H_\ell\left(\varepsilon^D(v)\right)\left(\varepsilon^D(\partial_\mu v), \varepsilon^D(\partial_\mu v)\right)\eta^2\, dx \\
&\qquad = -\int_{B_{2r}(z)} D^2 H_\ell\left(\varepsilon^D(v)\right)\left(\varepsilon^D(\partial_\mu v), V^{(\mu)}\right)\eta\, dx\,, \\
V_{ij}^{(\mu)} :=\ & \partial_i \eta \left(\partial_\mu v^j - \kappa_\mu^j\right) + \partial_j \eta\left(\partial_\mu v^i - \kappa_\mu^i\right) - \nabla\eta\cdot(\partial_\mu v - \kappa_\mu)\,\delta_{ij}\,.
\end{aligned}$$

The r.h.s. of (4.1) is estimated with the help of the Cauchy–Schwarz inequality applied to the bilinear form $D^2 H_\ell\left(\varepsilon^D(v)\right)$, and by letting

$$\Phi_\ell := D^2 H_\ell\left(\varepsilon^D(v)\right)\left(\varepsilon^D(\partial_\mu v), \varepsilon^D(\partial_\mu v)\right)^{1/2}$$

we deduce from (4.1)

$$\begin{aligned}
\int_{B_r(z)} \Phi_\ell^2\, dx &\le c \int_{B_{2r}(z)} \Phi_\ell \left(D^2 H_\ell\left(\varepsilon^D(v)\right)\left(V^{(\mu)}, V^{(\mu)}\right)\right)^{1/2} dx \\
&\le \frac{c}{r}\int_{B_{2r}(z)} \Phi_\ell \left|D^2 H_\ell\left(\varepsilon^D(v)\right)\right|^{1/2} |\nabla v - \kappa|\, dx\,.
\end{aligned}$$

Recall that h satisfies (1.15), thus (see Lemma 3.1 a)) we have the same inequality with exponent ω and uniform constant for each function h_ℓ, which yields

$$\left|D^2 H_\ell\left(\varepsilon^D(v)\right)\right|^{1/2} \le c\left(1 + \left|\varepsilon^D(v)\right|^2\right)^{\omega/4}\sqrt{\frac{h_\ell'(|\varepsilon^D(v)|)}{|\varepsilon^D(v)|}} =: \widetilde{\Psi}_\ell\,,$$

and we arrive at

$$(4.2) \qquad \int_{B_r(z)} \Phi_\ell^2\, dx \le \frac{c}{r}\int_{B_{2r}(z)} \Phi_\ell \widetilde{\Psi}_\ell |\nabla v - \kappa|\, dx\,.$$

Note that (4.2) corresponds to (2.4) in [BFZ] and as outlined there we get from (4.2) by choosing $\gamma = 4/3$

$$(4.3) \qquad \fint_{B_r(z)} \Phi_\ell^2\, dx \le c\left(\fint_{B_{2r}(z)}\left(\Phi_\ell\widetilde{\Psi}_\ell\right)^\gamma dx\right)^{1/\gamma}\frac{1}{r}\left(\fint_{B_{2r}(z)} |\nabla v - \kappa|^4\, dx\right)^{1/4}\,,$$

$\fint$ denoting the mean value. According to the version of the Sobolev-Poincaré inequality established in the Appendix we can select κ_μ in such a way that

$$\|\partial_\mu v - \kappa_\mu\|_{L^4(B_{2r}(z))} \le c\left\|\varepsilon^D(\partial_\mu v)\right\|_{L^\gamma(B_{2r}(z))} = c\left\|\partial_\mu\left(\varepsilon^D(v)\right)\right\|_{L^\gamma(B_{2r}(z))}\ .$$

Inserting this estimate into (4.3) we find

$$\text{(4.4)}\qquad \fint_{B_r(z)} \Phi_\ell^2\,dx \le c\left(\fint_{B_{2r}(z)}\left(\Phi_\ell\widetilde{\Psi}_\ell\right)^\gamma\,dx\right)^{1/\gamma}\left(\fint_{B_{2r}(z)}\left|\nabla\varepsilon^D(v)\right|^\gamma\,dx\right)^{1/\gamma}\ ,$$

and from the definition of the functions Φ_ℓ, $\widetilde{\Psi}_\ell$ it is immediate that $\left|\nabla\varepsilon^D(v)\right| \le \Phi_\ell\widetilde{\Psi}_\ell$ holds. Thus (4.4) yields

$$\text{(4.5)}\qquad \fint_{B_r(z)} \Phi_\ell^2\,dx \le c\left(\fint_{B_{2r}(z)}\left(\Phi_\ell\widetilde{\Psi}_\ell\right)^\gamma\,dx\right)^{2/\gamma}\ .$$

Note that (4.5) corresponds to (2.6) in [BFZ], and if we abbreviate $d := 2/\gamma$, $f := \Phi_\ell^\gamma$, $g := \widetilde{\Psi}_\ell^\gamma$, then (4.5) takes the form

$$\text{(4.6)}\qquad \left(\fint_{B_r(z)} f^d\,dx\right)^{1/d} \le c\fint_{B_r(z)} fg\,dx\ .$$

Returning to (3.3) and recalling the uniform local higher integrability of $\nabla v = \nabla u_\ell$ for any finite exponent, we see that f is in $L^d_{\mathrm{loc}}(\Omega)$ uniformly w.r.t. the approximation parameter ℓ. In order to apply Lemma 1.2 of [BFZ] to inequality (4.6) it remains to check that $\exp(\beta\widetilde{\Psi}_\ell^2) = \exp(\beta g^d)$ belongs to $L^1_{\mathrm{loc}}(\Omega)$ (uniformly in ℓ) for any $\beta > 0$. To this purpose we let

$$\Psi_\ell := \int_0^{|\varepsilon^D(v)|}\sqrt{\frac{h_\ell'(t)}{t}}\,dt$$

and deduce from the inequality stated after (3.13) that

$$\sup_\ell \int_{\Omega^*}\left|\nabla\Psi_\ell\right|^2\,dx \le c(\Omega^*) < \infty$$

is true for any subdomain $\Omega^* \Subset \Omega$, moreover, we clearly have

$$\int_\Omega \Psi_\ell^2\,dx \le c\int_\Omega h_\ell\left(|\varepsilon^D(v)|\right)\,dx \le c\int_\Omega h_\ell\left(\left|\varepsilon^D(u_0)\right|\right)\,dx \le c\int_\Omega h\left(\left|\varepsilon^D(u_0)\right|\right)\,dx\ ,$$

hence

$$\text{(4.7)}\qquad \sup_\ell \|\Psi_\ell\|_{W^1_2(\Omega^*)} \le c(\Omega^*) < \infty$$

for $\Omega^* \Subset \Omega$. Trudinger's inequality (see [GT], Theorem 7.15) in combination with (4.7) implies the bound

$$\text{(4.8)}\qquad \int_{B_\rho} \exp(\beta_0\Psi_\ell^2)\,dx \le c(\rho) < \infty$$

for discs $B_\rho \Subset \Omega$ and for some exponent β_0. But as outlined after (2.10) in [BF] the estimate (4.8) in combination with (1.15) gives

$$\int_{B_\rho} \exp\left(\beta \widetilde{\Psi}_\ell^2\right)\, dx \le c(\beta,\rho)$$

for any $\beta > 0$. Returning to (4.6) and applying Lemma 1.2 of [BFZ] we get for all $\beta > 0$

$$\int_{B_\rho} \Phi_\ell^2\, \ell n^{c_0\beta}(e+\Phi_\ell)\, dx \le c(\beta,\rho) \tag{4.9}$$

with a suitable constant c_0. Estimate (4.9) corresponds to (2.10) in [BFZ] and as in this reference we deduce from (4.9) that $\left(\sigma_\ell := DH_\ell\left(\varepsilon^D(v)\right)\right)$

$$\int_{B_\rho} |\nabla \sigma_\ell|^2\, \ell n^\alpha\left(e + |\nabla\sigma_\ell|\right)\, dx \le c(\alpha,\rho)$$

holds for all exponents $\alpha > 0$ and all discs $B_\rho \Subset \Omega$, which gives the uniform continuity of the tensors σ_ℓ due to results of Kauhanen, Koskela and Malý [KKM]. But then we have the uniform continuity of the tensors $\varepsilon^D(u_\ell)$ so that each partial derivative $\partial_\gamma u_\ell$ solves an elliptic system of the form

$$0 = \int_\Omega A_x\left(\varepsilon^D(w), \varepsilon^D(\varphi)\right)\, dx\,, \quad \varphi \in C_0^\infty(\Omega;\mathbb{R}^2)\,,$$

where $A_x : \mathbb{S}^2 \times \mathbb{S}^2 \to [0,\infty)$ is a coercive bilinear form depending continuously on x. From [Sc] we get $\partial_\gamma u_\ell \in C^{0,\nu}(\Omega;\mathbb{R}^2)$ uniformly in ℓ for any $\nu \in (0,1)$, so that $u \in C^{1,\nu}(\Omega;\mathbb{R}^2)$ is established. □

Appendix. A Sobolev–Poincaré type inequality involving ε^D

Here we are going to prove the inequality stated after (4.3), i.e. we claim

LEMMA A.1. *Let B denote the unit disc in $\mathbb{R}^2$. Then, for each $p \in (1,2)$, there exists a constant $c(p) \in (0,\infty)$ such that with $p^* := \frac{2p}{2-p}$ the estimate*

$$\|f-g\|_{L^{p^*}(B)} \le c(p)\|\varepsilon^D(f)\|_{L^p(B)} \tag{A1}$$

holds for any function $f \in W^1_p(\Omega;\mathbb{R}^2)$, where g is a suitable holomorphic function $B \to \mathbb{C}$ depending on f.
If we allow values $p \in (1,\infty)$, then we obtain a variant of (A1) with p^* being replaced by p.

Proof: Suppose that Lemma A.1 holds for functions in $C^\infty(\overline{B};\mathbb{R}^2)$. For $f \in W^1_p(B;\mathbb{R}^2)$ we then consider $f_n \in C^\infty(\overline{B};\mathbb{R}^2)$ with corresponding holomorphic functions g_n such that $\|f - f_n\|_{W^1_p(B)} \longrightarrow 0, n \to \infty$, and

$$\text{(A2)} \qquad \|f_n - g_n\|_{L^{p^*}(B)} \le c(p)\|\varepsilon^D(f_n)\|_{L^p(B)} \ ,$$

Sobolev's theorem implies

$$\sup_n \|f_n\|_{L^{p^*}(B)} < \infty \ ,$$

so that by (A2)

$$\sup_n \|g_n\|_{L^{p^*}(B)} < \infty \ .$$

We may therefore pass to a subsequence such that e.g.

$$g_n \to: g \text{ in } L^{p^*}(B')$$

for each subregion $B' \Subset B$, g denoting a holomorphic function. But then again by (A2)

$$\|f - g\|_{L^{p^*}(B')} \le \limsup_{n\to\infty} \|f_n - g_n\|_{L^p(B)} \le c(p)\|\varepsilon^D(f)\|_{L^p(B)} \ ,$$

and we get (A1) for the Sobolev function f by passing to the limit $B' \nearrow B$. Returning to the smooth case we observe the validity of the representation formula (see, e.g. [Hö], p.3, or [Sa], p.234)

$$\text{(A3)} \qquad f(z) = \frac{1}{2\pi i}\int_{\partial B}\frac{f(\xi)}{\xi - z}d\xi - \frac{1}{\pi}\int_B \frac{\partial_{\overline{z}} f(\xi)}{\xi - z}\, d\mathcal{L}^2(\xi) \ ,$$

which follows from Gauß' s theorem.
Here z is any point from the disc B, and on the r.h.s. of (A3) the term $\int_{\partial B}$ denotes the complex line integral taken over the circle, whereas $\int_B$ has to be calculated w.r.t. Lebesgue measure $\mathcal{L}^2$. Finally, $\partial_{\overline{z}} f$ is the Wirtinger derivative $\frac{1}{2}(\partial_x f + i\partial_y f)$, $z = x + iy$, so that f is holomorphic if and only if $\partial_{\overline{z}} f = 0$. Clearly $g(z) := \frac{1}{2\pi i}\int_{\partial B}\frac{f(\xi)}{\xi - z}\, d\xi$ is holomorphic in B, and from (A3) we obtain

$$\text{(A4)} \qquad |f(z) - g(z)| \le cV_{1/2}\left(|\varepsilon^D(f)|\right)(z) \ ,$$

where $V_{1/2}$ is the Riesz potential of $|\varepsilon^D(f)|$ defined in [GT], formula (7.31), for the choices $\mu = 1/2$, $n = 2$. According to [St], Theorem 1 on p.119, or to Lemma 7.12 of [GT] and the comments given after this lemma we have the continuity of $V_{1/2}$ from $L^p(B)$ into $L^q(B)$ for any q such that $\frac{1}{p} - \frac{1}{q} \le \frac{1}{2}$, i.e. for any $q \le \frac{2p}{2-p} = p^*$. Combining this argument with (A4), the desired inequality (A1) follows. $\square$

References

[Ad] Adams, R. A., Sobolev spaces. Academic Press, New York-San Francisco-London (1975).

[BI] Bartnik, R., Isenberg, J., The constraint equation. In: Chruściel, P. T., Friedrich, H. (eds.), The Einstein equations and large scale behaviour of gravitational fields, pp.1–38. Birkhäuser Verlag, Basel-Boston-Berlin (2004).

[BF] Bildhauer, M., Fuchs, M., Differentiability and higher integrability results for local minimizers of splitting type variational integrals in 2D with applications to nonlinear Hencky–materials. Calc. Var. (DOI 10.1007/s 00526-009-0257-y).

[BFZ] Bildhauer, M., Fuchs, M., Zhong, X., A lemma on the higher integrability of functions with applications to the regularity theory of two dimensional generalized Newtonian fluids. Manus. Math. 116 (2), 135–156 (2005).

[BYZ] Byun, S.-S., Yao, F., Zhou, S., Gradient estimates in Orlicz space for nonlinear elliptic equations. J. Functional Anal. 255, 1851–1873 (2008).

[Da] Dain, S., Generalized Korn's inequality and conformal Killing vectors. Calc. Var. 25 (4), 535–540 (2006).

[Fu] Fuchs, M., A note on non–uniformly elliptic Stokes–type systems in two variables. J. Mat. Fluid Mech. (DOI 10.1007/s 00021-008-0285y)

[FS] Fuchs, M., Schirra, O., An application of a new coercive inequality to variational problems studied in general relativity and in Cosserat elasticity giving the smoothness of minimizers. (to appear in Archiv d. Math.).

[Gi] Giaquinta, M., Multiple integrals in the calculus of variations and nonlinear elliptic systems. Princeton University Press (1983).

[GT] Gilbarg, D., Trudinger, N. S., Elliptic partial differential equations of second order. Springer-Verlag, Berlin-Heidelberg-New York (1998).

[Hö] Hörmander, L., An introduction to complex analysis in several variables. North–Holland Math. Library (1973).

[JLW] Jia, H., Li, D., Wang, L., Regularity theory in Orlicz spaces for elliptic equations in Reifenberg domains. J. Math. Anal. Appl. 334, 804–817 (2007).

[KKM] Kauhanen, J., Koskela, P., Malý, J., On functions with derivatives in a Lorentz space. Manus. Math. 100, 87–101 (1999).

[Me] Meyers, N. G., An L^p–estimate for the gradient of solutions of second order elliptic divergence equation. Ann. S.N.S Pisa III (17), 189–206 (1963).

[RR] Rao, M. M., Ren, Z. D., Theory of Orlicz spaces. Marcel Dekker, New York-Basel-Hong Kong (1991).

[Sa] Sauvigny, F., Partielle Differentialgleichungen der Geometrie und der Physik. Band 1 Grundlagen und Integraldarstellungen. Springer Verlag, Berlin-Heidelberg-New York (2003).

[Sc] Schirra, O., PhD-thesis (2009).

[St] Stein, E. M., Singular integrals and differentiability properties of functions. Princeton University Press, Princeton (1970).

[YSZ] Yao, F., Sun, Y., Zhou, S., Gradient estimates in Orlicz spaces for quasilinear elliptic equation. Nonlinear Analysis 69, 2553–2565 (2008).

[Ze] Zeidler, E., Nonlinear functional analysis and its applications, vol.IV, Springer Verlag, Berlin-Heidelberg-New York (1987).