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Abstract

In [Br] we prove partial regularity (and full regularity in 2D) for minimizers of splitting-type variational problems under general growth conditions, which is the corresponding generalization of the results due to splitting-type problems with power growth conditions from Bildhauer and Fuchs [BF1], [BF2]. In this article we extend the statements from [Br] for an additional x-dependence without severe restrictions.

1 Introduction

The study of regularity properties of minimizers $u : \Omega \to \mathbb{R}^N$ of energies

$$I[u, \Omega] := \int_\Omega F(\nabla u)\, dx, \tag{1.1}$$

where Ω denotes an open set in $\mathbb{R}^n$ and where $F : \mathbb{R}^{nN} \to [0, \infty)$ satisfies an anisotropic growth condition, i.e.

$$C_1|Z|^p - c_1 \le F(Z) \le C_2|Z|^q + c_2, \qquad Z \in \mathbb{R}^{nN} \tag{1.2}$$

with constants $C_1, C_2 > 0$, $c_1, c_2 \ge 0$ and exponents $1 < p \le q < \infty$, was pushed by Marcellini (see [Ma1] and [Ma2]). The research of Espotisto Leonetti and Mingione [ELM] shows that the statements do not stay true if one allows an additional x-dependence and considers minimizers of functionals

$$J[u, \Omega] := \int_\Omega F(\cdot, \nabla u)\, dx, \tag{1.3}$$

for $F : \Omega \times \mathbb{R}^{nN} \to [0, \infty)$. This is not only a technical extension of the autonomous situation and additional assumptions are often necessary.
In the autonomous case it is already well-known, that we have no hope for regularity of minimizers of (1.1), if p and q are too far apart (compare the counterexamples of [Gi] and [Ho]). To get better results one needs additional assumptions. Therefore Bildhauer, Fuchs and Zhong consider decomposable integrands, which means we have

$$F(Z) = f(\widetilde{Z}) + g(Z_n)$$

for $Z = (Z_1, ..., Z_n)$ with $Z_i \in \mathbb{R}^N$ and $\widetilde{Z} = (Z_1, ..., Z_{n-1})$. They assume power growth conditions for the C^2-functions f and g and get a very general theory in the case $p \geq 2$ (see [BF1], [BF2] and [BFZ]). In [Br] we have generalized this statements under the assumption

$$f(\widetilde{Z}) = a(|\widetilde{Z}|) \quad \text{and} \quad g(Z_n) = b(|Z_n|)$$

for N-functions a and b. Thereby the main assumptions are (h stands for a or b)

$$\frac{h'(t)}{t} \approx h''(t)$$

and superquadratic growth of h. The results of [Br] (where higher integrability theorems from [BF3] built the basic) are

- full $C^{1,\alpha}$-regularity for $n = 2$;
- partial $C^{1,\alpha}$-regularity in general vector case, if

  $$b(t) \leq ct^{\omega}a(t) \quad \text{and} \quad a(t) \geq \vartheta t^{\frac{\omega}{2}(n-2)} \tag{1.4}$$

  for an $\omega \leq 2$ and big values for t;
- full $C^{1,\alpha}$-regularity for $N = 1$ if $b(t) \leq ct^2 a(t)$ and $a(t) \leq ct^2 b(t)$ for $t \gg 1$.

If one has a look at the statements in the power growth situation you see that the conditions quoted above are natural generalizations to the case of N-functions (except of the case $N = 1$, see [BF1], [BF2] and [BFZ]).
From now on we consider minimizers of

$$\mathcal{T}[w] := \int_{\Omega} \Big[a(\cdot, |\widetilde{\nabla} w|) + b(\cdot, |\partial_n w|) \Big] \, dx. \tag{1.5}$$

where a and b are of class $C^2(\overline{\Omega} \times [0, \infty), [0, \infty))$ with the properties ($h = a$ or $h = b$):

$$\begin{gathered} h(x, \cdot) \text{ ist strictly increasing and convex with} \\ \lim_{t \to 0} \frac{h(x,t)}{t} = 0 \text{ and } \lim_{t \to \infty} \frac{h(x,t)}{t} = \infty. \end{gathered} \tag{A1}$$

for all $x \in \overline{\Omega}$. Furthermore we assume for all $t \geq 0$:

$$\widehat{\epsilon}\, \frac{h'(x,t)}{t} \leq h''(x,t) \leq \widehat{h}\, \frac{h'(x,t)}{t} \tag{A2}$$

uniformly in $x \in \overline{\Omega}$, with constants $\widehat{\epsilon}, \widehat{h} > 0$. Let

$$a(x,t) \leq c_1 b(x,t) \text{ for all } x \in \overline{\Omega} \text{ and big } t \tag{A3}$$

for a $c_1 > 0$. For having superquadratic growth we suppose

$$\frac{h'(x,t)}{t} \geq h_0 > 0 \text{ for all } t \geq 0 \tag{A4}$$

and all $x \in \overline{\Omega}$. To handle the terms involving derivatives after the spatial variable we need:

$$|\partial_\gamma h'(x,t)| \leq c_2 h'(x,t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+ \tag{A5}$$

and all $\gamma \in \{1, ..., n\}$ with a constant $c_2 \geq 0$.

Remark 1.1

- *The conditions (A1)-(A4) are the generalizations from those of [Br] for a x-dependence. So it is possible to show a (p,q)-growth condition as in (1.2) for the function F.*
- *A simple example is given by ($(x,Z) \in \overline{\Omega} \times \mathbb{R}^{nN}$)*

$$F(x,Z) := \alpha(x) a(|\widetilde{Z}|) + \beta(x) b(|Z_n|)$$

for functions a and b of class $C^2([0,\infty),[0,\infty))$ satisfying the autonomous assumptions from [Br] and strictly positive functions $\alpha, \beta \in C^1(\overline{\Omega})$.

A first step is to get results on higher integrability, where no results are known until now. We have

THEOREM 1.1 *Higher integrability:*
Suppose (A1)–(A5) and consider a local minimizer $u \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega, \mathbb{R}^N)$ of (1.5), then:

(a) $b(\cdot, |\partial_n u|)|\partial_n u|^2$ belongs to the space $L^1_{loc}(\Omega)$

(b) If we have

$$b(x,t) \leq c t^\omega a(x,t) \text{ for large } t \tag{A6}$$

and an $\omega \leq 2$, then $a(\cdot, |\widetilde{\nabla} u|)|\widetilde{\nabla} u|^2$ belongs to the space $L^1_{loc}(\Omega)$. Furthermore we have $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$.

Remark 1.2 • *The main problem in the proof of Theorem 1.1 is the regularization procedure: if we work with the ordinary regularization in this topic (see [BF1] for example), we do not have a convergence $u_\delta \to u$ (u_δ is the minimizer of the regularized problem) because of the x-dependence (it is the same problem described in [BF4] and [Br2]). The approach of [Br2] using a regularization from below with a function $h_M \leq h$ ($h = a$ or $h = b$, $M \gg 1$) does not solve the problem because it is not possible to get a uniform variant of (A2) for the function h_M. Therefore we use a variant of regularization described in [BF5].*

- *Note that in the non-autonomous situation superquadratic growth is already needed for higher integrability different from the autonomous case (compare [BF3]).*

- *In comparison to [BF3] we need (A6) to get higher integrability. The reason for this is that the assumption*

$$b(x,t) \leq ct^2 a(x,t^2) \quad \textit{(for large } t\textit{)}$$

stated in [BF3] does not extend to the regularized functions a_M and b_M.

Analogous to the proof from [Br] we need further assumptions in the general vector case ($x \in \overline{\Omega}$ arbitrary, $h = a$ or $h = b$):

$$\frac{h'(x,t)}{t} \leq h''(x,t) \text{ for } t \geq 0, \text{ if } \omega < 1, \tag{A7}$$

as well as

$$a(x,t) \geq \vartheta t^{\frac{\omega}{2}(n-2)} \text{ for large } t \tag{A8}$$

for an $\vartheta > 0$, where ω is defined in (A6).

THEOREM 1.2 *Partial $C^{1,\alpha}$-regularity:*

(a) *Assume (A1)-(A6) for an $\omega < 2$, (A7) and (A8). Furthermore we suppsoe for all $B \Subset \Omega$*

$$\operatorname{argmin}_{y \in B} a(y,t) \textit{ is independent of } t \textit{ and} \tag{A9}$$

$$a(x,t) \leq \theta_1\, t^{\theta_2 |x-y|} a(y,t) \textit{ for all } t \gg 1 \textit{ and all } x, y \in B \tag{A10}$$

with constants $\theta_1 > 0$ and $\theta_2 \geq 0$. Then for any local minimizer $u \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega, \mathbb{R}^N)$ of (1.5) exists an open subset Ω_0 of Ω such that $\mathcal{L}^n(\Omega_0 - \Omega) = 0$ and $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ for all $\alpha < 1$.

(b) *If* $n = 2$ *then we have* $\Omega_0 = \Omega$ *without (A3), (A6)-(A10) and the assumption* $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$.

(c) *If we have (A1), (A2) and (A4)–(A6) for an* $\omega \leq 2$ *as well as* $N = 1$, *then any local minimizer* $u \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega)$ *of (1.5) belongs to the space* $C^{1,\alpha}(\Omega)$ *for all* $\alpha < 1$, *provided we assume*

$$a(x,t) \leq ct^2 b(x,t) \text{ for large } t \tag{A11}$$

uniformly in $x \in \overline{\Omega}$.

Remark 1.3 • *The results about partial regularity from [Br] extend to the case of non-autonomous with the only restriction that we have to assume* $b(x,t) \leq ct^\omega a(x,t)$ *for an* ω *really smaller than 2. The reason for this is that we can not prove a uniform variant of* $b(t) \leq ct^\omega a(x, t^\omega)$ *to our regularization (see section 2).*

- *The results for* $n = 2$ *or* $N = 1$ *extend completely.*
- *As mentioned in [Br], section 4, we can remove the assumption* $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$ *if* $n = 2$.

Remark 1.4 • *From (A9) we get the existence of* $y^* \in B$ *such that* $a(y^*, t) \leq a(y, t)$ *for all* $(y, t) \in B \times [0, \infty)$. *This is necessary to prove the continuous growth condition in the iteration of the blow up. If we have a look at interesting examples for densities (see [Br2]), (A9) and (A10) are natural conditions for a* x*-dependence.*

- *In [ELM] sharp conditions for regularity of minimizers of non-autonomous anisotropic variational integrals are provided. The authors use a condition of the form (A9) (see (74)) and so we can proceed that this assumption is necessary to get regularity.*
- *Note that we are not able to consider minimizers of*

$$\int_\Omega \left[(1 + |\widetilde{\nabla} w|^2)^{\frac{p(x)}{2}} + (1 + |\partial_n w|^2)^{\frac{p(x)}{2}} \right] dx$$

for $p, q \in W^{1,\infty}_{loc}(\Omega, [2, \infty))$, *since the functions*

$$a(x,t) := \left(1 + t^2\right)^{\frac{p(x)}{2}} - 1 \quad \text{and} \quad b(x,t) := \left(1 + t^2\right)^{\frac{q(x)}{2}} - 1$$

do not satisfy condition (A5).

2 Preparations and higher integrability

First we define the regularization. Let ($h = a$ or $h = b$ and $t \geq 0$)

$$h_M(x,t) := \int_0^t s g_M(x,s)\, ds$$

where $M \gg 1$ and

$$g_M(x,t) := g(x,0) + \int_0^t \eta(s) g^{'}(x,s)\, ds,\;\; g(x,t) := \frac{h^{'}(x,t)}{t}.$$

Here $\eta \in C^1([0,\infty))$ denotes a cut-off function with the properties $0 \leq \eta \leq 1$, $\eta^{'} \leq 0$, $|\eta^{'}| \leq c/M$, $\eta \equiv 1$ on $[0, 3M/2]$ and $\eta \equiv 0$ on $[2M, \infty)$.

Lemma 2.1 *For the sequence* (h_M) *we have:*

- $h_M \in C^2(\overline{\Omega} \times [0,\infty))$, $h_M(x,t) = h(x,t)$ *for all* $t \leq 3M/2$ *and*

  $$\lim_{M\to\infty} h_M(x,t) = h(x,t) \;\; \textit{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+;$$

- $h_M \leq h$, $g_M \leq g$ *and from (A2) follows* $h_M^{''} \leq c(M)$ *on* $\overline{\Omega} \times \mathbb{R}_0^+$;
- *condition (A1) implies the same for* h_M;
- *By (A2) we get*

  $$\overline{\epsilon}\, \frac{h_M^{'}(x,t)}{t} \leq h_M^{''}(x,t) \leq \overline{h}\, \frac{h_M^{'}(x,t)}{t}$$

  uniformly in M;
- *inequality (A3) extends uniformly to* a_M *and* b_M:

  $$a_M(x,t) \leq \overline{c}_1 b_M(x,t) \textit{ for all } x \in \overline{\Omega} \textit{ and large } t;$$

- *By (A4) we deduce the same inequality for* h_M *uniform in* M:

  $$\frac{h_M^{'}(x,t)}{t} \geq \overline{h}_0 > 0 \textit{ for all } t \geq 0$$

  if we assume additionally (A2);
- *(A5) extends to* h_M *uniformly in* M:

  $$|\partial_\gamma h_M^{'}(x,t)| \leq \overline{c}_2 h_M^{'}(x,t) \textit{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+$$

  and all $\gamma \in \{1,...,n\}$;

- *if we have*

$$b(x,t) \le ct^{\omega} a(x,t^{\omega}) \text{ for big } t,$$

then the same is true for a_M and b_M uniformly in M.

Proof: By definition of h_M we get part 1 and the the first two statements of part 2. For the rest we need the equity

$$\frac{h_M'(x,t)}{t} = g_M(x,t) = \eta(t)\frac{h'(x,t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} h'(x,s)\, ds \tag{2.1}$$

for $(x,t) \in \overline{\Omega} \times \mathbb{R}_0^+$. By definition of g we get $g(x,0) = h''(x,0)$ and therefore

$$\begin{aligned} g_M(x,t) &= h''(x,0) + \int_0^t \eta(s) \left\{\frac{h''(x,s)}{s} - \frac{h'(x,s)}{s^2}\right\} ds \\ &= \eta(t)\frac{h'(x,t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} h'(x,s)\, ds. \end{aligned}$$

We have

$$h_M''(x,t) = g_M(x,t) + t g_M'(x,t)$$

and so we obtain

$$t g_M'(x,t) = t\eta(t) g'(x,t) = \eta(t)\left[h''(x,t) - \frac{h'(x,t)}{t}\right].$$

By (2.1) and (A2) follows for $\overline{\epsilon} := \min\{1, \widehat{\epsilon}\}$

$$\begin{aligned} h_M''(x,t) =& \eta(t) h''(x,t) + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} h^{'}(x,s)\, ds \\ \ge& \overline{\epsilon}\left[\eta(t)\frac{h'(x,t)}{t} + \int_0^t \left\{-\frac{\eta^{'}(s)}{s}\right\} h'(x,s)\, ds\right] \\ =& \overline{\epsilon} g_M(x,t) = \overline{\epsilon}\frac{h_M'(x,t)}{t}. \end{aligned}$$

By (A2) and (2.1) we get for $\overline{h} := \max\left\{1, \widehat{h}\right\}$

$$\begin{aligned} h_M''(x,t) &= \frac{h_M'(x,t)}{t} + \eta(t)\left[h''(t) - \frac{h'(x,t)}{t}\right] \\ &\le \frac{h_M'(x,t)}{t} + \left[\widehat{h} - 1\right]\eta(t)\frac{h'(x,t)}{t} \le \overline{h}\,\frac{h_M'(x,t)}{t} \end{aligned}$$

which proves part 4. Now one sees

$$h_M^{''}(x,t) \le cg_M(x,t) \le cg(x,0) + c\int_0^{2M} |g^{'}(x,s)|\, ds \le c(M).$$

By $h_M(x,0) = 0$ we receive

$$\lim_{t\to 0} \frac{h_M(x,t)}{t} = h_M'(x,0) = 0.$$

Furthermore we obtain

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t s g_M(x,s)\, ds = \lim_{t\to\infty} t g_M(x,t) = \infty,$$

noting

$$\lim_{t\to\infty} g_M(x,t) = \int_{3M/2}^{2M} \{-\eta'(s)\}\, g(x,s)\, ds > 0$$

which follows by (2.1) and monotonicity of h. Using (A3) we deduce $a^{'}(x,t) \le cb^{'}(x,t)$ für $t \ge t_0$ from (A2). So we have for $t \ge t_0$ by (2.1)

$$\begin{aligned}
\frac{a_M'(x,t)}{t} &= \eta(t)\frac{a'(x,t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} a'(x,s)\, ds \\
&\le c\left[\eta(t)\frac{b'(x,t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} b'(x,s)\, ds\right] \\
&= c\,\frac{b_M'(x,t)}{t},
\end{aligned}$$

if we assume $3M/2 \ge t_0$. Part 6: für $t \le 3M/2$ we deduce from (A1) and (A4)

$$h_M''(x,t) \ge \overline{h}_0.$$

In case $3M/2 < t < 2M$ follows

$$h_M''(x,t) \ge \overline{\epsilon} g_M(x,t) \ge \overline{\epsilon}\left[h_0\eta(t) + h_0\int_{3M/2}^t \{-\eta'(s)\}\ ds\right] = h_0\overline{\epsilon}$$

and for $t > 2M$ we get

$$h_M''(x,t) \ge \overline{\epsilon} h_0 \int_{3M/2}^{2M} \{-\eta'(s)\}\ ds = h_0\overline{\epsilon}.$$

The proof of the estimation for $\partial_\gamma h_M$ can be found in [BF5] (p. 14). For the last part we deduce from (A6) and (A2)

$$b'(x,t) \le ct^\omega a'(x,t) \text{ for } t \ge t_0.$$

By (2.1) this delivers for $t \ge t_0$ assuming $3M/2 \ge t_0$ (note $\eta^{'}(t) = 0$ for $t \le 3M/2$)

$$\begin{aligned} \frac{b_M'(x,t)}{t} &= \eta(t)\frac{b'(x,t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} b'(x,s)\,ds \\ &\le ct^\omega \left[\eta(t)\frac{a'(x,t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} a'(x,s)\,ds\right] \\ &= ct^\omega \frac{a_M'(x,t)}{t} \text{ for all } t \ge t_0. \end{aligned}$$

Remark 2.2

- *By [BF4] (Lemma A.1) (A1) and (A2) show*

$$h(x,2t) \le 2^{\widehat{h}+1} h(x,t) \textit{ for all } t \ge 0. \tag{2.2}$$

  Thus we get by Lemma 2.1 (part 3 and 4) an uniform Δ_2-condition for h_M. From the same quotation we deduce

$$h'(x,2t) \le 2^{\widehat{h}} h'(x,t) \textit{ for all } t \ge 0,$$

  such that this extends to h_M uniformly, too.

- *By monotonicity of h' (A1) and (A2) imply for $\mu := 2^{\widehat{h}+1}$*

$$\mu^{-1} t h'(x,t) \le h(x,t) \le t h'(x,t) \textit{ for all } t \ge 0$$

  which extends to h_M uniformly.

After these preparations we define u_M as the unique minimizer of ($B := B_R(x_0) \Subset \Omega$ arbitrary)

$$\mathcal{T}_M[w] := \int_B F_M(\cdot, \nabla w)\,dx := \int_B \left[a_M(\cdot, |\widetilde{\nabla} w|) + b_M(\cdot, |\partial_n w|)\right]\,dx$$

in $u + W_0^{1,2}(B, \mathbb{R}^N)$. The regularization u_M has the following properties:

Lemma 2.3 *Suppose (A1)-(A5). Then we have:*

- *u_M belongs to the space $W^{2,2}_{loc}(B, \mathbb{R}^N)$;*

- *$a_M(\cdot, |\nabla\widetilde{u}_M|)|\widetilde{\nabla}u_M|^2$ and $b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2$ are elements of $L^1_{loc}(B)$;*
- *if $n = 2$ or $N = 1$, then we obtain $u_M \in W^{1,\infty}_{loc}(B, \mathbb{R}^N)$;*
- *for $\gamma \in \{1, ..., n\}$ $\partial_\gamma u_M$ solves*

$$\int_B D^2_P F_M(\cdot, \nabla u_M)(\nabla w, \nabla\varphi)\,dx + \int_B \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla\varphi\,dx = 0 \text{ for all } \varphi \in W^{1,2}_0(B, \mathbb{R}^N)$$

with $\operatorname{spt}(\varphi) \Subset B$.

- *u_M is in $W^{1,2}(B, \mathbb{R}^N)$ uniformly bounded and we have*

$$\sup_M \int_B F_M(\cdot, \nabla u_M)\,dx < \infty;$$

- *if we have $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$ then $\sup_M \|u_M\|_\infty < \infty$.*

The first part follows from [BF4] (Lemma 2.5) and part 3 is proved in [BF5], Thm. 1.1 (ii) and (iii), with $p = q = 2$. For part 5 we quote [Br2] Lemma 1.2.
Part 2: Minimizing $\mathcal{T}_M$ is a variational problem with splitting condition and power growth conditions with $p = q = 2$. As remarked in [BF3] (Remark 3 b)) it is no problem to extend the approach from [BF3], Thm. 1, to the non-autonomous situation and we get $\nabla u_M \in L^4_{loc}(B, \mathbb{R}^{nN})$. By quadratic growth of a_M and b_M we receive the required statement.
Surely $\partial_\gamma u_M$ is the solution if we only allow test-functions $\varphi \in C^\infty_0(B, \mathbb{R}^N)$. But we have boundedness of $D^2_P F_M(\cdot, \nabla u_M)$ (compare Lemma 2.1, part 2) and $\partial_\gamma D_P F_M(\cdot, \nabla u_M) \in L^2(B, \mathbb{R}^{nN})$. The latter follows from Lemma 2.1 (part 2 and 4) in combination with (A5). Now we get part 4 by approximation.
Uniform boundedness of u_M is obtained by the maximum-principle of [DLM].

Proof of Theorem 1.1: Let

$$\Gamma_M := 1 + |\nabla u_M|^2, \quad \widetilde{\Gamma}_M := 1 + |\widetilde{\nabla}u_M|^2 \quad \text{and} \quad \Gamma_{n,M} := 1 + |\partial_n u_M|^2.$$

We want to bound

$$\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2\,dx$$

independent from M like in [BF3]. Thereby we consider $\eta \in C_0^\infty(B)$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$ for $r < R$ and $|\nabla \eta| \le c/(R-r)$. After integrating by parts and using the uniform bound on u_M (see Lemma 2.3) the only term of interest is

$$\int_B \eta^{2k} |\partial_n \left[b_M(\cdot, |\partial_n u_M|) \right] | |\partial_n u_M| \, dx. \tag{2.3}$$

Here one can see

$$\begin{aligned} T_2 \le & c \int_B \eta^{2k} |\partial_n b_M(\cdot, |\partial_n u_M|)| |\partial_n u_M| \, dx \\ & + c \int_B \eta^{2k} b_M'(\cdot, |\partial_n u_M|) |\partial_n u_M| |\partial_n \partial_n u_M| \, dx \\ :=& c\, T_2^1 + c\, T_2^2. \end{aligned}$$

By Lemma 2.1 (part 7) follows

$$|\partial_n b_M(x,t)| = \left| \int_0^t \partial_n b_M'(x,s) \, ds \right| \le c\, b_M(x,t)$$

and thereby with Young's inequality

$$\begin{aligned} T_2^1 \le & \tau \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\partial_n u_M|^2 \, dx \\ & + c(\tau) \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) \, dx. \end{aligned}$$

Furthermore we get the inequality

$$\begin{aligned} T_2^2 \le & \tau \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\partial_n u_M|^2 \, dx \\ & + c(\tau) \int_B \eta^{2k} \frac{b_M'(\cdot, |\partial_n u_M|)}{|\partial_n u_M|} |\partial_n \partial_n u_M|^2 \, dx. \end{aligned}$$

using Remark 2.2. If we absorb the τ-terms in (2.3) we get

$$\begin{aligned} & \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\partial_n u_M|^2 \, dx \\ \le & c(r) + c \int_B \eta^{2k} \frac{b_M'(\cdot, |\partial_n u_M|)}{|\partial_n u_M|} |\partial_n \partial_n u_M|^2 \, dx, \end{aligned} \tag{2.4}$$

where $c(r)$ is a constant with $c(r) \to \infty$ for $r \to R$, but independent from M. Estimating the integral on the r.h.s. of (2.4) we need a Caccioppoli-type inequality as in [BF3] and the only term which needs a comment is

$$-\int_B \partial_n D_P F_M(\cdot, \nabla u_M) : \nabla \left\{ \eta^{2k} \partial_n u_M \right\} \, dx.$$

A first estimation shows the bound

$$\begin{aligned} & c \int_B |a_M^{'}(\cdot, |\widetilde{\nabla} u_M|)||\widetilde{\nabla} \left\{ \eta^{2k} \partial_n u_M \right\}| \, dx \\ + & c \int_B |b_M^{'}(\cdot, |\partial_n u_M|)||\partial_n \left\{ \eta^{2k} \partial_n u_M \right\}| \, dx \\ & := c \left[\mathcal{W}_1 + \mathcal{W}_2 \right] \end{aligned}$$

by Lemma 2.1, part 7. Now we consider both terms separately:

$$\begin{aligned} \mathcal{W}_1 & \leq c \int_B \eta^{2k-1} a_M^{'}(\cdot, |\widetilde{\nabla} u_M|) |\nabla \eta| |\partial_n u_M| \, dx \\ & + c \int_B \eta^{2k} a_M^{'}(\cdot, |\widetilde{\nabla} u_M|) |\partial_n \widetilde{\nabla} u_M| \, dx \\ & := c \left[\mathcal{W}_1^1 + \mathcal{W}_1^2 \right]. \end{aligned}$$

By Young's inequality we get

$$\begin{aligned} \mathcal{W}_1^2 \leq \; & \tau \int_B \eta^{2k} \frac{a_M^{'}(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_n \widetilde{\nabla} u_M|^2 \, dx \\ & + c(\tau) \int_B \eta^{2k} a_M^{'}(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} u_M| \, dx \end{aligned}$$

which can be handled as in [BF4] (section 3). As an upper bound for $\mathcal{W}_1^1$ we obtain

$$\int_B \eta^{2k} a_M^{'}(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} u_M| \, dx + \int_B \eta^{2k-2} |\nabla \eta|^2 \frac{a_M^{'}(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_n u_M|^2 \, dx.$$

We can estimate the second integral exactly as in [BF1] (section 3) because all assumptions for a and b extend uniformly a_M and b_M. If we use Remark 2.2 and Lemma 2.3 (part 5) we can estimate the first one independent from M. So we get

$$\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) |\partial_n u_M|^2 \, dx \leq c(r). \tag{2.5}$$

Now we want to bound

$$\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} u_M|^2 \, dx. \tag{2.6}$$

As before, after integrating by parts, the only difference to the calculations of [BF4] is the integral

$$\int_B u_M \eta^{2k} \partial_\gamma \left[a_M(\cdot, |\widetilde{\nabla} u_M|) \right] \partial_\gamma u_M \, dx.$$

Here we estimate

$$
\begin{aligned}
U_2 \leq c &\int_B \eta^{2k} |\partial_\gamma a_M(\cdot, |\widetilde{\nabla} u_M|)| |\widetilde{\nabla} u_M| \, dx \\
&+ c \int_B \eta^{2k} a'_M(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} u_M| |\partial_\gamma \partial_\gamma u_M| \, dx \\
:=& c \, U_2^1 + c \, U_2^2 .
\end{aligned}
$$

Using 2.2 and Lemma 2.1 (part 7) we receive

$$
\begin{aligned}
U_2^1 \leq \tau &\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} u_M|^2 \, dx \\
&+ c(\tau) \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \, dx
\end{aligned}
$$

as well as

$$
\begin{aligned}
U_2^2 \leq \tau &\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} u_M|^2 \, dx \\
&+ c(\tau) \int_B \eta^{2k} \frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_\gamma \widetilde{\nabla} u_M|^2 \, dx.
\end{aligned}
$$

We absorb the first term in (2.6) and for the second one we need a Caccioppoli-type inequality as in [BF3]. Thereby we only have to consider

$$
\int_B \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla \left\{ \eta^{2k} \partial_\gamma u_M \right\} \, dx.
$$

By Lemma 2.1 (part 7) we obtain the upper bound

$$
\begin{aligned}
c \, &\int_B a'_M(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} \left\{ \eta^{2k} \partial_\gamma u_M \right\} | \, dx \\
+ c \, &\int_B b'_M(\cdot, |\partial_n u_M|) |\partial_n \left\{ \eta^{2k} \partial_\gamma u_M \right\} | \, dx \\
:=& c \left[\mathcal{U}_1 + \mathcal{U}_2 \right] .
\end{aligned}
$$

It follows

$$
\begin{aligned}
\mathcal{U}_1 \leq c &\int_B \eta^{2k-1} a'_M(\cdot, |\widetilde{\nabla} u_M|) |\nabla \eta| |\partial_\gamma u_M| \, dx \\
&+ c \int_B \eta^{2k} a'_M(\cdot, |\widetilde{\nabla} u_M|) |\partial_\gamma \widetilde{\nabla} u_M| \, dx \\
:=& \, c \left[\mathcal{U}_1^1 + \mathcal{U}_1^2 \right] .
\end{aligned}
$$

By Young's inequality one sees

$$\mathcal{U}_1^2 \le \tau \int_B \eta^{2k} \frac{a_M'(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_\gamma \widetilde{\nabla} u_M|^2 \, dx$$
$$+c(\tau) \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) \, dx$$

which can be handled conventionally. Furthermore we get

$$\mathcal{U}_2 \le \int_B \eta^{2k-1} |\nabla \eta| b_M'(\cdot, |\partial_n u_M|) |\widetilde{\nabla} u_M| \, dx$$
$$+ \int_B \eta^{2k} b_M'(\cdot, |\partial_n u_M|) |\partial_\gamma \partial_n u_M| \, dx.$$

For the second integral we deduce from Young's inequality and Remark 2.2 the upper bound

$$\tau \int_B \eta^{2k} \frac{b_M'(\cdot, |\partial_n u_M|)}{|\partial_n u_M|} |\partial_\gamma \partial_n u_M|^2 \, dx + c(\tau) \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) \, dx$$

which is uncritical. For the observation of the first one we see

$$\int_B \eta^{2k-1} |\nabla \eta| b_M'(\cdot, |\partial_n u_M|) |\widetilde{\nabla} u_M| \, dx \le \int_B \eta^{2k-2} |\nabla \eta|^2 \frac{b_M'(\cdot, |\partial_n u_M|)}{|\partial_n u_M|} |\widetilde{\nabla} u_M|^2 \, dx$$
$$+ \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|) \, dx$$

which can be bound as in [BF3] (section 3). Note that we need therefore $b_M(x,t) \le ct^2 a_M(x,t^2)$, but we have the stronger inequality $b_M(x,t) \le ct^2 a_M(x,t)$. So we get

$$\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|) |\widetilde{\nabla} u_M|^2 \, dx \le c(r). \tag{2.7}$$

By Lemma 2.1 (part a and 6) we receive

$$\int_B \eta^2 |\nabla^2 u_M|^2 \, dx \le \int_B D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \partial_\gamma \nabla u_M) \, dx.$$

Using a Caccioppoli-type inequality as in [BF3] we can bound this independent from M (note that the r.h.s. of this inequality was bound in the rest of the proof). So we obtain uniform boundedness of u_M in $W^{2,2}_{loc}(B, \mathbb{R}^N)$ (remember Lemma 2.3, part 5) and as in [Br2] (end of section 2) we deduce

$$\begin{aligned} &u_M \rightharpoonup u \text{ in } W^{2,2}_{loc}(B, \mathbb{R}^N), \\ &\nabla u_M \to \nabla u \text{ in } L^2_{loc}(B, \mathbb{R}^{nN}), \\ &\nabla u_M \to \nabla u \text{ a.e.} \end{aligned} \tag{2.8}$$

for $M \to \infty$. This implies $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$ and using Fatou's Lemma (2.5) and (2.7) points out the claim of Theorem 1.1. $\square$

3 Partial $\mathbf{C^{1,\alpha}}$-regularity

We define the excess as

$$E(x,r) := \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2 \, dy + \fint_{B_r(x)} \overline{a}(\cdot, |\nabla u - (\nabla u)_{x,r}|) \, dy$$

for $\overline{a}(x,t) := a(x,t)t^\omega$ with the $\omega \in (0,2)$ from (A6). If $r \le R_0$ such that $\chi + \theta_2 R_0 \le 2$, then Theorem 1.1 guarantees together with (A10) the existence of $E(x,r)$. To show this we estimate

$$\begin{aligned}
\fint_{B_r(x)} \overline{a}(\cdot, |\nabla u - (\nabla u)_{x,r}|) \, dy &\le \fint_{B_r(x)} \fint_{B_r(x)} \overline{a}(y, |\nabla u(y) - \nabla u(z)|) \, dydz \\
&\le c \fint_{B_r(x)} \fint_{B_r(x)} \overline{a}(y, |\nabla u(y)|) \, dydz \\
&+ c \fint_{B_r(x)} \fint_{B_r(x)} \overline{a}(y, |\nabla u(z)|) \, dydz.
\end{aligned}$$

For the second term we use (A10) and assume w.l.o.g. $|\nabla u(z)| \ge 1$:

$$\begin{aligned}
\overline{a}(y, |\nabla u(z)|) &\le c |\nabla u(z)|^{\theta_2 |y-z|} |\nabla u(z)|^\omega a(z, |\nabla u(z)|) \\
&\le |\nabla u(z)|^2 a(z, |\nabla u(z)|).
\end{aligned}$$

If we distinguish into an integral over $[|\partial_n u| \le |\widetilde{\nabla} u|]$ and the complement we see the existence of the excess.

LEMMA 3.1 *Assume (A1)-(A10) for an $\omega < 2$ and fix an $L > 0$. Then there is a $C^*(L)$, such that for every $\tau \in (0, 1/4)$ exists an $\epsilon = \epsilon(\tau, L) > 0$ with the following property: if*

$$|(\nabla u)_{x,r}| \le L \text{ and } E(x,r) + r^{\gamma^*} \le \epsilon \tag{3.1}$$

for a ball $B_r(x) \Subset \Omega$ this implies

$$E(x, \tau r) \le C^* \tau^2 [E(x,r) + r^{\gamma^*}] \tag{3.2}$$

where $\gamma^* \in (0,2)$ is arbitrary.

Proof: Now we extend the ideas of [Br]. For $z \in B_1 := B_1(0)$ let

$$u_m(z) := \frac{1}{\lambda_m r_m}\Big(u(x_m + r_m z) - a_m - r_m A_m z\Big),$$
$$a_m := (u)_{x_m,r_m} \text{ and } A_m := (\nabla u)_{x_m,r_m}.$$

Thereby $(f)_{x,r}$ denotes the mean value of the function f over the ball $B_r(x)$. For $\lambda_m^2 := E(x_m, r_m) + r_m^{\gamma^*}$ we deduce from (3.1)

$$|A_m| \leq L, \; \fint_{B_1} |\nabla u_m|^2 \, dz + \lambda_m^{-2} \fint_{B_1} \overline{a}(x_m + r_m z, \lambda_m |\nabla u_m|)dz + \lambda_m^{-2} r_m^{\gamma^*} = 1 \quad (3.3)$$

Whereas (3.2) reads after scaling as

$$\begin{aligned} &\fint_{B_\tau} |\nabla u_m - (\nabla u_m)_{0,\tau}|^2 \, dz \\ &+\lambda_m^{-2} \fint_{B_\tau} \overline{a}(x_m + r_m z, \lambda_m |\nabla u_m - (\nabla u_m)_{0,\tau}|)dz > C_* \tau^2. \end{aligned} \quad (3.4)$$

Using (3.3) we have after passing to subsequences

$$A_m \to: A, \;\; u_m \rightharpoonup: \overline{u} \;\text{ in }\; W^{1,2}(B_1, \mathbb{R}^N), \;\; (\overline{u})_{0,1} = 0, \;\; (\nabla \overline{u})_{0,1} = 0 \quad (3.5)$$

$$\lambda_m \nabla u_m \to 0 \;\text{ in }\; L^2(B_1, \mathbb{R}^{nN}) \text{ and a.e. on } B_1. \quad (3.6)$$

If we have a look at the proof in [Br] it is no problem to verify the limit equation and so we have to show

$$\nabla u_m \to \nabla \overline{u} \text{ in } L^2_{loc}(B), \quad (3.7)$$

$$\lim_{m\to\infty} \lambda_m^{-2} \fint_{B_r} \overline{a}(x_m + r_m z, \lambda_m |\nabla u_m - (\nabla u_m)_{0,r}|)dz = 0 \text{ for all } r < 1. \quad (3.8)$$

to end up the proof of Lemma 3.1. If we want to establish a Caccioppoli-type inequality as in Lemma 2.1 from [Br] we have to bound additionally to the estimations there the integral ($P \in \mathbb{R}^{nN}$ is arbitrary)

$$\int_B \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla \left\{\eta^2 \left[\partial_\gamma u_M - P\right]\right\} \, dx.$$

Using Lemma 2.1 (part 7) leads us to the terms

$$T_M^1 := \int_B a_M'(\cdot, |\widetilde{\nabla} u_M|)|\partial_\gamma \widetilde{\nabla} u_M| \eta^2 \, dx,$$

$$T_M^2 := \int_B a'_M(\cdot, |\widetilde{\nabla} u_M|)|\nabla u_M - P|\eta|\nabla\eta|\, dx,$$
$$T_M^3 := \int_B b'_M(\cdot, |\partial_n u_M|)|\partial_\gamma \partial_n u_M|\eta^2\, dx,$$
$$T_M^4 := \int_B b'_M(\cdot, |\partial_n u_M|)|\nabla u_M - P|\eta|\nabla\eta|\, dx.$$

From Young's inequality we deduce for $\tau > 0$ from Remark 2.2

$$T_M^1 \leq \tau \int_B \frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_\gamma \widetilde{\nabla} u_M|^2 \eta^2\, dx + c(\tau) \int_B a_M(\cdot, |\widetilde{\nabla} u_M|)\eta^2\, dx.$$

For T_M^2 the same arguments show the upper bound

$$c(\eta) \int_B a'_M(\cdot, |\widetilde{\nabla} u_M|)\, dx + c(\eta) \int_B a_M(\cdot, |\widetilde{\nabla} u_M|)\, dx$$
$$+c(\eta) \int_{B \cap \operatorname{spt} \eta} \frac{a'_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|} |\partial_n u_M|^2\, dx.$$

Similarly we get

$$T_M^3 \leq \tau \int_B \frac{b'_M(\cdot, |\partial_n u_M|)}{|\partial_n u_M|} |\partial_\gamma \partial_n u_M|^2 \eta^2\, dx + c(\tau) \int_B b_M(\cdot, |\partial_n u_M|)\eta^2,$$
$$T_M^4 \leq c(\eta) \int_B b'_M(\cdot, |\partial_n u_M|)\, dx + c(\eta) \int_B b_M(\cdot, |\partial_n u_M|)\, dx$$
$$+c(\eta) \int_{B \cap \operatorname{spt} \eta} \frac{b'_M(\cdot, |\partial_n u_M|)}{|\partial_n u_M|} |\widetilde{\nabla} u_M|^2\, dx.$$

After absorption of the τ-integrals we have to justify that we can interchange limes and integral for $M \to \infty$ in the remaining terms. We follow the argumentation of [Br] and choose for an arbitrary $\kappa > 0$ subset $S \subset B$ such that $\nabla u_M \to \nabla \overline{u}$ uniformly on S and $\mathcal{L}^n(B - S) \leq \kappa$ (therefore we need (3.6) and Egorov's Theorem). Then we can show as in [Br] that the integrals over $B - S$ are smaller than $c\kappa^\mu$. Furthermore we have to establish the convergence a.e. from

$$\widetilde{\psi}_M := \int_0^{|\widetilde{\nabla} u_M|} \sqrt{\frac{a'_M(x,t)}{t}}\, dt, \;\; \psi_M^{(n)} := \int_0^{|\partial_n u_M|} \sqrt{\frac{b'_M(x,t)}{t}}\, dt$$

against $\widetilde{\psi}$ and $\psi^{(n)}$ (with a suitable definition). From the end of section 2 we know $\nabla u_M \to \nabla u$ a.e. and so we have to establish the a.e.-convergence of

$$\widetilde{\chi}_M(x,s) := \int_0^s \sqrt{\frac{a'_M(x,t)}{t}}\, dt, \;\; \chi_M^{(n)}(x,s) := \int_0^s \sqrt{\frac{b'_M(x,t)}{t}}\, dt.$$

By Lemma 2.1 (part 2) this follows by Lebesgue's theorem on majorized convergence. Note that additionally in our calculations we have the terms

$$\widetilde{\psi}_{M,x} := \int_0^{|\widetilde{\nabla} u_M|} \nabla_x \sqrt{\frac{a'_M(x,t)}{t}}\, dt, \quad \psi^{(n)}_{M,x} := \int_0^{|\partial_n u_M|} \nabla_x \sqrt{\frac{b'_M(x,t)}{t}}\, dt.$$

But by Lemma 2.1 (part 7) we can bound them by $\widetilde{\psi}_M$ und $\psi^{(n)}_M$ (and these can bound as in [Br]).
In the limit version of the essential Caccioppoli-type inequality we have to add

$$\begin{aligned}
T^1 &:= \int_B a(\cdot, |\widetilde{\nabla} u|)\eta^2\, dx, \\
T^2 &:= \int_B a'(\cdot, |\widetilde{\nabla} u|)|\nabla u - P|\eta|\nabla \eta|\, dx, \\
T^3 &:= \int_B b(\cdot, |\partial_n u|)\eta^2\, dx, \\
T^4 &:= \int_B b'(\cdot, |\partial_n u|)|\nabla u - P|\eta|\nabla \eta|\, dx
\end{aligned}$$

on the r.h.s. For the proof of (3.7) we get after scaling

$$\begin{aligned}
T^1_m &:= \frac{r_m^2}{\lambda_m^2}\int_{B_1} a(x_m + r_m z, |\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|)\eta^2\, dz, \\
T^2_m &:= \frac{r_m^2}{\lambda_m^2}\int_{B_1} a'(x_m + r_m z, |\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|)|\lambda_m \nabla u_m|\eta \frac{|\nabla \eta|}{r_m}\, dz, \\
T^3_m &:= \frac{r_m^2}{\lambda_m^2}\int_{B_1} b(x_m + r_m z, |A^{(n)}_m + \lambda_m \partial_n u_m|)\eta^2\, dz, \\
T^4_m &:= \frac{r_m^2}{\lambda_m^2}\int_{B_1} b'(x_m + r_m z, |A^{(n)}_m + \lambda_m \partial_n u_m|)|\lambda_m \nabla u_m|\eta \frac{|\nabla \eta|}{r_m}\, dz
\end{aligned}$$

which we have to bound uniformly in M. We separate into the sets $[|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m| \le K]$ and $[|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m| > K]$ and use uniform boundedness of $\lambda_m^{-2} r_m^2$:

$$T^1_m \le c(K) + c(K)\int_{B_1} \overline{a}(x_m + r_m z, \lambda_m\, |\nabla u_m|) dz \le c(K)$$

by (3.3). Similarly we get by (A6) the same result for T^3_m. From Remark 2.2 we deduce

$$T^2_m \le c(\eta, K)\int_{B_1} |\nabla u_m|\, dz + c(\eta, K)\int_{B_1 \cap [\ldots > K]} a(x_m + r_m z, \lambda_m\, |\nabla u_m|) dz$$

$$\leq c(\eta,K)+c(\eta,K)\int_{B_1}\overline{a}(x_m+r_mz,\lambda_m\,|\nabla u_m|)dz$$
$$\leq c(\eta,K).$$

where we use the L^2-bound for ∇u_m and (3.3). Analogously we estimate T_m^4 using (A6). Proving (3.8) we define

$$\widetilde{\psi}_m := \frac{1}{\lambda_m}\int_{|\widetilde{A}_m|}^{|\widetilde{A}_m+\lambda_m\widetilde{\nabla}u_m|}\sqrt{\frac{a'(x,t)}{t}}dt,\;\; \psi_m^{(n)} := \frac{1}{\lambda_m}\int_{|A_m^{(n)}|}^{|A_m^{(n)}+\lambda_m\partial_n u_m|}\sqrt{\frac{b'(x,t)}{t}}dt.$$

If we follow the argumentation of [Br] we get uniform $W^{1,2}_{loc}$-bounds again (additionally to the terms there we have $T_m^1,...,T_m^4$, which are uncritical) and can end up the proof of the blow up lemma just like in [Br]. Now we can iterate this lemma as in [BF6] for example. The only problem is the inequality

$$E(x_0,r)\leq c(\tau)E(x_0,\tau^kR)$$

for $\tau^{k+1}R\leq r\leq\tau^kR$. But by (A9) we can show

$$E(x_0,r)\leq c(\tau)E(x_0,\tau^kR)+c(\tau)r. \tag{3.9}$$

By convexity and Δ_2-condition of $\overline{a}$ we obtain

$$\begin{aligned}\fint_{B_r(x_0)}\overline{a}(y,|\nabla u(y)-(\nabla u)_{r,x_0}|)\,dy \leq{}& c\fint_{B_r(x_0)}\overline{a}(y,|\nabla u(y)-(\nabla u)_{\tau^kR,x_0}|)\,dy\\ &+c\fint_{B_r(x_0)}\overline{a}(y,|(\nabla u)_{\tau^kR,x_0}-(\nabla u)_{r,x_0}|)\,dy.\end{aligned}$$

For the first integral one directly sees the estimation

$$c(\tau)\fint_{B_{\tau^kR}(x_0)}\overline{a}(y,|\nabla u(y)-(\nabla u)_{\tau^kR,x_0}|)\,dy.$$

For the second one we use

$$y^*:=\operatorname{argmin}_{B_r(x_0)}a(y,t) \tag{3.10}$$

which is independent from t by (A9) and get the bound

$$\fint_{B_r(x_0)}\left|\overline{a}(y,|(\nabla u)_{\tau^kR,x_0}-(\nabla u)_{r,x_0}|)-\overline{a}(y^*,|(\nabla u)_{\tau^kR,x_0}-(\nabla u)_{r,x_0}|)\right|\,dy$$

$$+ \fint_{B_r(x_0)} \overline{a}(y^*, |(\nabla u)_{\tau^k R, x_0} - (\nabla u)_{r,x_0}|)\, dy.$$

The first term can be estimated by

$$\sup_{t\in[0,1]} \left|\nabla_x \overline{a}\left(y + t(y^* - y), |(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}|\right)\right| |y^* - y| \le c(\tau) r.$$

Remember (A5) and note the inequality

$$\begin{aligned}
|(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}| &\le \fint_{B_r(x_0)} |\nabla u - (\nabla u)_{\tau^k R,x_0}|\, dz \\
&\le c(\tau) \fint_{B_{\tau^k R}(x_0)} |\nabla u - (\nabla u)_{\tau^k R,x_0}|\, dz \\
&\le c(\tau) \left[E(x_0, \tau^k R) + 1\right] \le c(\tau),
\end{aligned}$$

since $E(x_0, \tau^k R) \le \epsilon$ (this is a consequence of the iteration of the blow up lemma, compare [BF6]). Jensen's inequality and (A9) lead us to

$$\begin{aligned}
\fint_{B_r(x_0)} \overline{a}(y^*, |(\nabla u)_{\tau^k R,x_0} - (\nabla u)_{r,x_0}|)\, dy &\le \fint_{B_r(x_0)} \overline{a}(y^*, |\nabla u(y) - (\nabla u)_{\tau^k R,x_0}|)\, dy \\
&\le \fint_{B_r(x_0)} \overline{a}(y, |\nabla u(y) - (\nabla u)_{\tau^k R,x_0}|)\, dy \\
&\le c(\tau) \fint_{B_{\tau^k R}(x_0)} \overline{a}(y, |\nabla u(y) - (\nabla u)_{\tau^k R,x_0}|)\, dy
\end{aligned}$$

by the choice of y^* and we receive (3.9). □

Proof of Theorem 1.2 b): As remarked in [Br] we can deduce the 2D-result from the proof of [BF5]. □

4 Regularity statements for N = 1

Let $N = 1$. Firstly we show

Lemma 4.1 *For all $t < \infty$ and all $B_\rho \Subset B$ we have*

$$\sup_M \|\nabla u_M\|_{L^t(B_\rho)} < \infty.$$

We want to estimate

$$\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+2}{2}}\, dx \tag{4.1}$$

for a cut-off function $\eta \in C_0^\infty(B)$ such that $\eta \equiv 1$ on $B_r(x_0)$ for a $\rho < R$ and $0 \le \eta \le 1$. If we follow the lines of [BF3] we get after integrating by parts (using uniform local bounds on u_M, see Lemma 2.3, part 6))

$$\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+2}{2}}\, dx \le c(\eta)\,[1 + I_1 + I_2 + I_3 + I_4] \tag{4.2}$$

where we have

$$\begin{aligned}
I_1 &:= \int_{\mathrm{spt}(\eta)} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx\\
I_2 &:= \int_{\mathrm{spt}(\eta)} \frac{a_M(\cdot, |\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|^2}\left[b_M(x,\cdot)^{-1}\left(\frac{a_M(\cdot, |\widetilde{\nabla} u_M|)}{\tau|\widetilde{\nabla} u_M|^2}\right)\right]^{\alpha+2} dx\\
I_3 &:= \int_B \eta^{2k}|\partial_n b_M(\cdot, |\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha+1}{2}}\, dx\\
I_4 &:= \int_B |\partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla[\partial_n u_M \eta^2 \Gamma_{n,M}^{\frac{\alpha}{2}}]|\, dx.
\end{aligned}$$

Note that the terms I_3 and I_4 are additionally to this one from [BF3] on account of the x-dependence. Since we have $a_M(x,t) \le ct^2 b_M(x,t)$ for large t (see Lemma 2.1, part 6) we can bound I_2 by

$$c(\tau)\left[1 + \int_{\mathrm{spt}(\eta)} a_M(\cdot, |\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\alpha}{2}}\, dx\right],$$

whereby we use a uniform Δ_2-condition for b_M^{-1}. This follows from the uniform version of (A2). We deduce from Lemma 1.1 (part 7) by Young's inequality

$$\begin{aligned}
I_2 \le\ & c\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+1}{2}}\, dx\\
\tau & \int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+2}{2}}\, dx + c(\tau)\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx
\end{aligned}$$

and absorb the first term on the r.h.s. in (4.1). Furthermore we obtain

$$\begin{aligned}
I_4 \le & \int_B |\eta^2 \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \partial_n \nabla u_M \Gamma_{n,M}^{\frac{\alpha}{2}}|\, dx\\
+2k & \int_B |\eta^{2k-1}\partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla\eta\, \partial_n u_M \Gamma_{n,M}^{\frac{\alpha}{2}}|\, dx
\end{aligned}$$

$$+\alpha \int_B |\eta^{2k}\partial_\gamma D_P F_M(\cdot,\nabla u_M) : \partial_n \nabla u_M \Gamma_{n,M}^{\frac{\alpha-2}{2}} \partial_n u_M^2|\, dx$$
$$:= I_4^1 + I_4^2 + I_4^3.$$

From splitting-structure and Lemma 2.1 (part 7) we deduce

$$I_4^1 \le c \int_B \eta^{2k} a_M'(\cdot,|\widetilde{\nabla} u_M|)|\partial_n \widetilde{\nabla} u_M|\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx$$
$$c \int_B \eta^{2k} b_M'(\cdot,|\partial_n u_M|)|\partial_n \partial_n u_M|\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx.$$

For the first integral we obtain by Remark 2.2 the upper bound

$$\tau \int_B \eta^{2k} \frac{a_M'(\cdot,|\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|}|\partial_n \widetilde{\nabla} u_M|^2 \Gamma_{n,M}^{\frac{\alpha}{2}}\, dx$$
$$+c(\tau)\int_B \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx.$$

For the second one we use the same arguments. The τ-terms can be absorbed in a Caccioppoli-type inequality (compare [BF3], section 5). Similarly we see

$$I_4^2 \le c \int_B \eta^{2k-2} a_M'(\cdot,|\widetilde{\nabla} u_M|)|\nabla\eta|\Gamma_{n,M}^{\frac{\alpha+1}{2}}\, dx$$
$$c \int_B \eta^{2k-1} b_M'(\cdot,|\partial_n u_M|)|\nabla\eta|\Gamma_{n,M}^{\frac{\alpha+1}{2}}\, dx.$$

By Remark 2.2 we receive for the first term the estimation

$$\int_B \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx + \int_B \eta^{2k-2}\frac{a_M'(\cdot,|\widetilde{\nabla} u_M|)}{|\widetilde{\nabla} u_M|}|\nabla\eta|^2\Gamma_{n,M}^{\frac{\alpha+2}{2}}\, dx.$$

The second one exactly corresponds to term $\mathcal{S}_3$ in [BF3] (section 3) and an estimation of this leads us to I_2. The second integral in the estimation of I_4^2 is bounded by $c(\eta)\,[1+I_1]$ (remember Lemma 2.3, part 5). Putting all this estimations together we finally obtain

$$\int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha+2}{2}}\, dx$$
$$\le c(\eta)\left[1 + \int_{\mathrm{spt}(\eta)} b_M(\cdot,|\partial_n u_M|)\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx + \int_{\mathrm{spt}(\eta)} a_M(\cdot,|\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\alpha}{2}}\, dx\right] \quad (4.3)$$
$$+c\int_B \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)\Gamma_{n,M}^{\frac{\alpha}{2}}\, dx$$

if we use Remark 2.2. Now we have to separate the mixed integrand in the last term $=: \mathcal{I}$. Therefore we define for $\tau > 0$ the N-function (we can neglect the case $\alpha = 0$)

$$\mathcal{K}_\tau(x,t) := \tau t^{\frac{\alpha+2}{\alpha}} b_M(x, t^{\frac{1}{\alpha}}) \tag{4.4}$$

and see the inequality

$$\mathcal{K}_\tau^*(x,s) \le s\widehat{b}_M(x,\cdot)^{-1}\left(\frac{s}{\tau}\right) \quad \text{with} \quad \widehat{b}_M(x,t) := t^{\frac{2}{\alpha}} b_M(x, t^{\frac{1}{\alpha}})$$

for the conjugate function $\mathcal{K}_\tau^*$. By Lemma (4.3) (part 8) we obtain for $t \ge 1$

$$\frac{a_M(x,t)}{\tau} \le \frac{ct^2 b_M(x,t)}{\tau} = \frac{c\widehat{b}_M(x,t^\alpha)}{\tau}. \tag{4.5}$$

Obviously we have

$$\begin{aligned}
&\widehat{b}_M(x,t) = \lambda_M(x, t^{\frac{1}{\alpha}}) \text{ for } \lambda_M(x,t) := t^2 b_M(x,t),\\
&\widehat{b}_M(x,\cdot)^{-1}(t) = \left[\lambda_M(x,\cdot)^{-1}(t)\right]^\alpha.
\end{aligned}$$

Using Lemma 2.1 (part 4) one can show a uniform Δ_2-condition for $\lambda_M(x,\cdot)^{-1}$ and thereby for $\widehat{b}_M(x,\cdot)^{-1}$. So (4.5) implies

$$\widehat{b}_M(x,\cdot)^{-1}\left(\frac{a_M(x,t)}{\tau}\right) \le c(\tau)t^\alpha.$$

By Young's inequality for N-functions we get

$$\begin{aligned}
\mathcal{I} &\le c\left[1 + \int_B \eta^{2k}\mathcal{K}_\tau(|\partial_n u_M|^\alpha)\,dx + \int_B \eta^{2k}\mathcal{K}_\tau^*(a_M(\cdot,|\widetilde{\nabla} u_M|))\,dx\right]\\
&\le c\left[1 + \tau\int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\partial_n u_M|^{\alpha+2}\,dx\right.\\
&\quad \left. + c(\tau)\int_B \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^\alpha\,dx\right].
\end{aligned}$$

Inserting this into (4.3) and absorb the τ-term we receive

$$\begin{aligned}
&\int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha+2}{2}}\,dx\\
&\le c(\eta)\left[1 + \int_{\mathrm{spt}(\eta)} b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha}{2}}\,dx + \int_{\mathrm{spt}(\eta)} a_M(\cdot,|\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\alpha}{2}}\,dx\right].
\end{aligned} \tag{4.6}$$

Note that the relation between a_M and b_M is symmetric and they have exact the same properties. Therefore we can show by the same arguments

$$\begin{aligned}&\int_B \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)|\widetilde{\Gamma}_M^{\frac{\beta+2}{2}}\,dx\\&\le c(\eta)\left[1+\int_B b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\beta}{2}}\,dx+\int_B a_M(\cdot,|\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\beta}{2}}\,dx\right].\end{aligned} \tag{4.7}$$

Now we iterate (4.6) and (4.7) and use for the start of the induction $\alpha = 0$ together with Lemma 2.3 (part 5). This gives the claim of Lemma 4.1.
Now we have to show

$$\sup_M \|\nabla u_M\|_{L^\infty(B_\rho)} < \infty \tag{4.8}$$

for $B_\rho \Subset B$ to follow the result of Theorem 1.2 c). Note that we have the growth estimates

$$\begin{aligned}\lambda|X|^2 \le D_P^2 F_M(x,Z)(X,X) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|X|^2,\\|\partial_\gamma D_P F_M(Z)| \le c(1+|Z|^2)^{\frac{q-1}{2}}\end{aligned}$$

for all $Z, X \in \mathbb{R}^{nN}$, all $x \in \overline{\Omega}$ and all $\gamma \in \{1,...,n\}$ uniformly in M. Using this and Lemma 4.1 we can follow (4.8) by the arguments of [Br2] (Lemma 5.4). □

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