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Martin Fuchs

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Martin Fuchs

Saarland University
Dep. of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

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Abstract

Let $f : B \to \mathbb{C}$ denote a Sobolev function of class W^1_p defined on the unit disc. We show that the distance of f to the class of all holomorphic functions measured in the norm of the space $W^1_p(B;\mathbb{C})$ is bounded by the L^p–norm of the Wirtinger derivative $\partial_{\overline{z}} f$. As a consequence we obtain a Korn type inequality for vectorfields $B \to \mathbb{R}^2$.

Let B denote the open unit disc in the complex plane. For numbers $1 \le p < \infty$ we consider the Sobolev space $W^1_p(B;\mathbb{C})$ of functions $f \in L^p(B;\mathbb{C})$ having first order weak partial derivatives belonging to the same Lebesgue class (see, e.g., [Ad] for details). Finally, we introduce the space $H(B)$ of all holomorphic functions $B \to \mathbb{C}$. If we write $z = x + iy$ for the variable $z \in B$, then the Wirtinger operator $\partial_{\overline{z}}$ is defined as $\partial_{\overline{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$ acting on weakly differentiable functions $f : B \to \mathbb{C}$. Note that f is holomorphic if and only if $\partial_{\overline{z}} f = 0$. Now we can state our

THEOREM: *For any $p \in (1,\infty)$ there is a constant $K = K(p)$ such that*

$$\inf_{h \in H(B)} \left[\|f-h\|_{L^p(B)} + \|\nabla f - \nabla h\|_{L^p(B)}\right] \le K \|\partial_{\overline{z}} f\|_{L^p(B)} \tag{1}$$

holds for all $f \in W^1_p(B;\mathbb{C})$. Moreover, the infimum on the left–hand side of (1) is attained. In the limit case $p = 1$ the following statement holds: for any number $q \in [1,2)$ there exists a constant $\widetilde{K} = \widetilde{K}(q)$ with the property that

$$\inf_{h \in H(B)} \|f-h\|_{L^q(B)} \le \widetilde{K} \|\partial_{\overline{z}} f\|_{L^1(B)} \tag{2}$$

is true for all $f \in W^1_1(B;\mathbb{C})$. Again, the infimum is attained.

REMARKS:

1.) For functions $g: B \to \mathbb{C}$ we use the symbol ∇g to denote the Jacobian matrix of g.

2.) For vectorfields $u: B \to \mathbb{R}^2$ we introduce the symmetric gradient $\varepsilon(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ and the deviatoric part $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{2}(\operatorname{div} u)\mathbf{1}$, $\mathbf{1}$ denoting the unit matrix. Observing that $|\varepsilon^D(u)| = \sqrt{2}|\partial_{\overline{z}} u|$ we obtain the following Korn type inequality: there is a constant $C(p)$, $1 < p < \infty$, such that for each field $u \in W^1_p(B;\mathbb{R}^2)$ we can find a holomorphic function $h \in H(B)$ with the property

$$\|u - h\|_{L^p(B)} + \|\nabla u - \nabla h\|_{L^p(B)} \leq C(p)\|\varepsilon^D(u)\|_{L^p(B)}\,.$$

This is the appropriate twodimensional variant of the Korn type estimates involving ε^D on domains in $\mathbb{R}^n$, $n \geq 3$, recently obtained by Dain [Da].

3.) We conjecture that (2) can be improved even under weaker hypothesis: to be precise, consider a function f from the space $L^1(B;\mathbb{C})$ such that $\partial_{\overline{z}} f$ is a complex measure of finite total variation $\int_B |\partial_{\overline{z}} f|$. We then claim the existence of a constant $\widetilde{C}$ being independent of f and of a function $h \in H(B)$ such that

$$\|f - h\|_{L^2(B)} \leq \widetilde{C}\int_B |\partial_{\overline{z}} f|$$

holds. In order to prove this result it will be necessary to construct a sequence $\{f_n\}$ of smooth functions $\overline{B} \to \mathbb{C}$ with the properties

$$\int_B |f_n - f|\,dx \to 0\,,\quad \int_B |\partial_{\overline{z}} f_n|\,dx \to \int_B |\partial_{\overline{z}} f|\,.$$

This would imply the above inequality at least for $L^q(B)$, $q < 2$, on the left–hand side.

4.) Suppose that an exponent $p \in (1,\infty)$ is given. Then, according to [St], Proposition 4, p.60, there exists a constant $M(p)$ with

$$\|\nabla f\|_{L^p(B)} \leq M(p)\|\partial_{\overline{z}} f\|_{L^p(B)}$$

for all functions f from the class $\overset{\circ}{W}{}^1_p(B;\mathbb{C})$ (compare also [FS] for a slightly different form of this inequality). This implies the validity of (1) for functions with zero trace.

5.) Obviously the Theorem extends to more general bounded domains in the plane having a smooth boundary curve.

Proof: Let us suppose first that f is of class $C^1(\overline{B};\mathbb{C})$. Then it holds (see [Hö] or [Sa])

$$f(z) = \frac{1}{2\pi i}\int_{\partial B}\frac{f(w)}{w - z}\,dw - \frac{1}{\pi}\int_B \frac{\partial_{\overline{z}} f(w)}{w - z}\,d\mathcal{L}^2(w)\,, \tag{3}$$

where $h(z) := \frac{1}{2\pi i}\int_{\partial B}\frac{f(w)}{w-z}\,dw$ is a complex line integral and where in $U(z) := -\frac{1}{\pi}\int_B \frac{\partial_{\overline{z}} f(w)}{w-z}\,d\mathcal{L}^2(w)$ we integrate with respect to the twodimensional Lebesgue measure $\mathcal{L}^2$. Clearly the function h belongs to the class $H(B)$, and for the potential $U(z)$ we have

$$\begin{aligned} U(z) &= -\frac{1}{\pi}\int_B \frac{1}{w-z}\left(\text{Re }\partial_{\overline{z}} f(w) + i\text{ Im }\partial_{\overline{z}} f(w)\right)\,d\mathcal{L}^2(w) \\ &= -\frac{1}{\pi}\int_B \frac{1}{|w-z|^2}(\overline{w}-\overline{z})\left(\text{Re }\partial_{\overline{z}} f(w) + i\text{ Im }\partial_{\overline{z}} f(w)\right)\,d\mathcal{L}^2(w) \\ &= -\frac{1}{\pi}\int_B \frac{1}{|w-z|^2}(w-z)\cdot\left(\text{Re }\partial_{\overline{z}} f(w),\ \text{Im }\partial_{\overline{z}} f(w)\right)\,d\mathcal{L}^2(w) \\ &\quad -\frac{1}{\pi}\, i\int_B \frac{1}{|w-z|^2}(w-z)\cdot\left(\text{Im }\partial_{\overline{z}} f(w), -\text{ Re }\partial_{\overline{z}} f(w)\right)\,d\mathcal{L}^2(w)\,, \end{aligned}$$

where on the right–hand side $(w-z)\cdot(\ldots,\ldots)$ denotes the scalar product in $\mathbb{R}^2$. Thus $U(z)$ is the sum of two quasi–potentials in the sense of Morrey [Mo], Definition 3.7.1, and from Theorem 3.7.1 in this reference we obtain for any $p \in (1,\infty)$ the estimate

$$\|U\|_{L^p(B)} + \|\nabla U\|_{L^p(B)} \le K(p)\|\partial_{\overline{z}} f\|_{L^p(B)}\,. \tag{4}$$

Returning to (3) and using (4) we get

$$\begin{aligned} &\|f-h\|_{L^p(B)} + \|\nabla f - \nabla h\|_{L^p(B)} \\ &\quad = \|U\|_{L^p(B)} + \|\nabla U\|_{L^p(B)} \le K(p)\|\partial_{\overline{z}} f\|_{L^p(B)}\,, \end{aligned} \tag{5}$$

which clearly implies (1) for smooth f. If f is in $W^1_p(B;\mathbb{C})$ with $1<p<\infty$, then we choose $f_n \in C^1(\overline{B};\mathbb{C})$ such that $\|f_n - f\|_{W^1_p(B)} := \|f_n - f\|_{L^p(B)} + \|\nabla f_n - \nabla f\|_{L^p(B)} \longrightarrow 0$. Let h_n denote the corresponding sequence in $H(B)$. Inequality (5) implies $\sup_n \|h_n\|_{W^1_p(B)} < \infty$, and after passing to a subsequence we find $h \in H(B)$ such that e.g.

$$\|h_n - h\|_{W^1_p(G)} \longrightarrow 0\,,\ n\to\infty\,,$$

for any subdomain $G \Subset B$. This implies

$$\|f-h\|_{W^1_p(G)} = \lim_{n\to\infty}\|f_n - h_n\|_{W^1_p(G)} \le \limsup_{n\to\infty}\|f_n - h_n\|_{W^1_p(B)} \overset{(5)}{\le} K(p)\|\partial_{\overline{z}} f\|_{L^p(B)}$$

and we arrive at estimate (1) by letting $G \nearrow B$.

For proving (2) we return to (3) and observe

$$|f(z) - h(z)| \le \frac{1}{\pi}\int_B \frac{1}{|w-z|}\ |\partial_{\overline{z}} f(w)|\ d\mathcal{L}^2(w)$$

at least for f of class $C^1(\overline{B};\mathbb{C})$. Noting that the right–hand side of the foregoing inequality is equal to $\frac{1}{\pi}V_{1/2}\left(|\partial_{\overline{z}} f|\right)(z)$, where $V_{1/2}$ is the Riesz–potential introduced in (7.31) of [GT] for the choices $\mu = 1/2$ and $n=2$, our claim follows from (7.34) in [GT] by choosing $p=1$. If f is just of class $W^1_1(B;\mathbb{C})$, then (2) is verified by approximation.

□

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