Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 249

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Saarbrücken 2009

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 249
submitted: October 12, 2009

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0 Introduction

A classical result in several complex variables going back to H. Cartan says that, over a Stein open set $U \subset \mathbb{C}^n$, each finitely generated ideal $(g_1, \ldots, g_r) \subset \mathcal{O}(U)$ is closed. In this situation, the operator-valued map $\alpha_g : U \to \mathcal{L}(\mathbb{C}^r, \mathbb{C}), \alpha_g(z)(x_i) = \sum_{i=1}^r g_i(z)x_i$, is analytic and the above cited result precisely means that the image of the induced multiplication operator

$$\alpha_g : \mathcal{O}(U, \mathbb{C}^r) \to \mathcal{O}(U, \mathbb{C}), (\alpha_g f)(z) = \alpha_g(z)f(z)$$

is closed. To indicate a proof, let us observe that, more generally, every analytic operator-valued map $\alpha \in \mathcal{O}(U, \mathcal{L}(\mathbb{C}^r, \mathbb{C}^s))$ induces a morphism $\mathcal{O}_U^{\mathbb{C}^r} \xrightarrow{\alpha} \mathcal{O}_U^{\mathbb{C}^s}$ between coherent analytic sheaves. But then the image sheaf $\mathrm{Im}\alpha \subset \mathcal{O}_U^{\mathbb{C}^s}$ is a coherent subsheaf and

$$0 \to \ker \alpha \hookrightarrow \mathcal{O}_U^{\mathbb{C}^r} \xrightarrow{\alpha} \mathrm{Im}\alpha \to 0$$

becomes an exact sequence of coherent analytic sheaves on U. Since coherent sheaves are acyclic on Stein open sets in $\mathbb{C}^n$, the induced sequence of section spaces over U remains exact. By Cartan's Abgeschlossenheitssatz (cf. Section V.4 in [7]) the subspace

$$\alpha\mathcal{O}(U, \mathbb{C}^r) = \alpha\Gamma(U, \mathcal{O}_U^r) = \Gamma(U, \mathrm{Im}\alpha) \subset \Gamma(U, \mathcal{O}_U^s) = \mathcal{O}(U, \mathbb{C}^s)$$

is closed in the usual Fréchet space topology.
Let X, Y be complex Banach spaces. In the present note we show that, for each Stein open set $U \subset \mathbb{C}^n$ and each operator-valued map $\alpha \in \mathcal{O}(U, \mathcal{L}(Y, X))$ with the property that $\dim X/\alpha(z)Y < \infty$ for every $z \in U$, the induced multiplication operator

$$\alpha : \mathcal{O}(U, Y) \to \mathcal{O}(U, X), (\alpha f)(z) = \alpha(z)f(z)$$

has closed range. To prove this result we use methods from Markoe [12] and Leiterer [10] to show that in the above situation the image sheaf $\mathrm{Im}\alpha \subset \mathcal{O}_U^X$ is a Banach coherent subsheaf in the sense of Leiterer [10]. But then

$$0 \to \ker \alpha \hookrightarrow \mathcal{O}_U^Y \xrightarrow{\alpha} \mathrm{Im}\alpha \to 0$$

becomes an exact sequence of Banach coherent sheaves. Since Banach coherent sheaves are acyclic on Stein open sets in $\mathbb{C}^n$, we obtain as in the finite-dimensional case the closedness of the subspace

$$\alpha\mathcal{O}(U, Y) = \alpha\Gamma(U, \mathcal{O}_U^Y) = \Gamma(U, \mathrm{Im}\alpha) \subset \Gamma(U, \mathcal{O}_U^X) = \mathcal{O}(U, X).$$

For single variable operator-valued functions of the form $\alpha(z) = z - T$, where $T \in \mathcal{L}(X)$ is a bounded operator on a Banach space, closed range theorems of the above type are known and have been applied in the local spectral theory of Banach-space operators (see [2, 13, 14]). For instance, for every operator saytisfying the finiteness condition $\dim(X/TX) < \infty$, one can show that the generalized range $\mathrm{R}^\infty(T) = \bigcap_{k \geq 1} T^k X$ of T has a representation of the form

$$\mathrm{R}^\infty(T) = X_T(\mathbb{C} \setminus U),$$

where U is a suitable open zero neighbourhood in $\mathbb{C}$. As an application of our results we show in the final part of the paper that an analogous formula holds for commuting multioperators.

1 Main Result

Recall that an analytic sheaf $\mathcal{F}$ on an analytic space $(X, \mathcal{O}_X)$ is said to be an analytic Fréchet sheaf if all section spaces $\mathcal{F}(U)$ ($U \subset X$ open) are Fréchet $\mathcal{O}(U)$-modules and the restriction maps

$$\mathcal{F}(U) \to \mathcal{F}(V), \; s \mapsto s|_V \quad (U, V \subset X \text{ open with } V \subset U)$$

are continuous. A continuous morphism $\varphi : \mathcal{F} \to \mathcal{G}$ between analytic Fréchet sheaves is a sheaf homomorphism such that for every open set $U \subset X$ the induced section map $\varphi : \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ is continuous. An analytic Fréchet sheaf $\mathcal{F}$ on X is called Banach coherent (in the sense of Leiterer [10]) if, for each point $x \in X$ and each integer $n \geq 0$, there are an open neighbourhood $U \subset X$ of x and a Banach-free resolution for $\mathcal{F}|_U$ of length n, that is, an exact sequence of continuous morphisms

$$\mathcal{O}^{E_n}|_U \longrightarrow \ldots \longrightarrow \mathcal{O}^{E_1}|_U \longrightarrow \mathcal{O}^{E_0}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0,$$

where $E_0, \ldots, E_n$ are suitable Banach spaces. Examples of Banach coherent analytic Fréchet sheaves are coherent analytic sheaves and sheaves of the form $\mathcal{O}_X^E$, where E is a Banach space. The reader will find more details in Chapter 4 of [4].

In the following let X, Y be complex Banach spaces and let $\alpha : D \to \mathcal{L}(Y, X)$ be a holomorphic map on a fixed open set $D \subset \mathbb{C}^n$.

We start by considering some special cases of our main result. In particular we prove a local version which shows that the range of $\alpha : \mathcal{O}(U, Y) \to \mathcal{O}(U, X)$ is closed if $U \subset D$ is a small Stein open neighbourhood of a point $z \in D$ such that $\dim(X/\alpha(z)Y) < \infty$.

First we consider the case that the Banach space X on the right-hand side is finite dimensional, i.e., we have $Y = \mathbb{C}^p$ for some $n \in \mathbb{N}$.

Lemma 1.1 *If $X = \mathbb{C}^p$ for some $p \in \mathbb{N}$ and D is Stein, then*

$$\alpha\mathcal{O}(D, Y) \subset \mathcal{O}(D, \mathbb{C}^p)$$

is a closed subspace.

Proof. By a result of Markoe [12] (Proposition 5) the image sheaf $\alpha\mathcal{O}_D^Y \subset \mathcal{O}_D^{\mathbb{C}^p}$ is a coherent subsheaf. Therefore the second and the third sheaf in the short exact sequence

$$0 \longrightarrow \ker\alpha \hookrightarrow \mathcal{O}_D^Y \xrightarrow{\alpha} \alpha\mathcal{O}_D^Y \longrightarrow 0$$

of analytic Fréchet sheaves are Banach coherent in the sense of Leiterer [10] (cf. Section 4.5 in [4]). But then also the first sheaf $\ker\alpha$ is Banach coherent (Proposition 4.5.7 in [4]). Since Banach coherent analytic Fréchet sheaves are quasi-coherent and quasi-coherent analytic Fréchet sheaves are acyclic on Stein open subsets (Theorem 4.5.2 and Proposition 4.3.3 (b) in [4]), the induced sequence of section spaces

$$0 \longrightarrow \Gamma(D, \ker\alpha) \hookrightarrow \Gamma(D, \mathcal{O}_D^Y) \xrightarrow{\alpha} \Gamma(D, \alpha\mathcal{O}_D^Y) \longrightarrow 0$$

remains exact. By the 'Abgeschlossenheitssatz' for coherent sheaves (Chapter V §6.4 in [7]) it follows that

$$\alpha\mathcal{O}(D, Y) = \alpha\Gamma(D, \mathcal{O}_D^Y) = \Gamma(D, \alpha\mathcal{O}_D^Y) \subset \Gamma(D, \mathcal{O}_D^{\mathbb{C}^p}) = \mathcal{O}(D, \mathbb{C}^p)$$

is a closed subspace.
Now we consider the case that X is an arbitrary Banach space, but we assume in addition that the kernel of the operator $\alpha(z)$ is continuously projected in Y.

Lemma 1.2 *Let $0 \in D$ and $\dim(X/\alpha(0)Y) < \infty$. If $\ker\alpha(0) \subset Y$ is continuously projected, then there is an open set $V \subset \mathbb{C}^n$ with $0 \in V \subset D$ such that*

$$\alpha\mathcal{O}(U, Y) \subset \mathcal{O}(U, X)$$

is closed for every Stein open subset $U \subset V$.

Proof. We use a construction due to Markoe [12] to reduce the assertion to the case where X is finite dimensional.
Define $T = \alpha(0) \in \mathcal{L}(Y, X)$ and choose closed subspaces $M \subset Y$ and $N \subset X$ such that

$$Y = M \oplus \ker T \quad \text{and} \quad X = TY \oplus N.$$

Note that N is finite dimensional. Let us write $i_N : N \hookrightarrow X$ for the inclusion mapping.

For $z \in D$, the operator $\alpha(z) : M \oplus \ker T \to TY \oplus N$ possesses a matrix representation of the form

$$\alpha(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

with suitable analytic operator-valued functions a, b, c, d. Since both the maps $a(0) \in \mathcal{L}(M, TY)$ and $(\alpha(0), i_N) \in \mathcal{L}(M \oplus N, X)$ are invertible, we can choose an open neighbourhood $V \subset D$ of 0 in such a way that $a(z) \in \mathcal{L}(M, TY)$ and $(\alpha(z), i_N) \in \mathcal{L}(M \oplus N, X)$ are invertible for every $z \in V$.

Fix an open subset $U \subset V$. For the holomorphic operator-valued function

$$u : U \to \mathcal{L}(\ker T, N), \ u(z) = d(z) - c(z)a(z)^{-1}b(z),$$

it was shown by Eschmeier [3] that the identity

$$u\mathcal{O}(U, \ker T) = \mathcal{O}(U, N) \cap \alpha\mathcal{O}(U, Y)$$

holds. Therefore the map

$$\varphi_U : \mathcal{O}(U, N)/u\mathcal{O}(U, \ker T) \ \to \ \mathcal{O}(U, X)/\alpha\mathcal{O}(U, Y), \quad [f] \mapsto [f]$$

is a well-defined one-to-one linear map which is continuous if both sides are equipped with their canonical quotient topologies.

To show that φ_U is in fact a homoeomorphism, we construct a continuous right inverse. For this purpose, we first observe that by construction the operators

$$r(z) = \big(r_Y(z), r_N(z)\big) : X \overset{(\alpha(z), i_N)^{-1}}{\longrightarrow} M \oplus N \hookrightarrow Y \oplus N$$

depend analytically on $z \in V$ and satisfy

$$\alpha(z) r_Y(z) x + r_N(z) x = x \quad (x \in X, \ z \in V).$$

Thus for $f \in \mathcal{O}(U, X)$ we have

$$f = \alpha r_Y f + r_N f.$$

In particular we find that

$$r_N \ \alpha\mathcal{O}(U, Y) \subset \mathcal{O}(U, N) \cap \alpha\mathcal{O}(U, Y) = u \ \mathcal{O}(U, \ker T).$$

Hence we obtain a well-defined continuous linear mapping

$$\Psi_U : \mathcal{O}(U, X)/\alpha\mathcal{O}(U, Y) \ \to \ \mathcal{O}(U, N)/u\mathcal{O}(U, \ker T), \ [f] \mapsto [r_N f].$$

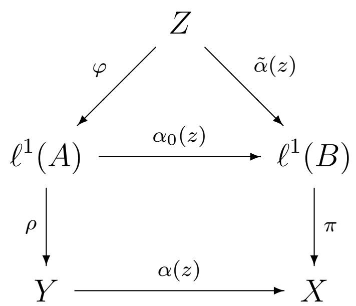
Since $f - r_N f = \alpha r_Y f \in \alpha\mathcal{O}(U,Y)$ for all $f \in \mathcal{O}(U,X)$, it follows that Ψ_U is a right inverse for φ_U. Hence both maps are topological isomorphisms.
To complete the proof it suffices to observe that by Lemma 1.1 the range space of Ψ_U is Hausdorff, whenever $U \subset V$ is a Stein open set.
Our next aim is to show that Lemma 1.2 remains true without the condition that $\ker \alpha(0) \subset Y$ is continuously projected. To prove this we use an idea due to Kaballo [8].

Theorem 1.3 *(Closed-Range Theorem, local version) Suppose that* $0 \in D$ *and that* $\dim(X/\alpha(0)Y) < \infty$. *Then there is an open set* $V \subset \mathbb{C}^n$ *with* $0 \in V \subset D$ *such that*

$$\alpha\mathcal{O}(U,Y) \subset \mathcal{O}(U,X)$$

is closed for every Stein open subset $U \subset V$.

Proof. After shrinking D if necessary, we may suppose that $\dim(X/\alpha(z)Y) < \infty$ for all $z \in D$. As shown by Kaballo [8] (see 1.1, 1.2 and 1.3) there is a diagram



with surjective continuous linear operators π, ρ, φ and holomorphic functions

$$\tilde{\alpha} : D \to \mathcal{L}(Z, \ell^1(B)) \quad \text{and} \quad \alpha_0 : D \to \mathcal{L}(\ell^1(A), \ell^1(B))$$

such that $\ker \tilde{\alpha}(0)$ is continuously projected, $\dim(\ell^1(B)/\tilde{\alpha}(z)Z) < \infty$ for every $z \in D$ and such that the intertwining relations

$$\pi \circ \alpha_0(z) = \alpha(z) \circ \rho \quad \text{and} \quad \pi \circ \tilde{\alpha}(z) = \alpha(z) \circ \rho \circ \varphi$$

hold for all $z \in D$. More explicitly (see also [8]) the space Z can be chosen as the direct sum

$$Z = \ell^1(A) \oplus \ker \pi$$

and the mappings φ and $\tilde{\alpha}(z)$ can be defined by $\varphi(x,y) = x$ and $\tilde{\alpha}(z)(x,y) = \alpha_0(z)x + y$.
We consider now an arbitrary open set $U \subset D$. Then

$$\pi_U : \mathcal{O}(U, \ell^1(B)) \to \mathcal{O}(U,X), \; f \mapsto \pi f$$

is a surjective continuous linear operator between Fréchet spaces (see e.g. Appendix 1 in [4]). We claim that

$$\overset{-1}{\pi_U}\left(\alpha\mathcal{O}(U,Y)\right) = \tilde{\alpha}\mathcal{O}(U,Z).$$

To check this, note first that $\pi\tilde{\alpha}f = \alpha\,(\rho\varphi f)$ for $f \in \mathcal{O}(U,Z)$. To prove the opposite inclusion, consider a function $f \in \mathcal{O}(U,\ell^1(B))$ such that $\pi_U(f) = \alpha g$ for some $g \in \mathcal{O}(U,Y)$. Using the surjectivity of ρ we can choose a function $h \in \mathcal{O}(U,\ell^1(A))$ with $g = \rho h$. It follows that

$$\pi f = \alpha\rho h = \pi\alpha_0 h$$

and hence that

$$f = \alpha_0 h + (f - \alpha_0 h) = \tilde{\alpha}\,(h, f - \alpha_0 h) \in \tilde{\alpha}\mathcal{O}(U,Z).$$

The formula just proved implies that the map π_U induces a well-defined linear isomorphism

$$\hat{\pi_U} : \mathcal{O}(U,\ell^1(B))/\tilde{\alpha}\mathcal{O}(U,Z) \;\longrightarrow\; \mathcal{O}(U,X)/\alpha\mathcal{O}(U,Y), \quad [f] \mapsto [\pi_u f]\,.$$

Obviously this map is continuous if both sides are equipped with their quotient topologies. We will show that also the inverse of $\hat{\pi_U}$ is continuous. Since $\pi_U : \mathcal{O}(U,\ell^1(B)) \to \mathcal{O}(U,X)$ is a surjective continuous linear operator between Fréchet spaces, there exists a continuous, not necessarily linear, right inverse $r_U : \mathcal{O}(U,X) \to \mathcal{O}(U,\ell^1(B))$ for π_U (Satz 7.1 in [11]). For $f, g \in \mathcal{O}(U,X)$ with $f - g \in \alpha\mathcal{O}(U,Y)$, we have

$$\pi_U\,(r_U f - r_U g) = f - g \in \alpha\mathcal{O}(U,Y).$$

But then $r_U f - r_U g \in \tilde{\alpha}\mathcal{O}(U,Z)$. Hence r_U induces a well-defined continuous mapping

$$\hat{r_U} : \mathcal{O}(U,X)/\alpha\mathcal{O}(U,Y) \to \mathcal{O}(U,\ell^1(B))/\tilde{\alpha}\mathcal{O}(U,Z),\ [f] \mapsto [r_U f].$$

By construction $\hat{r_U}$ is the inverse of $\hat{\pi_U}$ (in particular, $\hat{r_U}$ is linear) and hence $\hat{\pi_U}$ is a topological isomorphism.
Since by Lemma 1.2 there is an open neighbourhood $V \subset D$ of 0 such that

$$\tilde{\alpha}\mathcal{O}(U,Z) \subset \mathcal{O}(U,\ell^1(B))$$

is closed for every Stein open subset $U \subset V$, the assertion follows.
Let X, Y be Banach spaces and let $\alpha \in \mathcal{O}\left(D, \mathcal{L}(Y,X)\right)$ be a holomorphic map on an open neighbourhood D of 0 in $\mathbb{C}^n$ such that $\dim\left(X/\alpha(0)Y\right) < \infty$.

Corollary 1.4 *There exist an open neighbourhood $V \subset D$ of 0, a Banach space E, a finite-dimensional subspace $N \subset X$ and a holomorphic map $u : V \to \mathcal{L}(E, N)$ such that*

$$\rho_U : \mathcal{O}(U, N)/u\mathcal{O}(U, E) \to \mathcal{O}(U, X)/\alpha\mathcal{O}(U, Y),\ [f] \mapsto [f]$$

are well-defined topological isomorphisms for each open subset $U \subset V$.

Proof. We assume that $\dim(X/\alpha(z)Y) < \infty$ for all $z \in D$ and keep the notation from the proof of Theorem 1.3. As shown there, for each open subset $U \subset D$, the mapping

$$\hat{\pi_U} : \mathcal{O}(U, \ell^1(B))/\tilde{\alpha}\mathcal{O}(U, Z) \longrightarrow \mathcal{O}(U, X)/\alpha\mathcal{O}(U, Y),\ [f] \mapsto [\pi_U f]$$

is a well-defined topological isomorphism. By the proof of Lemma 1.2 there are an open neighbourhood $V \subset D$ of 0, a Banach space E, a finite-dimensional subspace $N \subset \ell^1(B)$ and a holomorphic mapping $u : V \to \mathcal{L}(E, N)$ such that, for every open subset $U \subset V$, the mapping

$$\varphi_U : \mathcal{O}(U, N)/u\mathcal{O}(U, E) \to \mathcal{O}(U, \ell^1(B))/\tilde{\alpha}\mathcal{O}(U, Z), \quad [f] \mapsto [f]$$

is a well-defined topological isomorphism. Note that

$$\sigma_U : \mathcal{O}(U, N)/u\mathcal{O}(U, E) \to \mathcal{O}(U, \pi N)/(\pi u)\mathcal{O}(U, E),\ [f] \mapsto [\pi f]$$

and

$$\rho_U : \mathcal{O}(U, \pi N)/(\pi u)\mathcal{O}(U, E) \to \mathcal{O}(U, X)/\alpha\mathcal{O}(U, Y),\ [f] \mapsto [f]$$

are well-defined continuous linear maps with $\rho_U \circ \sigma_U = \hat{\pi_U} \circ \varphi_U$ for every open subset $U \subset V$. Since the maps σ_U are surjective, it follows easily, that the maps ρ_U are topological isomorphisms.
In [3] a version of Corollary 1.4 was used to show that on the open subset

$$D_0 = \{z \in D;\ \dim(X/\alpha(z)Y) < \infty\} \subset D$$

the quotient sheaf $\mathcal{O}^X_{D_0}/\alpha\mathcal{O}^Y_{D_0}$ is coherent. Indeed it follows from Corollary 1.4 that locally on D_0 there are sheaf isomorphisms of the form

$$\rho^V : \mathcal{O}^N_V/u\mathcal{O}^E_V \xrightarrow{\sim} \mathcal{O}^X_V/\alpha\mathcal{O}^Y_V,\ [(f, U)_z] \mapsto [(f, U)_z],$$

where $N \subset X$ is a finite-dimensional subspace, E is a suitable Banach space and $u : V \to \mathcal{L}(E, N)$ is an analytic map. Since by a result of Markoe (see [12], Proposition 5) the quotient sheaf on the left is coherent and since

coherence is a local property, the coherence of the analytic sheaf $\mathcal{O}_{D_0}^X/\alpha\mathcal{O}_{D_0}^Y$ follows.
It is well known (cf. Section 4.1 in [4]) that the section spaces of a coherent analytic sheaf carry a canonical nuclear Fréchet space topology which, for the sheaf $\mathcal{O}_U$, is given by the identification $\Gamma(U, \mathcal{O}_U) \cong \mathcal{O}(U)$. Morphisms of coherent sheaves induce continuous linear maps between section spaces. Hence, in the situation explained above, we obtain induced topological isomorphisms

$$\Gamma\left(U,\ \mathcal{O}_V^N/u\mathcal{O}_V^E\right) \xrightarrow{\sim} \Gamma\left(U,\ \mathcal{O}_V^X/\alpha\mathcal{O}_V^Y\right) \quad (U \subset V \text{ open})$$

between nuclear Fréchet spaces. Furthermore the quotient map $\mathcal{O}_V^N \to \mathcal{O}_V^N/u\mathcal{O}_V^E$ induces continuous linear maps

$$\mathcal{O}(U,N)/u\mathcal{O}(U,E) \longrightarrow \Gamma\left(U, \mathcal{O}_V^N/u\mathcal{O}_V^E\right) \quad (U \subset V \text{ open}).$$

Using the commutativity of the diagrams

$$\begin{array}{ccccc} \Gamma\left(U, \mathcal{O}_V^X\right) & \xrightarrow{\sim} & \mathcal{O}(U,X) & \longrightarrow & \mathcal{O}(U,X)/\alpha\mathcal{O}(U,Y) \\ \downarrow & & & & \downarrow (\rho_U)^{-1} \\ \Gamma\left(U, \mathcal{O}_V^X/\alpha\mathcal{O}_V^Y\right) & \xleftarrow{\sim} & \Gamma\left(U, \mathcal{O}_V^N/u\mathcal{O}_V^E\right) & \longleftarrow & \mathcal{O}(U,N)/u\mathcal{O}(U,E) \end{array}$$

we deduce that, for $U \subset V$ open, the section map $\Gamma\left(U, \mathcal{O}_V^X\right) \xrightarrow{q_U} \Gamma\left(U, \mathcal{O}_V^X/\alpha\mathcal{O}_V^Y\right)$ is continuous.
From now on let us make the assumption that $\dim(X/\alpha(z)Y) < \infty$ for all $z \in D$. By the above cited result of Markoe [12] the image sheaf $\alpha\mathcal{O}_D^Y$ is coherent when X is finite dimensional. More generally, for an arbitrary Banach space X the image sheaf $\alpha\mathcal{O}_D^Y$ is still Banach coherent.

Corollary 1.5 *Let $\alpha \in \mathcal{O}(D, \mathcal{L}(Y,X))$ be a holomorphic operator-valued function on an open set $D \subset \mathbb{C}^n$ such that $\dim(X/\alpha(z)Y) < \infty$ for all $z \in D$. Then the quotient map*

$$\mathcal{O}_D^X \xrightarrow{q} \mathcal{O}_D^X/\alpha\mathcal{O}_D^Y$$

is a continuous morphism between Banach coherent analytic Fréchet sheaves. In particular, the image sheaf $\alpha\mathcal{O}_D^Y$ is a Banach coherent analytic Fréchet sheaf.

Proof. Let $U \subset D$ be an arbitrary open subset. By the remarks following Corollary 1.4 we can choose an open cover $U = \bigcup_{k \in \mathbb{N}} U_k$ such that all the section maps

$$\Gamma\left(U_k, \mathcal{O}_D^X\right) \xrightarrow{q_{U_k}} \Gamma\left(U_k, \mathcal{O}_D^X/\alpha\mathcal{O}_D^Y\right) \quad (k \in \mathbb{N})$$

are continuous. A straightforward application of the closed graph theorem for Fréchet spaces shows that also the section map

$$\Gamma\left(U, \mathcal{O}_D^X\right) \xrightarrow{q_U} \Gamma\left(U, \mathcal{O}_D^X / \alpha \mathcal{O}_D^Y\right)$$

is continuous. Thus we have shown that $q : \mathcal{O}_D^X \longrightarrow \mathcal{O}_D^X / \alpha \mathcal{O}_D^Y$ is a continuous morphism between analytic Fréchet sheaves. For each open set $U \subset D$, the space

$$\Gamma\left(U, \alpha \mathcal{O}_D^Y\right) = \ker q_U$$

is a Fréchet space as a closed subspace of the Fréchet space $\Gamma\left(U, \mathcal{O}_D^X\right)$. Therefore the image sheaf $\alpha \mathcal{O}_D^Y$ is an analytic Fréchet sheaf and

$$0 \longrightarrow \alpha \mathcal{O}_D^Y \hookrightarrow \mathcal{O}_D^X \xrightarrow{q} \mathcal{O}_D^X / \alpha \mathcal{O}_D^Y \longrightarrow 0$$

is an exact sequence of continuous morphisms between analytic Fréchet sheaves. Since the second and the third sheaf in this sequence are Banach coherent, by the result of Leiterer (Proposition 4.5.7 in [4]) used before, the same is true for the first sheaf. This observation completes the proof.

Since Banach coherent analytic sheaves are acyclic on Stein open sets, Corollary 1.5 allows us to deduce our main result by a repetition of the arguments that lead to a proof in the case of finite-dimensional image spaces.

Theorem 1.6 *(Closed-Range Theorem) Let X, Y be Banach spaces. Suppose that $\alpha \in \mathcal{O}\left(D, \mathcal{L}(Y, X)\right)$ is an analytic operator-valued map on an open set $D \subset \mathbb{C}^n$ such that $\dim\left(X/\alpha(z)Y\right) < \infty$ for all $z \in D$. Then*

$$\alpha \mathcal{O}(U, Y) \subset \mathcal{O}(U, X)$$

is closed for every Stein open subset $U \subset D$.

Proof. By Corollary 1.5 the second and the third sheaf in the exact sequence of analytic Fréchet sheaves

$$0 \longrightarrow \ker \alpha \hookrightarrow \mathcal{O}_D^Y \xrightarrow{\alpha} \alpha \mathcal{O}_D^Y \longrightarrow 0$$

are Banach coherent. The result of Leiterer cited above (Proposition 4.5.7 in [4]) implies that the same is true for the subsheaf $\ker \alpha \subset \mathcal{O}_D^Y$.

Since Banach coherent analytic sheaves are acyclic on Stein open subsets (Theorem 4.5.2 and Proposition 4.3.3 (b) in [4]), the induced sequence of section spaces

$$0 \longrightarrow \Gamma(U, \ker \alpha) \hookrightarrow \Gamma(U, \mathcal{O}_D^Y) \xrightarrow{\alpha} \Gamma(U, \alpha \mathcal{O}_D^Y) \longrightarrow 0$$

remains exact on each Stein open subset $U \subset D$. Hence

$$\alpha \mathcal{O}(U, Y) = \alpha \Gamma(U, \mathcal{O}_D^Y) = \Gamma(U, \alpha \mathcal{O}_D^Y) \subset \Gamma(U, \mathcal{O}_D^X) = \mathcal{O}(U, X)$$

is a closed subspace for each Stein open subset $U \subset D$.

2 An application

Let $T = (T_1, \dots, T_n) \in \mathcal{L}(X)^n$ be a commuting tuple of bounded linear operators on a complex Banach space X such that $\dim\left(X/\sum_{i=1}^n T_i X\right) < \infty$. Then there is an open neigbourhood D of the origin $z = 0 \in \mathbb{C}^n$ such that

$$\dim\left(X/\sum_{i=1}^{n}(z_i - T_i)X\right) < \infty \quad (z \in D),$$

and Theorem 1.6 applied to the analytic operator-valued map

$$\alpha_T : D \to \mathcal{L}(X^n, X),\ \alpha_T(z) = (z_1 - T_1, \dots, z_n - T_n)$$

shows that $\sum_{i=1}^n (z_i - T_i)\mathcal{O}(U, X) \subset \mathcal{O}(U, X)$ is closed for each Stein open set $U \subset D$. Hence the spectral subspaces

$$X_T\left(\mathbb{C}^n \setminus U\right) = X \cap \sum_{i=1}^{n}(z_i - T_i)\mathcal{O}(U, X) \subset X$$

are closed for every Stein open set $U \subset D$. In particular, the space

$$X_\infty = \bigcup_{U \in \mathcal{U}(0) \text{open}} X_T\left(\mathbb{C}^n \setminus U\right) = \bigcup_{k=1}^{\infty} X_T\left(\mathbb{C}^n \setminus B_{\frac{1}{k}}(0)\right) \subset X$$

is a countable union of closed linear subspaces of X. By Corollary 1.2 in [3], the quotient sheaf $\mathcal{H}_T = \mathcal{O}_D^X/\alpha_T \mathcal{O}_D^{X^n}$ is a coherent analytic sheaf. Hence its stalk at $z = 0$ is a finitely generated module over the local Noetherian ring $\mathcal{O}_0$. The spaces $M_k = \sum_{|\alpha|=k} T^\alpha X$ $(k \in \mathbb{N})$ form a decreasing sequence of finite-codimensional closed subspaces of X. Following the notation used in the one-variable case, we write $\mathrm{R}^\infty(T) = \bigcap_{k \geq 1} M_k$. Let $\mathfrak{m} \subset \mathcal{O}_0$ be the maximal ideal of $\mathcal{O}_0$. Then in [5], Lemma 3.22 (see also [6], Section 1.3 for the Hilbert-space case), it was shown that the canonical map $\Phi : X \to \mathcal{H}_0,\ x \mapsto [x]$, induces vector-space isomorphisms

$$\Phi_k : X/M_k \to \mathcal{H}_0/\mathfrak{m}^k\mathcal{H}_0,\ x + M_k \mapsto [x] + \mathfrak{m}^k\mathcal{H}_0 \quad (k \geq 1).$$

By Krull's intersection theorem (cf. [15], Theorem 4.19), applied to the Noetherian $\mathcal{O}_0$-module $\mathcal{H}_0$, the right vertical map in the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & \mathcal{H}_0 \\ \downarrow q_X & & \downarrow q_{\mathcal{H}} \\ \prod_{k=1}^{\infty} X/M_k & \xrightarrow{(\Phi_k)_{k\geq 1}} & \prod_{k=1}^{\infty} \mathcal{H}_0/\mathfrak{m}^k\mathcal{H}_0, \end{array}$$

is injective. Here the vertical maps are defined by $q_X(x) = (x + M_k)_{k\geq 1}$ and $q_{\mathcal{H}}(h) = \left(h + \mathfrak{m}^k \mathcal{H}_0\right)_{k\geq 1}$. Since the lower horizontal map is an isomorphism, we obtain that

$$X_\infty = \text{Ker } \Phi = \text{Ker } q_X = \text{R}^\infty(T)$$

is a closed subspace of X. By Baire's category theorem there is a natural number $k \geq 1$ such that $\text{R}^\infty(T) = X_T\left(\mathbb{C}^n \setminus B_{\frac{1}{k}}(0)\right)$. Thus we have obtained the following result.

Theorem 2.1 *Let $T = (T_1, \dots, T_n) \in \mathcal{L}(X)^n$ be a commuting tuple of bounded linear operators on a complex Banach space X such that* $\dim\left(X/\sum_{i=1}^n T_i X\right) < \infty$. *Then there is an open neigbourhood D of the origin $z = 0 \in \mathbb{C}^n$ such that*

$$\text{R}^\infty(T) = X_T\left(\mathbb{C}^n \setminus U\right) = \left\{x \in X;\; x \in \sum_{i=1}^n (z_i - T_i)\mathcal{O}_0^X\right\}.$$

For a single operator $T \in \mathcal{L}(X)$ with $\dim X/TX < \infty$, the preceding representation of the space $\text{R}^\infty(T) = \bigcap_{k=1}^\infty T^k X$ is well known (see for instance [9], Proposition 3.7.2). As an elementary application one obtains that the restriction of T to the invariant subspace $\text{R}^\infty(T) =$ is surjective. In [6] X. Fang asked whether, more generally, in the setting of Theorem 2.1 the identity $\text{R}^\infty(T) = \sum_{i=1}^n T_i \text{R}^\infty(T)$ holds. Since in the multivariable case there are examples of commuting tuples $T \in \mathcal{L}(X)^n$ which possess closed spectral subspaces $X_T(F)$ such that the surjectivity spectrum of $T|X_T(F)$ is not contained in F (see [1]), this question remains open at this time.

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