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Abstract

We discuss partial regularity results concerning local minimizers $u : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^3$ of variational integrals of the form

$$\int_\Omega \left\{ a(|\epsilon^D(w)|) + b(|\operatorname{div}(w)|) \right\} dx,$$

where a and b are N-functions of rather general type. We prove partial regularity results under quite natural conditions between a and b. Furthermore we can extend this to the non-autonomous situation which finally leads to the study of minimizers of the functional

$$\int_\Omega \left\{ (1 + |\epsilon^D(w)|^2)^{\frac{p(x)}{2}} + (1 + |\operatorname{div}(w)|^2)^{\frac{q(x)}{2}} \right\} dx,$$

where p and q are Lipschitz-functions.

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set describing an elastic body on which the displacement $u : \Omega \to \mathbb{R}^3$ is defined. In the case of linear elasticity the elastic energy of the deformation is defined by

$$J_1[u] = \int_\Omega \left[\frac{1}{2}\lambda(\operatorname{div}(u))^2 + \kappa|\varepsilon(u)|^2 \right], \tag{1.1}$$

where $\lambda, \kappa > 0$ denote physical constants and $\varepsilon(u) = \frac{1}{2}\left(\varepsilon(u) + \varepsilon(u)^T\right)$ is the symmetric gradient of u. Minimizing the functional given in (1.1) leads to a linear elliptic system which has solutions of class C^∞ (compare, e.g., [FS]). In order to model a nonlinear material behaviour, in particular the nonlinear Hencky material (see [Ze]), J_1 is replaced by

$$J_2[u] = \int_\Omega \left[\frac{1}{2}\lambda(\operatorname{div}(u))^2 + \varphi(|\varepsilon^D(u)|) \right] \tag{1.2}$$

for a suitable function φ. Here $\varepsilon^D(u) = \varepsilon(u) - \frac{1}{3}\operatorname{div}(u)I$ denotes the deviatoric part of the symmetric gradient. In the simplest case one assumes power growth conditions in the sense of $(F(\varepsilon) = \varphi(|\varepsilon|))$

$$\Lambda_1(1 + |\epsilon|^2)^{\frac{p-2}{2}} |\sigma|^2 \le D^2F(\epsilon)(\sigma, \sigma) \le \Lambda_2(1 + |\epsilon|^2)^{\frac{p-2}{2}} |\sigma|^2$$

for all $\epsilon, \sigma \in \mathbb{S}$ with positive constants Λ_1, Λ_2 and for an exponent $p > 1$. Here $\mathbb{S}$ denotes the space of symmetric 3×3-matrices. In case $p \in (1, 2]$ partial regularity is obtained in [Se1] and [Se2], whereas full regularity in the 2D case is a consequence of the work of Frehse/Seregin [FrS] with a little modification. Since the model above is used as an approximation for plasticity, the density usually is of nearly linear growth which means $\varphi(t) = t \ln(1+t)$ or $\varphi(t) = (1+t^2)^{\frac{p}{2}}$ for some $p > 1$ close to 1 (compare [Ka], [Kl] and [NH]).
In [BF1] the superquadratic case is studied for the first time with the result, that minimizers in two dimensions are of class $C^{1,\alpha}$, provided

$$p < 4. \tag{1.3}$$

This result is generalized in [BF2], where the authors consider functionals of the type

$$J_3[u] = \int_\Omega \left[a(|\operatorname{div}(u)|) + b(|\varepsilon^D(u)|) \right] \, dx, \tag{1.4}$$

where a and b are N-functions (see [Ad] for a definition) with superquadratic growth. In the 3D situation, according to our knowledge, no regularity results for the functional J_3 are available which motivates the studies in this work.
To be precise, we assume for $h \in \{a, b\}$ that $h : [0, \infty) \to [0, \infty)$ is a C^2-function satisfying:

$$\begin{gathered} h \text{ is strictly increasing and convex with} \\ \lim_{t \to 0} \frac{h(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{h(t)}{t} = \infty. \end{gathered} \tag{A1}$$

Furthermore we assume the existence of a positive number $\widehat{h}$ such that we have for all $t \geq 0$

$$\frac{h'(t)}{t} \leq h''(t) \leq \widehat{h} \, \frac{h'(t)}{t}. \tag{A2}$$

We further require that our problem is non-degenerate at the origin, i.e.

$$h''(0) > 0. \tag{A3}$$

A discussion of property (A2), as well as examples for functions which satisfy the conditions above, can be found in [BF3]. Finally we suppose the existence of a $c > 0$ such that

$$a(t) \leq c b(t) \text{ for large } t. \tag{A4}$$

Note that quite similar conditions are used in [BF2], [Fu1], [Fu2] and [BrF]. Let us give some comments on (A1-4):

i) We have $h(0) = h'(0)$, and by convexity h' is an increasing function with $h'(t) > 0$ for all $t > 0$.

ii) The inequality $\frac{h'(t)}{t} \leq h''(t)$ implies that the function $t \mapsto \frac{h'(t)}{t}$ is increasing, moreover we deduce the lower bound

$$h(t) \geq \frac{1}{2} h''(0) t^2, \; t \geq 0 \,. \tag{1.5}$$

(A1) shows that h is a N-function in the sense of Adams [Ad, Section 8.2].

iii) From (A2) it is to deduce that h and h' satisfy global ($\Delta 2$)-conditions (compare [BF3], Lemma A.1), and it is easy to see that

$$h(t) \leq c(t^q + 1) \quad (t \geq 0) \tag{1.6}$$

for a suitable exponent $q \geq 2$ and a constant c. Therefore the convexity of h implies that $h'(t)$ can be bounded in terms of t^{q-1}.

iv) From the (Δ_2)-condition of h and from the convexity of h we deduce the inequality

$$\overline{k}^{\,-1} h'(t)t \leq h(t) \leq t h'(t) \quad (t \geq 0) \,. \tag{1.7}$$

v) Let $F(\varepsilon) := a(|\operatorname{tr}(\varepsilon)|) + b(|\varepsilon^D|)$ then we conclude from (A2) the ellipticity condition

$$D^2 F(\varepsilon)(\tau, \tau) \approx \frac{a'(|\operatorname{tr}(\varepsilon)|)}{|\operatorname{tr}(\varepsilon)|} |\operatorname{tr} \tau|^2 + \frac{b'(|\varepsilon^D|)}{|\varepsilon^D|} |\tau^D|^2 \tag{1.8}$$

for all $\varepsilon, \tau \in \mathbb{S}$. Recalling iii) and using (see ii)) $\frac{h'(|Z|)}{|Z|} \geq h''(0)$, we get from (1.8)

$$\min \{a''(0), b''(0)\} \, |\tau|^2 \leq D^2 F(\varepsilon)(\tau, \tau) \leq C(1 + |\varepsilon|^2)^{\frac{q-2}{2}} \, |\tau|^2 \,, \tag{1.9}$$

and (1.9) means that F is of anisotropic $(2, q)$–growth.

vi) Condition (A4) can also stated in the form $b(t) \leq c a(t)$ for large t.

Finally we need an assumption limiting the range of anisotropy in terms of a and b. More precisely we suppose

$$b(t) \leq c t^{\omega} a(t) \quad \text{for large } t \text{ with } \omega \geq 0. \tag{A5}$$

Now we can state our main result in the autonomous setting:

THEOREM 1.1. *Let $u \in W^{1,a}_{loc}(\Omega)$ be a local minimizer of (1.4) under the assumptions (A1)-(A5) with $\omega < 4/3$. Then we have*

a) $\displaystyle \int_0^{|\operatorname{div}(u)|} \sqrt{\frac{a'(t)}{t}}\, dt, \; \int_0^{|\varepsilon^D(u)|} \sqrt{\frac{b'(t)}{t}}\, dt \in W^{1,2}_{loc}(\Omega);$

b) $a(|\operatorname{div}(u)|), \; b(|\varepsilon^D(u)|) \in L^3_{loc}(\Omega).$

THEOREM 1.2. *Let* $u \in W^{1,a}_{loc}(\Omega, \mathbb{R}^3)$ *be a local minimizer of (1.4) under the assumptions (A1)-(A5) with* $\omega < 4/3$. *Then there is an open subset* Ω_0 *of* Ω *with full Lebesgue measure such that* $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^3)$ *for any* $0 < \alpha < 1$.

An explicit description of the set Ω_0 is given after Lemma 3.1. Unfortunately we could not rule out the occurrence of singular points (for ∇u), but even if they exist, the solution itself is at least continuous. In fact, from Theorem 1.1 b) combined with (1.5) it follows that $|\varepsilon(u)| \in L^6_{loc}(\Omega)$ holds, and we deduce from Korn's inequality (see e.g. [FS] or [AM]) and Sobolev's embedding theorem

Corollary 1.1. *Under the assumptions of Theorem 1.1 and 1.2 any local minimizer of problem (1.4) is locally Hölder continuous with exponent 1/2.*

Remark 1.1.

- *A definition of the Orlicz-Sobolev space* $W^{1,a}(\Omega)$ *can be found in [Ad]. A solution* $u \in W^{1,a}(\Omega, \mathbb{R}^3)$ *of the global problem w.r.t. to boundary data in* $W^{1,a}(\Omega, \mathbb{R}^3)$ *can be generated as in [BrF] (Lemma 4.5) since* $a \le cb$.
- *It is easy to see that the result of Theorem 1.1 extends to higher dimensions, if we require* $\omega < 4/n$ *with integrability up to* $n/(n-2)$.
- *Our model clearly covers the functional in (1.2), where we obtain* $p < 2 + 4/3$ *which corresponds to (1.3) noting that we have* $n = 3$.
- *In [BrF] the first author and Fuchs consider minimizers of*

$$\int_\Omega h(|\varepsilon(u)|)\, dx$$

assuming $h''(t) \le c(1+t^2)^{\frac{\omega}{2}} \frac{h'(t)}{t}$ *(all other assumptions on h are the same). They prove partial regularity of local minimizers if* $\omega < 4/3$. *The r.h.s. is obviously much weaker than the one of (A2) but in our case the anisotropy is generated by (A5).*

- *A main tool for our approach is a Korn-type inequality in Orlicz spaces proved by Fuchs (see [Fu3]).*

Now we would like to consider the non-autonomous situation, i.e.

$$J_4[u] = \int_\Omega \left[a(x, |\operatorname{div}(u)|) + b(x, |\varepsilon^D(u)|)\right] dx. \tag{1.10}$$

Here we suppose the assumptions (A1)-(A5) uniformly in $x \in \Omega$. A first assumption to handle the x-dependence is requiring the natural condition ($h \in \{a, b\}$)

$$|\partial_\gamma h'(x,t)| \le c_2 h'(x,t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}^+_0 \tag{A6}$$

and all $\gamma \in \{1, ..., n\}$ with a constant $c_2 \ge 0$. Since the research of Esposito, Leonetti and Mingione [ELM] it is known that regularity results from the autonomous situation do not necessarily stay true if one allows an additonal x-dependece of the density (in case of anisotropic growth conditions). If we assume special structure conditions for the x-dependence the reslts should adjust (compare [Br1]). Therefore we suppose, following the ideas of [Br3], the existence of constants $\theta_1, \theta_2 \geq 0$ such that ($B \Subset \Omega$)

$$a(x,t) \leq \theta_1 \, t^{\theta_2 |x-y|} a(y,t) \text{ for all } t \gg 1 \text{ and all } x, y \in B \tag{A7}$$

as well as

$$\operatorname{argmin}_{y \in B} a(y,t) \text{ is independent of } t. \tag{A8}$$

Hence we obtain

THEOREM 1.3. *Let $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^3)$ be a local minimizer of (1.10) under the assumptions (A1)-(A5) with $\omega < 4/3$ uniformly in $x \in \Omega$ as well as (A6)-(A8). Then there is an open subset Ω_0 of Ω with full Lebesgue measure such that $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^3)$ for any $0 < \alpha < 1$.*

Remark 1.2.

- *Since the N-functions we are considering now depend on x we are not able to construct a solution in an Orlicz-space and so we work with a local $W^{1,2}$-minimizer which clearly exists.*
- *By (A7) and (A8) we are able to extend Theorem 1.1 and Theorem 1.2 to the case of x-dependent integrands. An easy way to obtain an example is*

  $$h(x,t) := \alpha(x) h(t),$$

  where h satisfies (A1)-(A4) and α is a strictly positive Lipschitz-function.
- *Since the standard regularization we use in the autonomous case does not converge, it is not trivial to extend the result from this situation. Another difficulty occurs in the blow-up procedure on account of our x-dependent excess function. Those are the same problems as in the classical variational setting of (1.3), see [Br2] and [Br3].*
- *Corollary 1.1 stays true in the fashion of Theorem 1.3.*

Finally we would like to consider minimizers of

$$J_5[u] = \int_\Omega \left\{ (1 + |\epsilon^D(w)|^2)^{\frac{p(x)}{2}} + (1 + |\operatorname{div}(w)|^2)^{\frac{q(x)}{2}} \right\} dx, \tag{1.2}$$

where p and q are Lipschitz-functions from $\Omega \to [2, \infty)$. It is easy to establish (A1)-(A3) as well as (A7) and (A8) for the functions

$$a(x,t) := (1 + t^2)^{\frac{p(x)}{2}} - 1 \quad \text{and} \quad b(x,t) := (1 + t^2)^{\frac{q(x)}{2}} - 1$$

but they do not fullfill (A6). Hence this energy is not covered by Theorem 1.3. In this case we obtain

THEOREM 1.4. *Let $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^3)$ be a local minimizer of (1.2) under the assumptions $p \le q$ and $\|p - q\|_\infty < 4/3$. Then there is an open subset Ω_0 of Ω with full Lebesgue measure such that $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^3)$ for any $0 < \alpha < 1$.*

Remark 1.3. *The differences between the proofs of Theorem 1.3 and Theorem 1.4 are exactly the same as the ones between the proofs in [Br3] and [Br4]. Therefore we only give a proof of Theorem 1.3.*

- *In order to obtain regularity results for minimizers of (1.2) in two dimensions we have to modify the proof of Theorem 1.3 in [BF2] in the same way as done in [Br4], Theorem 1.1 b), with the result full $C^{1,\alpha}$-regularity without any restriction between p and q.*

Remark 1.4.

- *Let us finally compare our results with the classical variational setting, which means*

$$J[u] = \int_\Omega \left[a(|\nabla_1 u|) + b(|\nabla_2 u|)\right]\, dx,$$

where $\nabla u = (\nabla_1 u, \nabla_2 u)$ is an arbitrary decomposition. In this case we have to suppose weaker $\omega < 2$ to obtain the same result (this is also true in case (1.10) and (1.2), see [Br2]-[Br4]). In our setting we do not have a maximum principle for minimizers, which is a crucial tool in in the papers [Br2]-[Br4]. Hence we need a completely different approach to obtain higher integrability.

- *A further problem is that we have to estimate ∇u by $\varepsilon(u)$ in terms of N-functions, which means we need Korn-type inequalities in Orlicz-spaces. In the blow-up procedure Lemma 4.7 from [BrF] is therefore a crucial tool whereas we use the Korn-inequality in $W^{1,h}_0$ from [Fu3] during the higher integrability proof.*

Our paper is organized as follows: In section 2 we prove Theorem 1.1, where we use the standard regularization working with Sobolev's inequaility. In section 3 we prove Theorem 1.2 via blow-up. For the proof of Theorem 1.3 in section 4 we work with a regularization which was intruduced in [BF2] and extends the techniques from the other sections for a x-dependence.

2 Higher integrability

In this section we prove Theorem 1.1. A first step is to approximate (1.4) locally by variational problems with sufficiently regular minimizers. Let

$$\delta := \delta(\rho) := \frac{1}{1 + \rho^{-1} + \|(\varepsilon((u)_\rho)\|^{2q}_{L^q(B)}},$$

$$H_\delta(\epsilon) := \delta\left(1 + |\epsilon|^2\right)^{\frac{q}{2}} + H(\epsilon)$$

for $\epsilon \in \mathbb{S}$ and for a small parameter $\rho > 0$. Here the parameter q is defined in (1.9) and $(u)_\rho$ denotes the mollification of u with radius ρ. For $B := B_{R_0}(x_0) \Subset \Omega$ we define u_δ as the unique minimizer of

$$\mathbb{J}_\delta \left[w, B\right] := \int_B H_\delta(\epsilon(w))dx \tag{2.1}$$

in $(u)_\rho + W^{1,q}_0(B, \mathbb{R}^3)$. Some elementary properties of u_δ are summarized in the following Lemma (see [BF1], Lemma 3.1, Lemma 4.1 and estimate (4.10), as well as the inequalities (12) and (13) from [Fu2] for part c). Note that our situation is easier since we do not have to work under the constraint $\operatorname{div} w = 0$):

Lemma 2.1. *Let the hypothesis of Theorem 1.1 hold. Then we have*

a) $u_\delta \in W^{2,2}_{loc}(B, \mathbb{R}^3)$,

b) $(1 + |\epsilon(u_\delta)|^2)^{\frac{q}{4}} \in W^{1,2}_{loc}(B)$,
and for all $\eta \in C^\infty_0(B)$, $Q \in \mathbb{R}^{3\times 3}$ *and* $\gamma \in \{1, ..., n\}$ *we obtain*

$$\int_B \eta^2 D^2 H_\delta(\epsilon(u_\delta))(\partial_\gamma \epsilon(u_\delta), \partial_\gamma \epsilon(u_\delta))\, dx$$
$$\leq c \int_B D^2 H_\delta(\epsilon(u_\delta))([\partial_\gamma u_\delta - Q] \odot \nabla\eta, [\partial_\gamma u_\delta - Q] \odot \nabla\eta)\, dx.$$

c) As $\rho \to 0$ *we have* $u_\delta \rightharpoondown u$ *in* $W^{1,2}(B, \mathbb{R}^3)$ *and*

$$\delta \int_B \left(1 + |\epsilon(u_\delta)|^2\right)^{\frac{q}{2}}\, dx \to 0.$$

d) The integrals $\int_B \left[a(|\operatorname{div}(u_\delta)|) + b(|\varepsilon^D(u_\delta)|)\right] dx$ *are bounded independent of* δ *and therefore the same is true for* $\int_B a(|\varepsilon(u_\delta)|)\, dx$ *on account of (A4).*

Furthermore we need the following statements, which can be proven exactly as Lemma 2.2 from [BrF]:

Lemma 2.2. *Under the assumptions of Theorem 1.1 it holds:*

a) u_δ *is uniformly bounded in* $W^{1,a}(B, \mathbb{R}^3)$.

b) The sequence $a(|u_\delta|)$ *is uniformly bounded in any space* $L^\chi(B)$, $\chi < 3$, *so that*

$$a(|u_\delta|)|u_\delta|^\mu \in L^1(B)$$

uniformly, provided $\mu < 4$.

After these preparations we start with the proof of Theorem 1.1 following the main ideas of [BrF], Theorem 1.1: In a first step we work with a cut-off function $\eta_1 \in C_0^\infty(B_{\widetilde{r}}(z))$ with $\eta_1 \equiv 1$ on $B_r(z)$, $0 \le \eta_1 \le 1$ and $|\nabla\eta_1| \le c/(\widetilde{r} - r)$, where $0 < r < R$ are such that $B_R(z) \Subset B$ and $\widetilde{r} := \frac{R+r}{2}$. We get by Sobolev's inequality

$$\begin{aligned}
&\int_{B_{r(z)}} a(|\operatorname{div}(u_\delta)|)^3\,dx \le \int_{B_{\widetilde{r}}(z)} \eta_1^6 a(|\operatorname{div}(u_\delta)|)^3\,dx\\
&\le c\left\{\int_{B_{\widetilde{r}}(z)} |\nabla\eta_1|^2 a(|\operatorname{div}(u_\delta)|)\,dx + \int_{B_{\widetilde{r}}(z)} \eta_1^2 \frac{[a'(|\operatorname{div}(u_\delta)|)]^2}{[a(|\operatorname{div}(u_\delta)|)]} |\nabla \operatorname{div}(u_\delta)|^2\,dx\right\}^3 .
\end{aligned}$$

Using (1.7), Lemma 2.1 d) and (A3) we obtain for a suitable positive number β (summation w.r.t. $\gamma \in \{1, 2, 3\}$)

$$\begin{aligned}
&\int_{B_{r(z)}} a(|\operatorname{div}(u_\delta)|)^3\,dx \le c(R-r)^{-\beta} + c\left\{\int_{B_{\widetilde{r}}(z)} \eta_1^2 \frac{a'(|\operatorname{div}(u_\delta)|)}{|\operatorname{div}(u_\delta|} |\nabla \operatorname{div}(u_\delta)|^2\,dx\right\}^3\\
&\le c(R-r)^{-\beta} + c\left\{\int_{B_{\widetilde{r}}(z)} \eta_1^2 D^2H(\varepsilon(u_\delta))(\partial_\gamma\varepsilon(u_\delta), \partial_\gamma\varepsilon(u_\delta))\,dx\right\}^3 .
\end{aligned}$$

A similar calculation shows the same inequality for $\int_{B_{r(z)}} b(|\varepsilon^D(u_\delta)|)^3\,dx$ and therefore

$$\begin{aligned}
&\int_{B_{r(z)}} a(|\operatorname{div}(u_\delta)|)^3\,dx + \int_{B_{r(z)}} b(|\varepsilon^D(u_\delta)|)^3\,dx\\
&\le c(R-r)^{-\beta} + c\left\{\int_{B_{\widetilde{r}}(z)} \eta_1^2 D^2H(\varepsilon(u_\delta))(\partial_\gamma\varepsilon(u_\delta), \partial_\gamma\varepsilon(u_\delta))\,dx\right\}^3 . \qquad (2.2)
\end{aligned}$$

In order to discuss the integral on the r.h.s. of (2.2) we apply the Caccioppoli-type inequality from Lemma 2.1 c): we have for all $\kappa > 0$

$$\begin{aligned}
&\int_{B_{\widetilde{r}}(z)} \eta_1^2 D^2H_\delta(\epsilon(u_\delta))(\partial_\gamma\epsilon(u_\delta), \partial_\gamma\epsilon(u_\delta))\,dx\\
&\quad\le c\int_{B_{\widetilde{r}}(z)} D^2H(\epsilon(u_\delta))(\partial_\gamma u_\delta \odot \nabla\eta_1, \partial_\gamma u_\delta \odot \nabla\eta_1)\,dx\\
&\quad + c\delta\int_{B_{\widetilde{r}}(z)} \left(1 + |\epsilon(u_\delta)|^2\right)^{\frac{q-2}{2}} |\nabla\eta_1|^2|\nabla u_\delta|^2\,dx. \qquad (2.3)
\end{aligned}$$

The δ-term can be estimated by $c(R-r)^{-2}$ using the same arguments as in [BrF]. The remaining term in (2.3) decomposes into

$$\begin{aligned}
&\int_{B_{\widetilde{r}}(z)} \frac{a'(|\operatorname{div}(u_\delta)|)}{|\operatorname{div}(u_\delta)|} |\nabla u_\delta|^2|\nabla\eta_1|^2\,dx\\
&+ \int_{B_{\widetilde{r}}(z)} \frac{b'(|\varepsilon^D(u_\delta)|)}{|\varepsilon^D(u_\delta)|} |\nabla u_\delta|^2|\nabla\eta_1|^2\,dx
\end{aligned}$$

on account of (1.8) which can be estimated by

$$c(R-r)^{-2}\left[\int_{B_{\tilde{r}}(z)} a(|\nabla u_\delta|)\,dx + \int_{B_{\tilde{r}}(z)} b(|\nabla u_\delta|)\,dx\right]$$
$$c(R-r)^{-2}\left[1+\int_{B_{\tilde{r}}(z)} a(|\nabla u_\delta|)|\nabla u_\delta|^\omega\,dx\right]$$

as a consequence of (A2), Lemma 2.2 a) and (A5). Following the lines of [BrF] gives us together with Lemma 2.2 b)

$$\int_{B_{\tilde{r}}(z)} a(|\nabla u_\delta|)|\nabla u_\delta|^\omega\,dx \le c(R-r)^{-\alpha}\left[1+\int_{B_R(z)} a(|\varepsilon(u_\delta)|)|\varepsilon(u_\delta)|^\omega\,dx\right]$$

for a positive exponent α. Combining this with (2.2) and (2.3) we arrive at (by enlarging β if necessary)

$$\begin{aligned}&\int_{B_r(z)} a(|\operatorname{div}(u_\delta)|)^3\,dx + \int_{B_r(z)} b(|\varepsilon^D(u_\delta)|)^3\,dx\\ &\le c(R-r)^{-\beta}\left[1+\left\{\int_{B_R(z)} a(|\epsilon(u_\delta)|)|\epsilon(u_\delta)|^\omega\,dx\right\}^3\right]. \qquad (2.4)\end{aligned}$$

In a final step we argue similarly to [BrF] to see ($\kappa > 0$ is arbitary)

$$\begin{aligned}&\int_{B_r(z)} a(|\operatorname{div}(u_\delta)|)^3\,dx + \int_{B_r(z)} b(|\varepsilon^D(u_\delta)|)^3\,dx\\ &\le c(\kappa)(R-r)^{-\nu} + \kappa\int_{B_R(z)} a(|\varepsilon(u_\delta)|)^3\,dx\end{aligned}$$

for a $\nu > 0$, where $\omega < 4/3$ is needed. On account of convexity of a, (Δ_2)-condition, (A4) and a suitable choice of κ we get

$$\begin{aligned}&\int_{B_r(z)} a(|\operatorname{div}(u_\delta)|)^3\,dx + \int_{B_r(z)} b(|\varepsilon^D(u_\delta)|)^3\,dx\\ &\le c(R-r)^{-\nu} + \frac{1}{2}\left[\int_{B_R(z)} a(|\operatorname{div}(u_\delta)|)^3\,dx + \int_{B_R(z)} b(|\varepsilon^D(u_\delta)|)^3\,dx\right]. \qquad (2.5)\end{aligned}$$

To inequality (2.5) we may apply Lemma 3.1, p. 161, of [Gi] in order to see that $a(|\operatorname{div}(u)|)^3$ and $b(|\varepsilon^D(u)|)$ are in the space $L^1_{loc}(\Omega)$ uniformly w.r.t. δ. This proves Theorem 1.1 a).
During our calculations we have shown that

$$D^2H(\epsilon(u_\delta))(\partial_\gamma\epsilon(u_\delta),\partial_\gamma\epsilon(u_\delta)) \;\in L^1_{loc}(B) \qquad (2.6)$$

holds uniformly w.r.t. the approximation parameter. From (1.5) and (1.9) in combination with (2.6) we deduce uniform $W^{2,2}_{loc}$-bounds on u_δ, hence for suitable subsequences it holds

$$u \in W^{2,2}_{loc}(\Omega,\mathbb{R}^3), \quad u_\delta \rightharpoondown u \text{ in } W^{2,2}_{loc}(B,\mathbb{R}^3),$$

$$\nabla u_\delta \to \nabla u \quad \text{a.e. on } B.$$

Moreover, we see that the functions

$$\psi_\delta^1 := \int_0^{|\operatorname{div}(u_\delta)|} \sqrt{\frac{a'(t)}{t}}\, dt, \quad \psi_\delta^2 := \int_0^{|\varepsilon^D(u_\delta)|} \sqrt{\frac{b'(t)}{t}}\, dt$$

are uniformly bounded in the space $W^{1,2}_{loc}(B)$, thus we have weak $W^{1,2}_{loc}(B)$-convergence of ψ_δ^1 and ψ_δ^2 with limits

$$\psi^1 := \int_0^{|\operatorname{div}(u)|} \sqrt{\frac{a'(t)}{t}}\, dt, \quad \psi^2 := \int_0^{|\varepsilon^D(u)|} \sqrt{\frac{b'(t)}{t}}\, dt,$$

which finally proves Theorem 1.1. Note that $D^2H(\epsilon(u_\delta))(\partial_\gamma\epsilon(u_\delta), \partial_\gamma\epsilon(u_\delta))$ is bounded from below by

$$\frac{a'(|\operatorname{div}(u)|)}{|\operatorname{div}(u)|}|\partial_\gamma \operatorname{div}(u)|^2 + \frac{b'(|\varepsilon^D(u)|)}{|\varepsilon^D(u)|}|\partial_\gamma\varepsilon^D(u)|^2$$

as a consequnce of the growth condition in (1.8) and we can sum up over $\gamma \in \{1,2,3\}$.□

Remark 2.1. *Returning to the Caccioppoli inequality stated in Lemma 2.1 - now with arbitrary matrix $Q \in \mathbb{R}^{3\times3}$ - it is easy to see that the appropriate variant of (2.3) after passing to the limit $\delta \to 0$ gives the inequality*

$$\int_B \eta^2 \left[|\nabla\psi^1|^2 + |\nabla\psi^2|^2\right]\, dx \le c\int_B |\nabla\eta|^2 |D^2H(\varepsilon(u))||\nabla u - Q|^2\, dx \tag{2.7}$$

valid for any $\eta \in C_0^\infty(B)$ and all $Q \in \mathbb{R}^{3\times3}$. Alternatively we may replace $|\nabla\psi^1|^2 + |\nabla\psi^2|^2$ by (or just $|\nabla\varepsilon(u)|^2$) in this inequality. The reader should note that the l.h.s. of the δ-version of (2.7) is treated via lower semicontinuity, whereas on the r.h.s. we use equi-integrability in order to pass to the limit $\delta \to 0$, we refer to [Br2] (section 2) for details in a related fashion.

3 Proof of Theorem 1.2

Now we prove the partial regularity theorem stated in Theorem 1.2, where we combine the arguements from [BrF] and [Br2]. Let u denote a local J_3-minimizer and suppose w.l.o.g. that $\omega \in [1, \frac{4}{3})$ in (A3). We further let

$$\widetilde{a}(t) := t^\omega a(t),\ t \ge 0\,,$$

and recall that $\widetilde{a}$ is a N-function. From Lemma 2.2 and Theorem 1.1 b) follows that u is an element of the space $\in W^1_{\widetilde{a},\mathrm{loc}}(\Omega;\mathbb{R}^3)$, hence the excess-function

$$E(x,r) := \fint_{B_r} |\epsilon(u) - (\epsilon(u))_{x,r}|^2\, dy + \fint_{B_r} \widetilde{a}(|\epsilon(u) - (\epsilon(u))_{x,r}|)\, dy$$

for balls $B_r(x) \Subset \Omega$ is well-defined. Here and in what follows $\fint f, (f)$ denote the mean value of a function f.

Lemma 3.1. *Fix $L > 0$ and a subdomain $\Omega' \Subset \Omega$. Then there is a constant $C_*(L)$ such that for every $\tau \in (0,1)$ one can find a number $\kappa = \kappa(L,\tau)$ with the following property: if $B_r(x) \subset \Omega'$ and if*

$$|(\epsilon(u))_{x,r}| \leq L,\ E(x,r) \leq \kappa\,, \tag{3.1}$$

then it holds

$$E(x,\tau r) \leq C_*(L)\tau^2 E(x,r)\,. \tag{3.2}$$

Once having established Lemma 3.1, it is standard (see, e.g. Giaquinta's textbook [Gi3]) to prove the desired partial regularity result. It turns out that the regular set Ω_0 is given by

$$\Omega_0 = \left\{ x \in \Omega : \sup_{r>0} |(\epsilon(u))_{x,r}| < \infty \text{ and } \liminf_{r\downarrow 0} E(x,r) = 0 \right\}\,,$$

i.e. Lemma 3.1 shows that the set on the r.h.s. is open and $\nabla u \in C^{0,\alpha}$ there for any $0 < \alpha < 1$. Obviously Ω_0 is a set of full Lebesgue measure.

Proof of Lemma 3.1: We argue by contradiction (compare [Fu1]). Let $L > 0$ and choose $C_* = C_*(L)$ as outlined below. Then, for some $\tau \in (0,1)$, there is a sequence of balls $B_{r_m}(x_m) \Subset \Omega'$ such that

$$\begin{aligned} |(\epsilon(u))_{x_m,r_m}| &\leq L,\ E(x_m,r_m) =: \lambda_m^2 \to 0,\ \text{as } m \to \infty\,, & (3.3)\\ E(x_m,\tau r_m) &> C_*\tau^2\lambda_m^2\,. & (3.4) \end{aligned}$$

Letting $A_m := (\epsilon(u))_{x_m,r_m}$ we define for $z \in B_1 := B_1(0)$

$$\begin{aligned} \widetilde{u}_m(z) &:= \frac{1}{\lambda_m r_m}\Big[u(x_m + r_m z) - r_m A_m z\Big], & (3.5)\\ u_m(z) &:= \widetilde{u}_m(z) - R_m(z), & (3.6) \end{aligned}$$

where R_m is the orthogonal projection of $\widetilde{u}_m$ into the space of rigid motions with respect to the $L^2(B,\mathbb{R}^3)$ inner product. We get from (3.3) using

$$\epsilon(u_m)(z) = \frac{1}{\lambda_m}\Big[\epsilon(u)(x_m + r_m(z)) - A_m\Big]$$

the relations

$$|A_m| \leq L,\ \fint_{B_1} |\epsilon(u_m)|^2\, dz + \lambda_m^{-2} \fint_{B_1} \widetilde{a}(\lambda_m|\epsilon(u_m)|)\, dz = 1\,. \tag{3.7}$$

On the other hand, (3.4) reads after scaling

$$\fint_{B_\tau} |\epsilon(u_m) - (\epsilon(u_m))_{0,\tau}|^2\, dz + \lambda_m^{-2} \fint_{B_\tau} \widetilde{a}(\lambda_m|\epsilon(u_m) - (\epsilon(u_m))_{0,\tau}|)\, dz > C_*\tau^2\,. \tag{3.8}$$

After passing to suitable subsequences we obtain from (3.7)

$$A_m \to: A,\ u_m \rightharpoondown: \overline{u} \quad \text{in} \quad W_2^1(B_1;\mathbb{R}^3)\,,$$

$$\lambda_m \epsilon(u_m) \to 0 \quad \text{in} \quad L^2(B_1;\mathbb{S}) \text{ and a.e.}\,, \tag{3.9}$$

where obviously $(\varepsilon(\overline{u}))_{0,1} = 0$. To prove the second convergence we need Korn's inequality (see for example [FS], Lemma 3.0.1 and 3.0.3 and in particular [AM], Proposition 2.6 (g) and Proposition 2.7 (c)) which gives by the choice of R_m

$$\|u_m\|_{W^{1,2}(B)} \le \|\epsilon(u_m)\|_{L^2(B)}\,.$$

If we argue as in [Br2] (note that our assumptions on a and b are a little bit stronger than the conditions supposed there and of course monotonicity of $a'(t)/t$ and $b'(t)/t$ simplifies the calculations), after (3.9), replacing ∇ by ε, $\widetilde{\widetilde{\nabla}}$ by div and ∂_n by ε^D we obtain the limit equation:

$$\int_{B_1} D^2 H(A)(\epsilon(\overline{u}), \epsilon(\varphi))\, dz = 0.$$

valid for any $\varphi \in C_0^\infty(B_1, \mathbb{R}^3)$. Quoting standard results on weak solutions of elliptic systems with constant coefficients involving the symmetric gradient (see, e.g., [GM] or [FS], Lemma 3.5, our situation is easier since we have no imcompressibility condition) will give a contradiction to (3.8) as soon as we can show

$$\epsilon(u_m) \to \varepsilon(\overline{u}) \text{ in } L^2_{\text{loc}}(B_1, \mathbb{S})\ , \tag{3.10}$$

$$\lambda_m^{-2} \fint_{B_r} \widetilde{a}\left(\lambda_m |\varepsilon(u_m)|\right) dz \to 0,\ r < 1. \tag{3.11}$$

For a detailed exposition of how to obtain the desired contradiction we refer to the comments given in "Step 2: Strong convergence of the scaled functions" in [Br2]. In order to prove (3.10) and (3.11) we return to (2.7) (with $|\nabla\varepsilon(u)|^2$ in place of $|\nabla\psi^1|^2 + |\nabla\psi^2|^2$ on the l.h.s.) and get after scaling and with appropriate choice of the testfunction η

$$\int_{B_t} |\nabla\epsilon(u_m)|^2\, dz \le C(s-t)^{-2} \int_{B_s} |D^2 H\left(\lambda_m \epsilon(u_m) + A_m\right)| |\nabla u_m|^2\, dz \tag{3.12}$$

valid for $0 < t < s < 1$. On $[\lambda_m|\epsilon(u_m)| \le K]$ we have

$$\left|D^2 H\left(A_m + \lambda_m \epsilon(u_m)\right)\right| |\nabla u_m|^2 \le c(K)|\nabla u_m|^2\,,$$

whereas on $[\lambda_m|\epsilon(u_m)| \ge K]$ it holds (K large enough)

$$\begin{aligned}
&\left|D^2 H\left(\lambda_m \epsilon(u_m) + A_m\right)\right| |\nabla u_m|^2 \\
\le\ & c(K)\left[1 + \frac{a'(\lambda_m|\operatorname{div}(u_m)|)}{\lambda_m|\operatorname{div}(u_m)|} + \frac{b'(\lambda_m|\varepsilon^D(u_m)|)}{\lambda_m|\varepsilon^D(u_m)|}\right] |\nabla u_m|^2 \\
\le\ & c(K)\left[|\nabla u_m|^2 + \lambda_m^{-2}\widetilde{a}\left(\lambda_m|\nabla u_m|\right)\right]\,.
\end{aligned}$$

Here we have used monotonicity of $a'(t)/t$ and $b'(t)/t$, (1.8), as well as $b(t) \le \widetilde{a}(t)$ for large t (compare (A5)). Therefore, (3.12) implies on account of $|\nabla^2 u_m| \le c|\nabla\varepsilon(u_m)|$

$$\int_{B_t} |\nabla^2 u_m|^2\, dz \le c(s-t)^{-2}\left[\int_{B_s} |\nabla u_m|^2\, dz + \lambda_m^{-2} \int_{B_s} \widetilde{h}\left(\lambda_m|\nabla u_m|\right)\, dz\right]. \tag{3.13}$$

If we follow the lines of [BrF] (compare the calculations after (3.13), here one has to know $\omega \geq 1$) we can bound the r.h.s. of (3.13) uniformly in m and therefore we obtain uniform L^2_{loc}-bounds on $\nabla^2 u_m$, which shows (3.10) after passing to a subsequence.
In order to prove our claim (3.11) we introduce the auxiliary functions

$$\Psi^1_m := \frac{1}{\lambda_m}\left\{\int_0^{|\lambda_m \operatorname{div}(u_m)+\operatorname{tr}(A_m)|}\sqrt{\frac{a'(t)}{t}}\,dt - \int_0^{|\operatorname{tr}(A_m)|}\sqrt{\frac{a'(t)}{t}}\,dt\right\},$$
$$\Psi^2_m := \frac{1}{\lambda_m}\left\{\int_0^{|\lambda_m\varepsilon^D(u_m)+A^D_m|}\sqrt{\frac{b'(t)}{t}}\,dt - \int_0^{|A^D_m|}\sqrt{\frac{b'(t)}{t}}\,dt\right\}.$$

After scaling we can follow from (2.7) ($0 < t < 1$)

$$\int_{B_t}\left[|\nabla\Psi^1_m|^2 + |\nabla\Psi^1_m|^2\right]\,dz \leq c(t) \tag{3.14}$$

since the r.h.s. of (3.13) is an upper bound. Following the lines of [Br2] (after (3.22)) (replacing $\widetilde{\nabla}$ by div and ∂_n by ε^D) we easily obtain L^2_{loc}-bounds on Ψ^1_m and Ψ^2_m and therefore together with (3.14) it is shown that

$$\|\Psi^1_m\|_{W^1_2(B_t)}, \|\Psi^2_m\|_{W^1_2(B_t)} \leq c(t) < \infty\,, 0 < t < 1. \tag{3.15}$$

With (3.15) we can repeat exactly the arguments presented after (3.17) in the paper [Br2] ending up with (3.11). Note that the condition

$$a(t) \geq t^{\frac{\omega}{2}(n-2)} \quad (t \gg 1)$$

required in [Br2] is clearly satisfied in our context as a consequence of the superquadratic growth of h and the hypothesis $\omega < 4/3$. This completes the proof of Lemma 3.1. □

4 Proof of Theorem 1.3

Now we assume that the function $h = h(x,t)$ satisfies (A1)-(A5) uniformly in $x \in \Omega$. Furthermore we suppose (A6)-(A8) to handle the x-dependence. The first step is to approximate the variational problem (1.10) by a sequence of problems with sufficient regular minimizers. The standard regularitzation which we have used in section 2 does not converge in case of x-dependence on account of the anisotropic behaviour of the two parts in the decomposition of the densitiy H. This problem was firstly discussed in [ELM], note that if a and b behave like powers there is no problem (see [BF4], Remark 3 b) and [BF5], proof of Lemma 2.1, those arguments work in our setting, too). In [BF2] the authors develop a regularization function h_M ($M \gg 1$) to approximate h (we clearly need a function a_M to approximate a and b_M to approximate b) with the following poperties (a proof is given in [Br3] and [BF2], for the (∇_2)-condition in part c) have a look at (5.1))

Lemma 4.1. *Suppose that the functions a and b satisfy (A1)-(A6) uniformly in $x \in \Omega$. Then we have*

a) $h_M \in C^2(\Omega \times [0,\infty))$, $h_M(x,t) = h(x,t)$ *for all* $t \leq 3M/2$ *and*

$$\lim_{M\to\infty} h_M(x,t) = h(x,t) \quad \textit{for all } (x,t) \in \Omega \times \mathbb{R}_0^+$$

as well as $h_M \leq h$.

b) The regularization functions a_M *and* b_M *satisfy (A1)-(A6) uniformly in* $x \in \Omega$ *and uniformly in* M.

c) a_M *and* b_M *satisfy uniformly* (Δ_2)*- and* (∇_2)*-conditions.*

d) We have for $h \in \{a, b\}$ *and a positive* μ *independent of* M *uniformly in* $x \in \Omega$

$$\mu h_M'(\cdot,t)t \leq h_M(\cdot,t) \leq h_M'(\cdot,t)t \quad \textit{for all } t \geq 0.$$

e) We have with $H_M(x,\varepsilon) := a_M(x, |\operatorname{tr}(\varepsilon)|) + b_M(x, |\varepsilon^D|)$

$$\lambda|\tau|^2 \leq D_\varepsilon^2 H_M(\cdot,\varepsilon)(\tau,\tau) \leq \Lambda_M|\tau|^2$$

for all $\varepsilon, \tau \in \mathbb{S}$ *with a uniform constant* λ *and a constant* Λ_M *depending on* M.

f) h_M *and* H_M *satisfiy the growth-conditions stated in (1.5)-(1.9) uniformly in* $x \in \Omega$ *and uniformly in* M.

After these preparations we define u_M as the unique minimizer of ($B := B_R(x_0) \Subset \Omega$, where $x_0 \in \Omega$ is arbitrary whereas we will choose R depending on ω very small, which does not restrict our argumentation)

$$J_M[w] := \int_B H_M(\cdot, \varepsilon(w))\,dx := \int_B \left[a_M(\cdot, |\operatorname{div}(w)|) + b_M(\cdot, |\varepsilon^D(w)|)\right]\,dx$$

in $u + W_0^{1,2}(B,\mathbb{R}^3)$. The regularization u_M has the following properties:

Lemma 4.2. *a)* u_M *belongs to the space* $W^{2,2}_{loc}(B,\mathbb{R}^3)$.

b) $a_M(\cdot, |\operatorname{div}(u_M)|)^3$ *and* $b_M(\cdot, |\varepsilon(u_M|)^3$ *are elements of* $L^1_{loc}(B)$.

c) for $\gamma \in \{1,...,n\}$ $\partial_\gamma u_M$ *solves*

$$\int_B D_\varepsilon^2 H_M(\cdot, \varepsilon(u_M))(\varepsilon(w), \varepsilon(\varphi))\,dx + \int_B \partial_\gamma D_\varepsilon H_M(\cdot, \varepsilon(u_M)) : \varepsilon(\varphi)\,dx = 0 \textit{ for all } \varphi \in W_0^{1,2}(B,\mathbb{R}^3)$$

with $\operatorname{spt}(\varphi) \Subset B$.

d) u_M is in $W^{1,2}(B,\mathbb{R}^3)$ uniformly bounded and we have

$$\sup_M \int_B H_M(\cdot, \varepsilon(u_M))\, dx < \infty.$$

On account of Lemma 4.1 e) we consider the minimizer of an isotropic problem with quadratic growth and can deduce part a) from [Br5] (Lemma 2.1, note that our situation is easier since we do not restrict us to the constraint $\operatorname{div}(w) = 0$). Part b) follows from part a) by Sobolev's inequality and the quadratic growth of h_M stated in Lemma 4.1 e) and f). Weak differentiablity of $D_\varepsilon H_M(\varepsilon(u_M))$ follows from [Br5] (Lemma 2.1) and so $\partial_\gamma u_M$ is clearly a solution if we restrict us to testfunctions $\varphi \in C_0^\infty(B,\mathbb{R}^3)$, hence part c) follows by approximation as a consequnce of the growth conditions of $D_\varepsilon^2 H_M(\cdot, \varepsilon(u_M))$ and $\partial_\gamma D_\varepsilon H_M(\cdot, \varepsilon(u_M))$ (see Lemma 4.1). We have by Lemma 2.1 a)

$$\sup_M \int_B H_M(\cdot, \varepsilon(u_M))\, dx \le \sup_M \int_B H_M(\cdot, \varepsilon(u))\, dx \le \int_B H(\cdot, \varepsilon(u))\, dx,$$

hence we receive d).
To end up with the preparations we have to show higher integrability of u_M uniformly in M similar to Lemma 2.2. Therefore we choose $\widehat{x} \in B$ such that

$$a(\widehat{x}, t) \le a(x,t) \quad \text{for all } (x,t) \in \overline{B} \times [0,\infty)$$

and define $\widehat{a}(t) := a(\widehat{x}, t)$ (remember (A8)). Hence we get $\varepsilon(u) \in L^{\widehat{a}}(\Omega)$, since

$$\int_B H(\cdot, \varepsilon(u))\, dx < \infty$$

and $a \le cb$.

Lemma 4.3. *a) We get for big t and all $x, y \in B$*

$$a_M(x,t) \le c t^{\theta|x-y|} a_M(y,t)$$

uniformly in M, which means (A7) extends to a_M.

b) We have $a_M(\cdot, |u_M|)|u_M|^\mu \in L^1(B)$ uniformly, provided $\mu < 4$.

Proof: Since $a(x,t)$ behaves like $ta'(x,t)$ and $a_M(x,t)$ behaves like $ta'_M(x,t)$ it is enough to show the inequality for a'_M instead of a_M. We have for $t \gg 1$ (see [Br3])

$$\begin{aligned}
\frac{a'_M(x,t)}{t} &= \eta(t)\frac{a'(x,t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} a'(x,s)\, ds \\
&\le c\eta(t) t^{\theta|x-y|}\frac{a'(y,t)}{t} + c\int_0^t \left\{-\frac{\eta'(s)}{s}\right\} s^{\theta|x-y|} a'(y,s)\, ds \\
&\le c t^{\theta|x-y|}\frac{a'_M(x,t)}{t},
\end{aligned}$$

where we used (A7) for a' (η is a cut off function with $\eta \equiv 1$ on $[0, 3M/2]$ and $\eta' \le 0$). This proves part a).
In order to prove part b) we define $\widehat{a}_M(t) := a_M(\widehat{x}, t)$ and receive for all $(x,t) \in \overline{B} \times [0,\infty)$

$$\widehat{a}_M(t) \le c a_M(x,t) \tag{4.1}$$

uniformly in M. To prove (4.1) we start with

$$\begin{aligned}\frac{\widehat{a}'_M(t)}{t} &= \frac{a'_M(\widehat{x},t)}{t} = \eta(t)\frac{a'(\widehat{x},t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} a'(\widehat{x},s)\,ds \\ &\le c\eta(t)\frac{a'(x,t)}{t} + c\int_0^t \left\{-\frac{\eta'(s)}{s}\right\} a'(x,s)\,ds = c\frac{a'_M(x,t)}{t}\end{aligned}$$

since $a(x,t) \approx ta'(x,t)$. By Lemma 4.1 d) we can conclude (4.1). We receive from (4.1) and Lemma 4.2 d) uniform boundedness of $\widehat{a}_M(|u_M|)^{\frac{1}{\beta}}$ in $L^\beta(B, R^3)$ for any $\beta \in (1,2)$. By Lemma 4.1 d) we have

$$\int_B |\nabla \widehat{a}_M(|u_M|)^{\frac{1}{\beta}}|^\beta\,dx \le c\int_B \widehat{a}'_M(|u_M|)|u_M|^{1-\beta}|\nabla u_M|^\beta\,dx.$$

Following the ideas of [BrF] we can control the r.h.s. by the $W^{1,\widehat{a}_M}$-norms of u_M (note that the constants in the calculations do not depend on M, since all estimates are uniform on account of Lemma 4.1). Hence we have to find a uniform bound for $\|\nabla u_M\|_{L^{\widehat{a}_M}}$. Using Lemma 5.1 a) we receive (note that $u_M - u \in W_0^{1,2}(\Omega, \mathbb{R}^3)$)

$$\|\nabla u_M\|_{L^{\widehat{a}_M}} \le c\left\{\|\varepsilon(u_M)\|_{L^{\widehat{a}_M}} + \|\nabla u\|_{L^{\widehat{a}_M}}\right\}$$

for a constant c, which does not depend on M (remember Lemma 4.1 c)). By (4.1) the first norm on the r.h.s. is uniformly bounded: we have

$$\int_B \widehat{a}_M(|\varepsilon(u_M)|)\,dx \le c\int_B a_M(x, |\varepsilon(u_M)|)\,dx \le c\int_B H_M(x, \varepsilon(u_M))\,dx \le c$$

independent of M (compare Lemma 4.2 d)). For the $L^{\widehat{a}_M}$ norms of ∇u we get by (4.1) and Lemma 5.2 c)

$$\|\nabla u\|_{L^{\widehat{a}_M}} \le \|\nabla u\|_{L^{\widehat{a}}} \le c\left\{\|\nabla u\|_2 + \|\varepsilon(u)\|_{L^{\widehat{a}}}\right\}.$$

In this calculation the first term is clearly finite because $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^3)$, whereas the same is true for the second one on account of

$$\int_B \widehat{a}(|\varepsilon(u)|)\,dx \le \int_B a(\cdot, |\varepsilon(u)|)\,dx \le c\int_B H(\cdot, \varepsilon(u))\,dx < \infty,$$

since u is a local minimizer and $a \le cb$. This means that $\widehat{a}_M(|u_M|)^{\frac{1}{\beta}}$ is bounded in $W^{1,\beta}(B, R^3)$ and we deduce from Sobolev's embedding theorem $\widehat{a}_M(|u_M|) \in L^\chi(B)$ uniformly in M for all $\chi < 3$. As a consequence of the superquadratic growth of a_M we obtain for all $\widetilde{\mu} < 4$

$$\widehat{a}_M(|u_M|)|u_M|^{\widetilde{\mu}} \in L^1(B).$$

For a given $\mu < 4$ we can choose $\mathrm{diam}(B)$ small enough such that $\theta|x-y| \le (4-\mu)/2$ which gives by part a) and the definition of $\widehat{a}_M$ (and (4.1))

$$a_M(\cdot, |u_M|)|u_M|^\mu \ \in L^1(B).$$

This proves Lemma 4.3. □

Now we are going to modify the arguments of section 2 to get higher integrability in case of x-dependence: we want to estimate

$$\int_{B_r(z)} a_M(\cdot|\,\mathrm{div}(u_M)|)^3\,dx.$$

We notice there that all estimates we used in section 2 are now independent of M by Lemma 4.1. If we follow the lines of the proof of Theorem 1.1 we have to add the term

$$\int_{B_{\widetilde{r}}(z)} \eta_1^2 |\nabla_x a_M(x, |\,\mathrm{div}(u)|)^{\frac{1}{2}}|^2\,dx$$

in the second estimation, which behaves like

$$\int_{B_{\widetilde{r}}(z)} \eta_1^2 a_M(x, |\,\mathrm{div}(u)|)\,dx$$

on account of Lemma 4.1 b) and is therefore uncritical. The second difference is that we have to add (remember Lemma 4.2) c))

$$-\int_B \partial_\gamma D_\varepsilon H_M(\cdot, \partial_\gamma \varepsilon(u_M)) : \varepsilon(\eta_1^2 \partial_\gamma u_M)\,dx$$

on the r.h.s. of the Caccioppoli-type inequality. Since $h_M'(t)/t$ behaves like $h_M''(t)$ (see Lemma 4.1 b)) we have no problems with this integral (compare the calculations after (2.4) in [Br3] for details). We arrive at

$$\begin{aligned} &\int_{B_{r(z)}} a_M(\cdot, |\,\mathrm{div}(u_M)|)^3\,dx + \int_{B_{r(z)}} b_M(\cdot, |\varepsilon^D(u_M)|)^3\,dx \\ &\le c(R-r)^{-\beta}\left[1 + \left\{\int_{B_{\widetilde{r}}(z)} a_M(\cdot, |\nabla u_M)|)|\nabla u_M|^\omega\,dx\right\}^3\right]. \end{aligned} \tag{4.2}$$

In order to replace ∇ by ε on the r.h.s. we make the following calculations:

$$\begin{aligned} \int_{B_{\widetilde{r}}(z)} a_M(\cdot, |\nabla u_M)|)|\nabla u_M|^\omega\,dx &\le c\int_{B_{\widetilde{r}}(z)} a_M(z, |\nabla u_M)|)|\nabla u_M|^{\omega+\widetilde{r}\theta}\,dx \\ &\le c\int_{B_R(z)} a_M(z, |\nabla(\eta_2 u_M)|)|\nabla(\eta_2 u_M)|^{\omega+\widetilde{r}\theta}\,dx \end{aligned}$$

$$\le c \int_{B_R(z)} a_M(z, |\varepsilon(\eta_2 u_M))|) |\varepsilon(\eta_2 u_M)|^{\omega+\tilde{r}\theta}\, dx$$

Here we have used Lemma 4.3 a) and Lemma 5.1 with uniform constants. By the uniform Δ_2-condition of a_M (see Lemma 4.1 c)) we can extract $\|\nabla\eta_2\|_\infty$ and obtain the upper bound (α is a positive exponent)

$$\int_{B_R(z)} a_M(z, |\varepsilon(u_M))|) |\varepsilon(u_M)|^{\omega+\tilde{r}\theta}\, dx + c(R-r)^{-\alpha} \int_{B_R(z)} a_M(z, |u_M|) |u_M|^{\omega+\tilde{r}\theta}\, dx$$
$$\le c \int_{B_R(z)} a_M(\cdot, |\varepsilon(u_M))|) |\varepsilon(u_M)|^{\omega+2R\theta}\, dx + c(R-r)^{-\alpha} \int_{B_R(z)} a_M(\cdot, |u_M|) |u_M|^{\omega+2R\theta}\, dx$$

on account of Lemma 4.3 a). If we choose R small enough we are going to get uniform bounds for the second integral by Lemma 4.3 b) and it follows from (4.2) for all $R \le R_0$

$$\int_{B_{r(z)}} a_M(\cdot, |\operatorname{div}(u_M)|)^3\, dx + \int_{B_{r(z)}} b_M(\cdot, |\varepsilon^D(u_M)|)^3\, dx$$
$$\le c(R-r)^{-\beta} \left[1 + \left\{ \int_{B_R(z)} a_M(\cdot, |\varepsilon(u_M)|) |\varepsilon(u_M)|^{\omega+2R_0\theta}\, dx \right\}^3 \right]. \tag{4.3}$$

By a suitable choice of R_0 we get $\omega + 2R_0\theta < 4/3$ since $\omega < 4/3$, hence we can end up as in section 2 to show

$$a(\cdot, |\operatorname{div}(u_M)|)^3,\ b(\cdot, |\varepsilon^D(u)|)^3 \in L^1_{loc}(B). \tag{4.4}$$

Furthermore we can show

$$\int_0^{|\operatorname{div}(u)|} \sqrt{\frac{a'(\cdot,t)}{t}}\, dt,\ \int_0^{|\varepsilon^D(u)|} \sqrt{\frac{b'(\cdot,t)}{t}}\, dt \in W^{1,2}_{loc}(\Omega) \tag{4.5}$$

where we use the fact hat the x-derivates of the functions above can be estimated by the functions itselfs by (A6). Moreover, we can deduce a Caccioppoli-type inequality similar to Remark 2.1 if we add (compare [Br3], section 3)

$$T^1 := \int_B a(\cdot, |\operatorname{div}(u)|)\eta^2\, dx,$$
$$T^2 := \int_B a'(\cdot, |\operatorname{div}(u)|) |\nabla u - P| \eta |\nabla\eta|\, dx,$$
$$T^3 := \int_B b(\cdot, |\varepsilon^D(u)|)\eta^2\, dx,$$
$$T^4 := \int_B b'(\cdot, |\varepsilon^D(u)|) |\nabla u - P| \eta |\nabla\eta|\, dx$$

on the r.h.s. Now we come to the blow up procedure: There are four main differences to the autonomous situation (the first two and the fourth are the same as in [Br3], where find more details can be found).

1) We have to "enlarge" the excess function, i.e. we define

$$\widetilde{a}(x,t) := t^{\omega+2R_0} a(x,t)$$

and R_0 is the biggest number we allow for the radius which we assume to be small. We get well-defineness of $E(x,r)$ similar to the situation in [Br3] on account of (4.4) together with (A5) and $\omega < 4/3$.

2) Instead of (3.1) and (3.2) we have

$$|(\epsilon(u))_{x,r}| \le L,\ E(x,r) + r^{\gamma^*} \le \kappa\,, \tag{4.6}$$

and

$$E(x,\tau r) \le C_*(L)\tau^2 \left[E(x,r) + r^{\gamma^*}\right]\ . \tag{4.7}$$

This is to guarantee that $\lambda_m^{-1} r_m$ converge to zero.

3) Now we can prove (3.3)-(3.9) as done before (note that $\widetilde{a}$ now depends on $x_m + r_m z$ in the scaled version). In order to show the strong convergence from (3.10) and (3.11) we have to bound

$$\begin{aligned}&\lambda_m^{-2} \int_{B_s} a\left(x_m + r_m z, \lambda_m|\nabla u_m|\right) |\lambda_m \nabla u_m|^{\omega}\, dz \\ &\le \lambda_m^{-2} \int_{B_s} a\left(x_m, \lambda_m|\nabla u_m|\right) |\lambda_m \nabla u_m|^{\omega+R_0}\, dz,\end{aligned}$$

where we used $r_m \le R_0$ and (A7). We obtain for $\widetilde{\widetilde{a}}(t) := a(x_m,t)t^{\omega+R_0}$ and $h_\lambda(t) := \lambda^{-2}h(\lambda t)$

$$\|\lambda_m \nabla u_m\|_{\widetilde{\widetilde{a}}_{\lambda_m}} \le c(s)\,\|\lambda_m u_m\|_{\widetilde{\widetilde{a}}_{\lambda_m}} + c\,\|\lambda_m \varepsilon(u_m)\|_{\widetilde{\widetilde{a}}_{\lambda_m}}$$

if we use Lemma 4.4 from [BrF]. In this calculation the first term can be bounded as done before and for the second integral we receive the estimate

$$\lambda_m^{-2} \int_{B_s} \widetilde{\widetilde{a}}(|\lambda_m \varepsilon(u_m)|)\, dz \le \lambda_m^{-2} \int_{B_s} \widetilde{a}(|\lambda_m \varepsilon(u_m)|)\, dz$$

as a consequence of (compare (A7))

$$\widetilde{\widetilde{a}}(t) \le a(x_m + r_m z, t)t^{\omega+2R_0} = \widetilde{a}(x_m + r_m z, t).$$

So we find uniform bounds by the obvious version of (3.7). This finally proves the strong convergence of the scaled functions. If we define functions ψ_m^1 and ψ_m^2 similar to section 3, we have to estimate additonal x-derivatives, but they are bounded by ψ_m^1 and ψ_m^2 itselfs on account of (A6).

4) In order to get the continuous growth condition after iterating the blow up lemma we have to use (A8). Details of this arguments are presented in [Br3], end of section 3.

5 Appendix

In this section we collect some auxiliary material concerning Korn-type inequalities, which are crucial tools in this work. We start with

Lemma 5.1. *a) Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^n$ and let φ denote a N-function of class $(\Delta_2) \cap (\nabla_2)$ (see, e.g., [RR] for a definition). Then there is a constant $C = C(n, \varphi, \Omega)$ such that*

$$\int_\Omega \varphi(|\nabla w|)\, dz \leq c \int_\Omega \varphi(|\varepsilon(w)|)\, dz$$

holds for any $w \in W_0^{1,\varphi}(\Omega, \mathbb{R}^n)$. C only depends on the (Δ_2)- and (∇_2)-condition of φ.

b) In the case that Ω is a ball $B_R(x_0)$ the constant C has the form

$$C = c(n, \varphi) R^{-\beta}$$

for a positive exponent β.

The proof of Lemma 5.1 a) is presented in [Fu3], part b) can easily be derived from this first inequality by scaling and using the (Δ_2)-property of φ. The proof in [Fu3] is based on a regularity theorem for Poisson equations in Orlicz spaces from Jia, Li and Whang [JLW]. From the calculations after (3.23) in [JLM] one can see that the constant depends on n, Ω, $\overline{k}$ and $\underline{k}$ where $\underline{k} > 1$ is the constant in

$$\varphi(2t) \geq \underline{k} \varphi(t), \quad t \geq 0.$$

□

Suppose now that h satisfyies(A1)-(A3). Then we have

$$th'(t) = \int_0^t \frac{d}{ds}\left[sh'(s)\right] ds = h(t) + \int_0^t sh''(s)\, ds \geq 2h(t),$$

and in conclusion

$$a(h) := \inf_{t>0} \frac{h'(t)t}{h(t)} \geq 2. \tag{5.1}$$

Therefore h is a N-function of (global) type (∇_2), which follows from Corollary 4 on p. 26 in [RR], and we have (assuming (A1)-(A3) in the following)

Lemma 5.2. *a) Let $u \in L^h(\Omega)$ be a function such that $\varepsilon(u) \in L^h(\Omega)$, then u belongs to the space $W^{1,h}(\Omega)$ and we have*

$$\|u\|_{W^{1,h}(\Omega)} \leq c \Big\{ \|u\|_{L^h} + \|\varepsilon(u)\|_{L^h(\Omega)} \Big\}$$

for a positive constant c only depending on the (Δ_2)-condition of h (and Ω and n).

b) Let $u \in L^2(\Omega)$ be a function such that $\varepsilon(u) \in L^h(\Omega)$, then we have

$$\|u-\gamma\|_{L^h(\Omega)} \leq c\,\|\varepsilon(u)\|_{L^h(\Omega)}$$

for a constant c depending on h and Ω. Thereby γ is the orthogonal projection of u into the space of rigid motions w.r.t. the $L^2(\Omega)$ inner product.

c) Let $u \in L^2(\Omega)$ be a function such that $\varepsilon(u) \in L^h(\Omega)$, then the we have

$$\|u\|_{L^h(\Omega)} \leq c\left\{\|u\|_{L^2(\Omega)} + \|\varepsilon(u)\|_{L^h(\Omega)}\right\}$$

for a constant c depending on h and Ω.

Proof: In [MM] the authors prove the representation

$$\nabla u = L_u(u) + L_\varepsilon(\varepsilon(u))$$

where the components of L_u and L_ε are singular integral operators whose continuity in $L^p(\Omega,\mathbb{R}^3)$ $(1<p<\infty)$ is established in [CZ]. This means we have for all $1<p<\infty$ and $L \in \{L_u, L_\varepsilon\}$

$$\|L(w)\|_p \leq c(p)\,\|L(w)\|_p$$

for all $w \in L^p(\Omega,\mathbb{R}^3)$. Now we have to find some exponents $p_1, p_2 \in (1,\infty)$ with the following properties: the function $h(t)/t^{p_1}$ increases and the function $h(t)/t^{p_2}$ decreases and we have positive constants k_1 and k_1 independent of t such that

$$\int_0^t \frac{h(s)}{s^{p_1}}\frac{ds}{s} \leq k_1\frac{h(t)}{t^{p_1}}, \tag{5.2}$$

$$\int_t^\infty \frac{h(s)}{s^{p_2}}\frac{ds}{s} \leq k_1\frac{h(t)}{t^{p_2}}. \tag{5.3}$$

If we have found them, we can quote the interpolation arguments of Torchinsky [To] to follow

$$\|L(w)\|_{L^h(\Omega)} \leq c\,\|L(w)\|_{L^h(\Omega)}$$

for all $w \in L^h(\Omega,\mathbb{R}^3)$. Thereby c depends on p_1, p_2, the norms of the operators L_u, L_ε in the space $L^p(\Omega)$ and the constants k_1, k_1 given in (5.2) and (5.3).
If we have a look at the proof of Lemma 4.3 in [BrF] (choose $\lambda = 1$ and $\omega = 0$), we see that every exponent $p_2 > \overline{k}$ is a possible choice. Let $p_1 \in (1,2)$ than we have

$$\left(\frac{h(t)}{t^{p_1}}\right)' = \frac{t^{p_1-1}(h'(t)t - p_1 h(t))}{t^{2p_1}} \geq \frac{t^{-1}(2-p_1)h(t)}{t^{p_1}} \geq 0$$

by (5.1). Now we prove (5.2): by (A3) and (1.7) we get

$$\int_0^t \frac{h(s)}{s^{p_1}}\frac{ds}{s} \leq \int_0^t \frac{h'(s)}{s}s^{1-p_1}\,ds \leq \frac{h'(t)}{t}\int_0^t s^{1-p_1}\,ds \leq \frac{\overline{k}}{2-p_1}\frac{h(t)}{t^{p_1}}.$$

Altogether, part a) is shown.
In order to prove part b) we use indirect arguments similar to Lemma A. 3.1 (3.4) in [FS]. Let

$$V^* := \left\{ v \in L^h(\Omega) : \varepsilon(v) \in L^h(\Omega), \int_\Omega vw \, dx = 0 \text{ for all } w \text{ in } V_* \right\}$$

where V_* is the space of rigid motions. Now we assume that the inequality

$$\|u\|_{L^h(\Omega)} \leq c \, \|\varepsilon(u)\|_{L^h(\Omega)} \tag{5.4}$$

for $u \in V^*$ is not right, than we find a sequence $(u_k) \subset L^h(\Omega)$ such that

$$\|u_k\|_{L^h(\Omega)} \geq k \, \|\varepsilon(u_k)\|_{L^h(\Omega)} \, . \tag{5.5}$$

We define

$$v_k := \frac{u_k}{\|u_k\|_{L^h(\Omega)}}$$

and deduce from part a) and (5.5) uniform boundedness of v_k in $W^{1,h}(\Omega)$. Reflexivity of this space, which follows from (Δ_2)- and (∇_2)-condition (see [Ad], Theorem 8.28), gives

$$v_k \rightharpoondown: v \in \ \text{ in } W^{1,h}(\Omega).$$

This suggests $v \in V^*$ (remember $L^h(\Omega) \hookrightarrow L^2(\Omega)$ is continuous). From [Ad], Theorem 8.32, we deduce compactness of the embedding $W^{1,h}(\Omega) \hookrightarrow L^h(\Omega)$ provided

$$\int_1^\infty \frac{h^{-1}(t)}{t^{\frac{n+1}{n}}} \, dt < \infty \tag{5.6}$$

(So we have (27) on p. 248 in [Ad] and (26) on p. 248 follows from superquadratic growth if $n \geq 3$). To prove the compactness we have to show according to Adams that h growth more slowely then h_* given by

$$h_*(t) = \int_0^t \frac{h^{-1}(\tau)}{\tau^{\frac{n+1}{n}}} \, d\tau = \int_0^{h^{-1}(t)} \frac{\tau h'(\tau)}{h(\tau)^{\frac{n+1}{n}}} \, d\tau \leq c \int_0^{h^{-1}(t)} \frac{1}{\tau^{\frac{2}{n}}} \, d\tau \tag{5.7}$$

near infinity (using (1.7) and (1.5)). This follows from

$$h_*^{-1}(h(t)) \leq ct^{1-\frac{2}{n}},$$

which is a consequence of (5.7). Hence

$$v_k \to v \ \text{ in } L^h(\Omega).$$

By definition of v_k we can follow

$$\|v\|_{L^h(\Omega)} = 1. \tag{5.8}$$

On the other hand we receive from (5.5)

$$\varepsilon(v_k) \to 0 \text{ in } L^h(\Omega)$$

which means we have $v \in V_x$ and therefore $v = 0$, which contradicts (5.8). Therefore (5.4) is right and by orthogonal projection we obtain part b), if h satisfies (5.6).
Let us have a look at the other situation, i.e.

$$\int_1^\infty \frac{h^{-1}(t)}{t^{\frac{n+1}{n}}}\, dt = \infty. \tag{5.9}$$

Then we deduce from [Ad], (39) on p. 253

$$\|u\|_\infty \le c\left\{\|u\|_{L^h(\Omega)} + \|\nabla u\|_{L^h(\Omega)}\right\} \le c\left\{\|u\|_{L^h(\Omega)} + \|\varepsilon(u)\|_{L^h(\Omega)}\right\} \tag{5.10}$$

where we used a) for the last inequality. In the following we estimate the L^h-norm of u: we have on account of (1.5) and (1.6) for any $\kappa > 0$

$$\begin{aligned} \|u\|_{L^h(\Omega)} &\le c\left\{\|u\|_2 + \|u\|_q\right\} \le c\left\{\|u\|_2 + \|u\|_\chi^\theta \|u\|_2^{1-\theta}\right\} \\ &\le \kappa \|u\|_\chi + c(\kappa)\|u\|_2 \le \kappa c(\Omega)\|u\|_\infty + c(\kappa)\|u\|_2 \end{aligned}$$

Here $\chi > q$ is an arbitrary exponent and $\theta \in (0,1)$ the suitable number from the interpolation inequality (compare [GT], 7.9, p. 146). Inserting this into (5.10) and absorbing the κ-term on the l.h.s. we get

$$\|u\|_\infty \le c\left\{\|u\|_2 + \|\varepsilon(u)\|_{L^h(\Omega)}\right\}, \tag{5.11}$$

which proves part c) if (5.9) is satisfied. If $u \in V^*$ we receive from (5.11)

$$\|u\|_{L^h(\Omega)} \le c\left\{\|\varepsilon(u)\|_2 + \|\varepsilon(u)\|_{L^h(\Omega)}\right\} \le c\,\|\varepsilon(u)\|_{L^h(\Omega)}$$

by [FS] Lemma 3.0.3 (ii) and (1.5). We receive part b) by orthogonal projection.
In a last step we show part c) if (5.6) is satisfied. In this case we have shown part b). Note that we have the representation

$$\gamma(z) = \sum_i \left(\int_\Omega u R_i\, dx\right) R_i(z),$$

where (R_i) is an orthonormal base of the space of rigid motions. So we obtain

$$\|\gamma\|_\infty \le c\,\|u\|_2$$

for a constant c depending on (R_i), hence we deduce part c) from part b). □

Remark 5.1. *Note that we only used the first inequality in (A2) to prove Lemma 5.1, the second one is not necessary if one supposes a global (Δ_2)-condition for h.*

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