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A note on inner functions and spherical isometries

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A commuting tuple $T = (T_1, \ldots, T_n) \in B(H)^n$ of bounded linear Hilbert-space operators is called a spherical isometry, if $\sum_{i=1}^n T_i^* T_i = 1_H$. In [12] Prunaru initiated the study of T-Toeplitz operators which he defines to be the solutions $X \in B(H)$ of the fixed-point equation $\sum_{i=1}^n T_i^* X T_i = X$. Using results of Aleksandrov on abstract inner functions we show that $X \in B(H)$ is a T-Toeplitz operator precisely when X satisfies $J^* X J = X$ for every isometry J in the unital dual operator algebra $\mathcal{A}_T \subset B(H)$ generated by T. As a consequence we deduce that a spherical isometry T has empty point spectrum if and only if the only compact T-Toeplitz operator is the zero-operator. Moreover we show that if $\sigma_p(T) = \emptyset$, then an operator which commutes modulo the finite-rank operators with $\mathcal{A}_T$ is a finite-rank perturbation of a T-Toeplitz operator.

§1 Introduction

A *spherical isometry* is a commuting tuple $T = (T_1, \ldots, T_n) \in B(H)^n$ of Hilbert-space operators satisfying

$$T_1^* T_1 + T_2^* T_2 + \ldots + T_n^* T_n = 1_H.$$

Since this condition is modelled after the defining function for the boundary $\partial\mathbb{B}_n$ of the Euclidean unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, one should expect that the operator theory of such a tuple T is closely related to the function theory on $\mathbb{B}_n$. The desired link is settled by a result of Athavale (Proposition 2 in [2]) saying that each spherical isometry is subnormal. More explicitly, for each spherical isometry $T \in B(H)^n$ there exist a Hilbert space $K \supset H$ and a spherical unitary, i.e. a commuting tuple $U \in B(K)^n$ of normal operators with Taylor-spectrum $\sigma(U) \subset \partial\mathbb{B}_n$, such that $T_i = U_i|H$ for $i = 1, \ldots, n$. Replacing K by $\bigvee_{\alpha \in \mathbb{N}_0^n} (U^*)^\alpha H$, if necessary, we may assume that U is the minimal normal extension of T which is known to be unique up to unitary equivalence.

Let $E(\cdot)$ denote the projection-valued spectral measure for U. If we fix a separating unit vector $z \in H$ for $W^*(U)$, then the probability measure $\mu = \langle E(\cdot)z, z\rangle$ is a scalar-valued spectral measure of U. Since μ is supported by the Taylor spectrum of U, we may regard μ as a measure on $\partial\mathbb{B}_n$. This measure will in the sequel be referred to as "the" scalar-valued spectral measure associated with T. (Note that μ is unique modulo mutually absolute continuity.) The multi-variable spectral theory for commuting tuples of normal operators asserts the existence of an isomorphism of von Neumann algebras

$$\Psi_U : L^\infty(\mu) \to W^*(U) \subset B(K)$$

mapping the coordinate functions z_i to the operators U_i $(i = 1, \ldots, n)$. If we define $H^\infty(\mu)$ to be the dual subalgebra

$$H^\infty(\mu) = \overline{\{[p|\partial\mathbb{B}_n]_\mu : p \in \mathbb{C}[z]\}}^{w^*} \subset L^\infty(\mu),$$

where $z = (z_1, \ldots, z_n)$, and write

$$\mathcal{A}_T = \overline{\{p(T) : p \in \mathbb{C}[z]\}}^{w^*} \subset B(H)$$

for the unital dual operator algebra generated by the components of T, then it is well known that the map Ψ_U induces a dual algebra isomorphism (i.e. isometric isomorphism and weak* homeomorphism)

$$\gamma_T : H^\infty(\mu) \to \mathcal{A}_T, \qquad f \mapsto \Psi_U(f)|H$$

extending the polynomial functional calculus of T in a unique way (see Conway [4], Proposition 1.1). It is a simple and well-known fact that this isomorphism yields a correspondence between so-called *μ-inner functions*, i.e. elements

$$\theta \in H^\infty(\mu) \quad \text{with} \quad |\theta| = 1 \quad \mu\text{-a.e. on } \partial\mathbb{B}_n,$$

and isometric operators related to T. For completeness sake, we give a proof.

1.1 Lemma. *Let $T \in B(H)^n$ be a spherical isometry with associated spectral measure $\mu \in M^+(\partial\mathbb{B}_n)$. An operator $J \in \mathcal{A}_T$ is an isometry if and only if $J = \gamma_T(\theta)$ with some μ-inner function $\theta \in H^\infty(\mu)$.*

Proof. Note that, for every $x \in H$ and every $\theta \in H^\infty(\mu)$, we have

$$\|\gamma_T(\theta)x\|^2 = \|\Psi_U(\theta)x\|^2 = \langle \Psi_U(|\theta|^2)x, x\rangle.$$

If θ is inner, we can extend the preceding line by $\ldots = \|x\|^2$ proving that $\gamma_T(\theta)$ is an isometry. For the reverse direction, remember that $\mu = \langle E(\cdot)z, z\rangle$ with $z \in H$ being a separating vector for U. So if $J \in \mathcal{A}_T$ is assumed to be an isometry and $J = \gamma_T(\theta)$ with $\theta \in H^\infty(\mu)$, then we obtain by the above equality applied to $x = z$ that

$$0 = \|z\|^2 - \langle \Psi_U(|\theta|^2)z, z\rangle = \langle \Psi_U(1 - |\theta|^2)z, z\rangle = \int_{\partial\mathbb{B}_n} (1 - |\theta|^2)d\mu.$$

The identity $\|\theta\|_{\infty,\mu} = \|J\| = 1$ shows that the integrand $1 - |\theta|^2$ is non-negative, hence it must be zero (μ-a.e.), proving that θ is indeed inner. □

Via this correspondence, richness and approximation results for μ-inner functions immediately yield informations about the structure of the dual algebra $\mathcal{A}_T$. The aim of the next section is therefore to collect and extend some known approximation devices for μ-inner functions.

§2 Approximation by inner functions

The results in this section will be formulated in a more general setting than it is needed for spherical isometries. In the sequel we may–without any additional effort–replace the sphere $\partial\mathbb{B}_n$ by a general compact subset $K \subset \mathbb{C}^n$. We write $M^+(K)$ for the set of all finite positive regular Borel measures on K and $C(K)$ for the space of all $\mathbb{C}$-valued continuous functions on K.

Given a unital closed subalgebra $A \subset C(K)$ containing the polynomials $\mathbb{C}[z]|K$ in n-complex variables $z = (z_1, \ldots, z_n)$ and a measure $\mu \in M^+(K)$, the triple (A, K, μ) is called *regular* in the sense of Aleksandrov [1], if for every function $\varphi \in C(K)$ with $\varphi > 0$, there exists a sequence (f_k) in A with $|f_k| < \varphi$ for all k and $\lim_{k\to\infty} |f_k| = \varphi$ μ-a.e. on K. Obviously, if $\nu \in M^+(K)$ is absolutely continuous with respect to μ and (A, K, μ) is regular, then so is (A, K, ν). For a regular triple (A, K, μ) the support of μ is necessarily contained in the Shilov boundary $S(A)$ of A. Remember that $S(A)$ is the smallest closed set $S \subset K$ with the property that $\|f\|_{\infty,K} = \|f\|_{\infty,S}$ for every $f \in A$.

To give an example of a regular triple relevant for spherical isometries, let

$$A(\mathbb{B}_n) = \{f \in C(\overline{\mathbb{B}}_n) : f|\mathbb{B}_n \text{ is holomorphic}\} \subset C(\overline{\mathbb{B}}_n)$$

be the ball algebra. Setting $K = \partial\mathbb{B}_n$ and $A = A(\mathbb{B}_n)|K$, which is isomorphic to $A(\mathbb{B}_n)$ by the maximum-modulus principle, Aleksandrov showed in [1] that the triple (A, K, μ) is regular for every finite regular Borel measure $\mu \in M^+(\partial\mathbb{B}_n)$.

Before we can formulate Aleksandrov's existence result of inner functions for regular triples, we have to gather some more notation. Recall that a measure $\mu \in M^+(K)$ is said to be *continuous*, if $\mu(\{\zeta\}) = 0$ for every $\zeta \in K$ and *discrete*, if there is a countable set $\Delta \subset K$ with $\mu(K \setminus \Delta) = 0$. Note that, for each measure $\mu \in M^+(K)$, the set $\Delta = \{\zeta \in K : \mu(\{\zeta\}) > 0\}$ is countable, and μ possesses a decomposition $\mu = \mu_d + \mu_c$ $(\mu_d \perp \mu_c)$ into a discrete and a continuos part defined by

$$\mu_d(\omega) = \mu(\omega \cap \Delta) \quad \text{and} \quad \mu_c(\omega) = \mu(\omega \cap \Delta^c)$$

for every Borel subset $\omega \subset K$. The elements of Δ are called (one-point) atoms of μ.

In the sequel, we write $H^\infty_A(\mu)$ for the weak*-closure of A in $L^\infty(\mu)$ and $H^2_A(\mu)$ for the norm closure of A in $L^2(\mu)$. We write

$$I_\mu = \{\theta \in H^\infty_A(\mu) : |\theta| = 1 \quad \mu\text{-a.e.}\} \subset L^\infty(\mu)$$

for the set of *μ-inner functions*. The following approximation theorem (see Corollary 29 in [1]) shows that, for a regular triple based on a continuous measure, the algebra $H^\infty_A(\mu)$ has a rich supply of inner functions.

2.2 Proposition. (Aleksandrov) *Let $\mu \in M^+(K)$ be a continuous measure such that the triple (A, K, μ) is regular. Then the weak* closure of the set*

$$I_\mu = \{\theta \in H^\infty_A(\mu) : |\theta| = 1 \quad \mu\text{-almost everywhere}\}$$

contains the $L^\infty(\mu)$-equivalence classes of all functions $f \in A$ with $|f| \leq 1$ on K. This immediately implies that $H^\infty_A(\mu) = \overline{LH}^{w^*} I_\mu$. □

Since $L^1(\mu)$ is separable, the weak*-topology on the closed unit ball B of $L^\infty(\mu)$ is metrizable. Since $I_\mu \subset B$, the weak*-closure and the weak*-sequential closure of I_μ in $L^\infty(\mu)$ coincide. Thus, to every $f \in A$ with $\|f\|_\infty, K \leq 1$ there is even a *sequence* of inner functions $\theta_k \in I_\mu$ $(k \in \mathbb{N}_0)$ converging to f (weak*). As a consequence, the weak*-closure in the above density relations may be replaced by the sequentially weak*-closure.

If $\mu = \mu_c + \mu_d$ is the decomposition of an arbitrary measure into its discrete and continuous part (as descriebd above) then there are natural isometric isomorphisms

$$L^2(\mu_c) \oplus L^2(\mu_d) \xrightarrow{\sigma_2} L^2(\mu) \qquad \text{and} \qquad L^\infty(\mu_c) \oplus L^\infty(\mu_d) \xrightarrow{\sigma_\infty} L^\infty(\mu)$$

mapping $(f, g) \mapsto f\chi_\Delta + g\chi_{K\setminus\Delta}$ with inverse given by $[f]_\mu \mapsto ([f]_{\mu_d}, [f]_{\mu_c})$. Concerning the induced splitting results for the subspaces $H^p_A(\mu) \subset L^p(\mu)$, the regularity allows one to show that the $H^p_A(\mu_d)$-part is indeed an L^p-summand.

2.3 Proposition. *In case of a regular triple (A, K, μ), the decompositions*

$$H^2_A(\mu) = L^2(\mu_d) \oplus H^2_A(\mu_c) \qquad \text{and} \qquad H^\infty_A(\mu) = L^\infty(\mu_d) \oplus H^\infty_A(\mu_c)$$

hold.

Proof. Fix an arbitrary $\zeta \in \Delta$ and choose a sequence of functions $g_k \in A$ corresponding to $\varepsilon = 1/k$ in the following lemma. It is not hard to show (cp. Eschmeier [9] for the H^2-case) that g_k converges to $\chi_{\{\zeta\}}$ both in the $L^2(\mu)$-norm and the weak*-topology of $L^\infty(\mu)$. This settles the non-trivial inclusion "$\supset$". □

2.4 Lemma. *Let (A, K, μ) be a regular triple, and $\{\zeta\} \subset K$ be a one-point atom of μ. Then, for each $\varepsilon > 0$, there exist an open neighborhood U of ζ with $\mu(U \setminus \{\zeta\}) < \varepsilon$ and a function $g \in A$ satisfying*

$$|g(\zeta) - 1| < \varepsilon, \qquad |g| < \varepsilon \quad (\text{on } K \setminus U) \qquad \text{and} \qquad |g| \le 1 \quad \text{on } K.$$

Proof. The outer-regularity of the Borel measure μ allows us to choose an open neighborhood U of ζ in such a way that $\mu(U) < \mu(\{\zeta\}) + \varepsilon$. By the Urysohn-Tietze extension theorem there is a positive continuous function $\varphi : K \to [\varepsilon, 1]$ with $\varphi(\zeta) = 1$ and $\varphi | K \setminus U = \varepsilon$. Since (A, K, μ) is regular, there exists a sequence of functions (g_j) in A such that $|g_j| < \varphi$ on K and $|g_j| \stackrel{j\to\infty}{\longrightarrow} \varphi$ μ-a.e. on K. Since $\{\zeta\}$ has positive μ-measure, this implies that $|g_j(\zeta)| \to 1$ as $j \to \infty$. Therefore the function $g = \alpha g_k$, where $k \in \mathbb{N}$ is sufficiently large and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ is suitably chosen, has all the desired properties. □

The above decomposition of $H^\infty_A(\mu)$ allows us to prove an approximation result similar to Proposition 2.2 without any continuity assumption on the underlying measure. Though weaker than the original one, the assertion is still suitable for applications in operator theory.

2.5 Proposition. *For every regular triple (A, K, μ) without any continuity assumption on μ, the assertion*

$$H^\infty_A(\mu) = \overline{LH}^{w^*} I_\mu$$

holds true. Moreover, the weak closure on the right hand side may be replaced by the sequentially weak* closure.*

Proof. It suffices to show that every $f \in A$ with $\|f\|_{\infty,K} \le 1$ belongs to the weak* closure on the right-hand side. The direct sum representation $H^\infty_A(\mu) = L^\infty(\mu_d) \oplus H^\infty_A(\mu_c)$ allows us to decompose $[f]_\mu = [g]_{\mu_d} \oplus [h]_{\mu_c}$ with $g = f\chi_\Delta$ and $h = f\chi_{K\setminus\Delta}$, where Δ denotes the set of all (one-point) atoms of μ. By the cited result of Aleksandrov (see Proposition 2.2), the continuous part $[h]_{\mu_c}$ can be approximated in the weak* topology of $L^\infty(\mu_c)$ by a sequence $(\theta_k)_{k\ge 0}$ of μ_c-inner functions $\theta_k \in I_{\mu_c} \subset H^\infty_A(\mu_c)$, $k \ge 0$. For later use, we choose a representative $h_k : K \to \mathbb{C}$ of $\theta_k \in L^\infty(\mu)$ with $h_k | \Delta = 0$ $(k \ge 0)$.

To treat the discrete part, observe that $|\mathrm{Re}\, g|, |\mathrm{Im}\, g| \le |f| \le 1$. So we may define four bounded measurable functions $u_+, u_-, v_+, v_- : K \to \mathbb{C}$ by the formulas

$$u_\pm = \mathrm{Re}\, g \pm i\sqrt{1 - (\mathrm{Re}\, g)^2}, \quad v_\pm = \mathrm{Im}\, g \pm i\sqrt{1 - (\mathrm{Im}\, g)^2}.$$

If u is any of these four functions $u_\pm$, $v_\pm$, then we have by construction that $u = 0$ on $K \setminus \Delta$ and $|u|^2 = u\overline{u} = 1$ on Δ. Hence, the equivalence classes $[u_\pm]_{\mu_d}$ and $[v_\pm]_{\mu_d}$ are μ_d-inner functions since they are unitary elements of $L^\infty(\mu_d) = H^\infty_A(\mu_d)$. Moreover, $g = \frac{1}{2}(u_+ + u_-) + \frac{i}{2}(v_+ + v_-)$.

To finally solve the approximation problem stated in the assertion, we define four sequences of inner functions in $H^\infty_A(\mu)$ by

$$\eta^1_k = [u_+]_{\mu_d} \oplus [h_k]_{\mu_c}, \quad \eta^2_k = [u_-]_{\mu_d} \oplus [h_k]_{\mu_c}, \quad \eta^3_k = [v_+]_{\mu_d} \oplus [h_k]_{\mu_c},$$

$$\eta_k^4 = [v_-]_{\mu_d} \oplus [(-h_k)]_{\mu_c} \qquad (k \geq 0),$$

where $\oplus$ refers to the direct sum $H_A^\infty(\mu) = L^\infty(\mu_d) \oplus H_A^\infty(\mu_c)$. Obviously, the sequence (f_k) with

$$f_k = \frac{1}{2}(\eta_k^1 + \eta_k^2) + \frac{i}{2}(\eta_k^3 + \eta_k^4) \qquad (k \geq 0)$$

belongs to $LH(I_\mu)$ and has f as its weak*-limit. □

2.6 Corollary. *In the situation of the preceding corollary, we also have*

$$L^\infty(\mu) = \overline{LH}^{w^*}\{\overline{\eta} \cdot \theta : \eta, \theta \in I_\mu\}.$$

Proof. Since the complex polynomials in $z = (z_1, \ldots, z_n)$ and $\bar{z}$ are weak*-dense in $L^\infty(\mu)$, it suffices to show that, for arbitrary multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, the monomials $p(z) = z^\alpha \bar{z}^\beta$ are in the weak* closure of the set $LH\{\overline{\eta} \cdot \theta : \eta, \theta \in I_\mu\}$. According to the previous proposition, there exist sequences (f_k) and (g_k) in $LH(I_\mu)$ such that $f_k \xrightarrow{k} z^\alpha$ and $g_k \xrightarrow{k} z^\beta$. Obviously, the sequence $(f_k \cdot \overline{g}_k)$ belongs to $LH\{\overline{\eta} \cdot \theta : \eta, \theta \in I_\mu\}$, and using the separate weak* continuity of the multiplication in $L^\infty(\mu)$ and the weak* continuity of the complex conjugation on $L^\infty(\mu)$, one can show that $f_k \overline{g}_k \xrightarrow{k} p$, as desired. □

§3 Applications to spherical isometries

In order to use the approximation results established in the preceding section for our purposes, we have to set there $K = \partial\mathbb{B}_n$ and $A = A(\mathbb{B}_n)|\partial\mathbb{B}_n$. Since the polynomials in $z = (z_1, \ldots, z_n)$ are norm-dense in $A(\mathbb{B}_n)$ the dual algebras $H_A^\infty(\mu)$ (from the last section) and $H^\infty(\mu)$ (defined in the introduction) coincide. Note that we freely make use of the symbols U, Ψ_U, γ_T defined in the introduction.

A) Reflexivity revisited. With every subset $\mathscr{S} \subset B(H)$ one associates a WOT-closed unital operator algebra called $\text{AlgLat}(\mathscr{S})$ as the set of all operators $C \in B(H)$ that leave invariant every closed $\mathscr{S}$-invariant subspace of H. The family $\mathscr{S}$ is called reflexive if the identity

$$\text{AlgLat}(\mathscr{S}) = \mathcal{W}_{\mathscr{S}}$$

holds with $\mathcal{W}_{\mathscr{S}}$ being the smallest WOT-closed subalgebra containing 1_H and $\mathscr{S}$. A commuting tuple $T \in B(H)^n$ is called reflexive, if $\text{AlgLat}(T) = \mathcal{W}_T$, where (in abuse of notation) we use the symbol T also to denote the subset $T \subset B(H)$ consisting of the different entries of the tuple T.

The fact that spherical isometries are reflexive has been shown by the author in [7]. The proof given there can be shortened by using Proposition 2.5. Note that this proposition and Lemma 1.1 guarantee that

$$\mathcal{A}_T = \overline{LH}^{w^*}\mathcal{I}_T$$

if we define $\mathcal{I}_T = \{J : J \text{ is an isometry in } \mathcal{A}_T\}$. As a family of commuting isometries, $\mathcal{I}_T$ is reflexive by a result of Bercovici (see Theorem 2.4 in [3]). Using this we obtain

$$\text{AlgLat}(T) = \text{AlgLat}(\mathcal{A}_T) = \text{AlgLat}(\mathcal{I}_T) = \mathcal{W}_{\mathcal{I}_T} \subset \overline{\text{ran}(\gamma_T)}^{WOT} = \mathcal{W}_T,$$

without a decomposition of T into a unitary and pure part as it was necessary in [7].

B) Toeplitz operators. Recall that a classical result of Brown and Halmos says that an operator $X \in B(H^2(\mathbb{D}))$ is a Hardy-space Toeplitz operator over the unit disc $\mathbb{D}$ if and only if $T^*XT = X$ where $T = M_z$ denotes the multiplication with z which is an isometric operator on $H^2(\mathbb{D})$. Generalizing this algebraic condition, Prunaru [12] defines the set of all Toeplitz operators with respect to a spherical isometry $T \in B(H)^n$ to be

$$\mathcal{T}(T) = \{X \in B(H) : \sum_{i=1}^{n} T_i^* X T_i = X\}.$$

Prunaru even extends this definition to commuting families of spherical isometries: Given an arbitrary index set Γ and natural numbers $n_\alpha \in \mathbb{N}$ $(\alpha \in \Gamma)$, a family $\mathscr{F} = (T_\alpha)_{\alpha\in\Gamma}$ of multi-operators is called a commuting family of spherical isometries, if each operator tuple $T_\alpha = (T_{\alpha,1}, \ldots, T_{\alpha,n_\alpha}) \in B(H)^{n_\alpha}$ is a spherical isometry and all operators $T_{\alpha,j}$ with $\alpha \in \Gamma$ and $1 \leq j \leq n_\alpha$ commute with each other. Following [12] the set of all $\mathscr{F}$-Toeplitz operators is then defined to be

$$\mathcal{T}(\mathscr{F}) = \bigcap_{\alpha\in\Gamma} \mathcal{T}(T_\alpha).$$

One of the main results of Prunaru [12] (see part (3) of Theorem 2.9 therein) is the identification of the set of all $\mathscr{F}$-Toeplitz operators as

$$\mathcal{T}(\mathscr{F}) = \{P_H Y|H : Y \in (\widehat{\mathscr{F}})'\}$$

where $\widehat{\mathscr{F}} = (N_\alpha) \subset B(K)$ denotes the minimal normal extension of the family $\mathscr{F} = (T_\alpha) \subset B(H)$, and $(\widehat{\mathscr{F}})' = \bigcap \big((N_{\alpha,j})' : \alpha \in \Gamma, 1 \leq j \leq n_\alpha\big) \subset B(K)$ denotes the commutant of $\widehat{\mathscr{F}}$.

Now let us return from this abstract setting to a concrete situation, namely the Hardy space $H = H^2(\mathbb{B}_n)$ over the unit ball in $\mathbb{C}^n$ (with respect to the surface measure on the sphere) on which the multiplication tuple $T = (M_{z_1}, \ldots, M_{z_n}) \in B(H)^n$ constitutes a spherical isometry. Guo and Wang found out (see Proposition 2.1 in [11]) that the T-Toeplitz operators are characterized by the condition that $M_\eta^* X M_\eta = X$ for every inner function $\eta \in H^\infty(\mathbb{B}_n)$. The following can be thought of as the abstract analogue of this result in the context of spherical isometries.

3.7 Proposition. *Let $T \in B(H)^n$ be a spherical isometry. An operator X is a T-Toeplitz-operator, i.e. satisfies $\sum_{i=1}^n T_i^* X T_i = X$, if and only if*

$$J^*XJ = X \qquad \textit{for every isometry } J \textit{ in the dual algebra } \mathcal{A}_T.$$

Proof. With the notations defined above consider the operator-families $\mathcal{I}_T = (\gamma_T(\theta))_{\theta\in I_\mu}$ on H and $\mathcal{I}_U = (\Psi_U(\theta))_{\theta\in I_\mu}$ on K. Note that $\mathcal{I}_T$ consists of commuting isometries, $\mathcal{I}_U$ consists of commuting unitary operators and $\mathcal{I}_U$ is a normal extension of $\mathcal{I}_T$. From the density assertion established in Corollary 2.6 we deduce that

$$W^*(U) = \Psi_U(L^\infty(\mu)) = \overline{LH}^{w^*}\{A^*B : A, B \in \mathcal{I}_U\}.$$

This implies that every subspace $M \subset K$ which is reducing for $\mathcal{I}_U$ also reduces U, and hence $\mathcal{I}_U$ is the minimal normal extension of $\mathcal{I}_T$. Moreover, Proposition 2.5 asserts that

$$\mathcal{A}_U = \Psi_U(H^\infty(\mu)) = \overline{LH}^{w^*}(\mathcal{I}_U),$$

proving the commutants $(U)' = (\mathcal{A}_U)' = (\mathcal{I}_U)'$ to be equal. By Theorem 2.9 in Prunaru [12] we obtain

$$\mathcal{T}(T) = P_H(U)'|H = P_H(\mathcal{I}_U)'|H = \mathcal{T}(\mathcal{I}_T).$$

Now a look at Lemma 1.1 finishes the proof. □

Our next aim is to show that the decomposition $\mu = \mu_c + \mu_d$ of the spectral measure $\mu \in M^+(\partial\mathbb{B}_n)$ associated with a spherical isometry $T \in B(H)^n$ into a continuous and a discrete part (see Section 2 for details) gives rise to an orthogonal decomposition of $H = H_c \oplus H_d$ which is reducing for every T-Toeplitz operator. Before proving this, we recall from the classical spectral theory of subnormal tuples that $\mu(\{\zeta\}) > 0$ for some $\zeta \in \mathbb{C}^n$ if and only if ζ is an eigenvalue of the minimal normal extension $U \in B(K)^n$ of T, and that in this case the operator $P_\zeta = \Psi_U(\chi_{\{\zeta\}}) \in B(K)$ is the orhogonal projection onto the joint eigenspace $\bigcap_{i=1}^n \ker(\zeta_i - U_i)$. (Here $\chi_{\{\zeta\}}$ denotes the characteristic function of the one-point atom $\{\zeta\}$.) Using the minimality of U it is not hard to show that U and T have the same eigenvalues, i.e.

$$\sigma_p(T) = \sigma_p(U) = \{\zeta \in \mathbb{C}^n : \mu(\{\zeta\}) > 0\}),$$

where σ_p stands for the point spectrum of the underlying tuple, and that the corresponding eigenspaces of T and U coincide. In the sequel, we abbreviate them by

$$H_d^\zeta = \bigcap_{i=1}^n \ker(\zeta_i - T_i) = \bigcap_{i=1}^n \ker(\zeta_i - U_i) \qquad (\zeta \in \sigma_p(T)),$$

and write $H_d = \bigoplus_{\zeta\in\sigma_p(T)} H_d^\zeta$, where the d in the notation refers to the discrete part. The orthogonal complement of H_d is denoted by $H_c = H \ominus H_d$.

A spherical isometry $T \in B(H)^n$ for which H_d is zero will be called *continuous*. So T is continuous if and only if μ is a continuous measure.

3.8 Proposition. *Let $T \in B(H)^n$ be a spherical isometry. The orthogonal decomposition $H = H_c \oplus \bigoplus_{\zeta\in\sigma_p(T)} H_d^\zeta$ reduces all T-Toeplitz operators and gives rise to a direct sum decomposition*

$$\mathcal{T}(T) = \mathcal{T}(T_c) \oplus \bigoplus_{\zeta\in\sigma_p(T)} B(H_d^\zeta),$$

where $T_c = T|H_c \in B(H_c)^n$ is a continuous spherical isometry.

Proof. Let $\zeta \in \mathbb{C}^n$ be an eigenvalue of T, i.e. a one-point atom of μ. Then the characteristic function $\chi_{\{\zeta\}}$ belongs to $H^\infty(\mu)$ (see Proposition 2.3) and hence to the restriction algebra. Thus $P_\zeta = \Psi_U(\chi_{\{\zeta\}})$ leaves H invariant, and thus commutes with $P_H \in B(K)$. $Q_\zeta = P_\zeta|H$ is the orthogonal projection from H onto H_d^ζ. So given any T-Toeplitz operator $X = P_H A|H$ with $A \in (U)'$ and an arbitrary vector $h \in H$ we find that

$$XQ_\zeta h = P_H A|HQ_\zeta h = P_H A P_\zeta h = P_\zeta P_H A|Hh = Q_\zeta X h,$$

proving the first part of the assertion as well as the inclusion "$\subset$". For the reverse inclusion, fix $X_c \in \mathcal{T}(T_c)$ and arbitrary operators $X_\zeta \in B(H_d^\zeta)$. Given $h \in H$, we decompose $h = h_c \oplus \bigoplus_\zeta h_\zeta$ to obtain

$$\sum_{i=1}^n T_i^* X T_i h = \sum_{i=1}^n (T_c)_i^* X_c (T_c)_i h_c \oplus \bigoplus_{\zeta\in\sigma_P(T)} \sum_{i=1}^n \overline{\zeta_i} X_\zeta \zeta_i h_\zeta = X_c h_c \oplus \bigoplus_\zeta X_\zeta h_\zeta = Xh.$$

So finally $X_c \oplus \bigoplus_\zeta X_\zeta \in \mathcal{T}(T)$. $\square$

3.9 Theorem. *A spherical isometry $T \in B(H)^n$ has empty point spectrum if and only if the zero-operator is the only compact T-Toeplitz operator.*

Proof. Suppose $\zeta \in \sigma_p(T) \neq \emptyset$ is an eigenvalue of T. Fix a rank-one operator $X_\zeta \in B(H_d^\zeta)$ on the corresponding eigenspace H_d^ζ. Setting $Xh = X_\zeta P_\zeta h$ $(h \in H)$ defines a non-zero compact Toeplitz operator by the previous proposition.

On the other hand, if $\sigma_p(T) = \emptyset$, then $T = T_c$ and the spectral measure μ associated with T is continuous. Thus there exists a weak* zero sequence (θ_k) of μ-inner functions by Aleksandrovs approximation theorem (cp. Proposition 2.2). Since the functional calculus $\gamma_T : H^\infty(\mu) \to B(H)$ is weak* continuous, the corresponding sequence of isometries $J_k = \gamma_T(\theta_k) \in B(H)$ is a weak* zero sequence. In particular, $J_k h \xrightarrow{k} 0$ weakly for every vector $h \in H$.

Now suppose X to be a compact T-Toeplitz operator. Using Proposition 3.7 we deduce that

$$\|Xh\| = \|J_k^* X J_k h\| \leq \|X J_k h\|$$

but the right-hand side tends to zero as $k \to \infty$ since X, as a compact operator, maps weak zero-sequences to norm zero-sequences. $\square$

As another application of Proposition 3.7 we prove a necessary condition for an operator $S \in B(H)$ to have finite-rank commutators

$$[S, A] = SA - AS \qquad \text{for every} \quad A \in \mathcal{A}_T.$$

Problems like this are inspired by a work of Davidson [5] who succeeded to identify the essential commutant of the set of all analytic Toeplitz operators on the Hardy space $H^2(\mathbb{D})$ over the unit disc. The corresponding generalization to the unit-ball case has been established by Guo in [10]. As a variation of this theme, Guo and Wang (see [11]) characterized all operators $S \in B(H^2(\mathbb{B}_n))$ having finite-rank commutators $[S, T_f]$ with all analytic Toeplitz operators $T_f \in B(H^2(\mathbb{B}_n))$, $f \in H^\infty(\mathbb{B}_n)$.

In the sequel we closely follow the work of Guo and Wang [11] and the corresponding ideas from Davidson [5].

3.10 Lemma. *(a) Let $(F_k)_{k\geq 1}$ be a sequence in $B(H)$ satisfying rank$(F_k) \leq M$ for $k \geq 1$ with some fixed natural number M. If (F_k) has a WOT-limit $F \in B(H)$, then rank$(F) \leq M$.*

(b) Let $\mathcal{A} \subset B(H)$ be a closed subspace. If $S \in B(H)$ has the property that $[S, A]$ is of finite rank for all $A \in \mathcal{A}$, then there exists a constant $M > 0$ such that rank$([S, A]) \leq M$ for all $A \in \mathcal{A}$.

Proof. The proof is almost a word-by-word repetition of that in [11]. For the reader's convenience, we state it here. Assuming rank$(F) > M$ one can find $N = M + 1$ vectors $x_1, \ldots, x_N \in H$ and an orthonormal system $\{y_1, \ldots, y_N\} \subset H$ such that $d = \det(\langle Fx_i, y_j\rangle_{ij}) \neq 0$. Since for every $k \in \mathbb{N}$ the set of vectors $\{F_k x_1, \ldots, F_k x_N\}$ is linearly dependent, we obtain $0 = \det(\langle F_k x_i, y_j\rangle_{ij}) \xrightarrow{k\to\infty} d \neq 0$, a contradiction which finishes the proof of part (a).

The hypothesis of part (b) guarantees that the Banach space $\mathcal{A}$ can be represented as the union $\mathcal{A} = \bigcup_{k=1}^\infty \Gamma_k$ of the norm-closed sets $\Gamma_k = \{A \in \mathcal{A} : \text{rank}([S, A]) \leq k\}$.

Baire's category theorem asserts the existence of an inner point A_0 in some Γ_N with $N \in \mathbb{N}$. Since $\Gamma_N - A_0$ is an open neighborhood of zero, every operator $C \in B(H)$ satisfies $C/\alpha \in \Gamma_N - A_0$ with some suitably chosen number $\alpha = \alpha(C) > 0$. Hence $C \in \alpha\Gamma_N - \alpha\Gamma_N$, implying that $\operatorname{rank}([S, A]) \leq 2N$ for every $A \in \mathcal{A}$. □

Theorem 3.1 in Guo-Wang [11] says that, for $T = M_z \in B(H^2(\mathbb{B}_n))^n$ and $n \geq 2$, an operator $S \in B(H^2(\mathbb{B}_n))$ commutes modulo the finite-rank operators with all analytic Toeplitz operators T_f ($f \in H^\infty(\mathbb{B}_n)$), if and only if $S = T_g + F$ with $g \in H^\infty(\mathbb{B}_n)$ and $\operatorname{rank}(F) < \infty$. Giving up the special structure of $H^2(\mathbb{B}_n)$ at least the following can be said.

3.11 Theorem. *Let $T \in B(H)^n$ be a spherical isometry with $\sigma_p(T) = \emptyset$. Given a bounded linear operator $S \in B(H)$ such that $[S, A]$ is of finite rank for every $A \in \mathcal{A}_T$. Then $S = X + F$ with $X \in \mathcal{T}(T)$ and a finite-rank operator $F \in B(H)$.*

Proof. By the remarks preceding Proposition 3.8 the scalar-valued spectral measure μ associated with T is continuous. According to Aleksandrov's approximation theorem (cp. Proposition 2.2) there exists a weak* zero sequence of μ-inner functions $(\theta_k)_{k\geq 1}$ in $H^\infty(\mu)$. In view of Lemma 1.1, the sequence $J_k = \gamma_T(\theta_k)$ ($k \geq 1$) is a weak* zero-sequence of isometries in $\mathcal{A}_T$. Passing to a subsequence, if necessary, we may assume that the bounded sequence $(J_k^* S J_k)_{k\geq 1}$ converges to an operator $X \in B(H)$ with respect to the weak* topology. By Lemma 3.10 (b) (applied to $\mathcal{A} = \mathcal{A}_T$) there is a constant $M > 0$ such that $\operatorname{rank}([S, J_k]) \leq M$ for every $k \geq 1$. Now applying part (a) of the cited lemma we deduce that the operator

$$F = S - X = \mathrm{w}^* - \lim_k S - J_k^* S J_k = \mathrm{w}^* - \lim_k J_k^*(J_k S - S J_k)$$

is at most of rank M. It remains to check that X is a Toeplitz operator. In order to verify the criterion established in Proposition 3.7, we fix an arbitrary isometry $V \in \mathcal{A}_T$ and write

$$V^* J_k^* S J_k V = J_k^* V^* S V J_k = J_k^* V^*(VS + SV - VS) J_k = J_k^* S J_k + J_k^*[S, V] J_k$$

Note that the last summand satisfies $|\langle J_k^*[S, V] J_k x, y\rangle| \leq \|[S, V] J_k x\| \cdot \|J_k y\|$ for arbitrary vectors $x, y \in H$. Using the fact that the compact operator $[S, V]$ maps the weak zero-sequence $(J_k x)_{k\geq 1}$ in H to a norm zero-sequence, we deduce that $J_k^*[S, V] J_k \xrightarrow{k} 0$ (WOT). Thus passing to WOT-limits in the above algebraic identity yields

$$V^* X V = X \qquad (V \text{ being an arbitrary isometry in } \mathcal{A}_T),$$

as desired. □

C) Spherical isometries of class $C_{\cdot 0}$. We finally take a look at spherical isometries $T \in B(H)^n$ whose functional calculus satisfies a certain additional continuity condition. Here "functional calculus" does not refer to the dual algebra isomorphism $\gamma_T : H^\infty(\mu) \to \mathcal{A}_T$ but to another natural extension of the polynomial functional calculus of T arising as follows: The von Neumann-type inequality

$$\|p(T)\| \leq \|p(U)\| \leq \|p\|_{\infty, \partial\mathbb{B}_n} \quad (p \in \mathbb{C}[z])$$

which is inherited by T from its spherical unitary extension U, yields a contractive $A(\mathbb{B}_n)$-functional calculus $\Phi_T : A(\mathbb{B}_n) \to B(H)$ for T. Obviously, Φ_T is related to γ_T via the formula

$$\Phi_T(f) = \gamma_T([f|\partial\mathbb{B}_n]_\mu) \qquad (f \in A(\mathbb{B}_n)).$$

A spherical isometry T is said to be of class $C_{\cdot 0}$ if Φ_T satisfies the additional continuity assumption that

$$\Phi_T(f_k)^* \overset{k\to\infty}{\longrightarrow} 0 \qquad (SOT)$$

whenever (f_k) is a Montel-sequence, i.e. a bounded sequence in $A(\mathbb{B}_n)$ such that $f_k|\mathbb{B}_n$ is a pointwise zero-sequence. It is well known that a sequence $(f_k)_k$ in $A(\mathbb{B}_n)$ is of this type precisely when $([f_k|\partial\mathbb{B}_n])_k$ is a weak*-zero sequence in $H^\infty(\sigma)$ (or, equivalently, $L^\infty(\sigma)$), where σ denotes the normalized surface measure on the sphere $\partial\mathbb{B}_n$.

If T is of class $C_{\cdot 0}$, then one easily deduces that the map $\Phi_T : A(\mathbb{B}_n) \to B(H)$ extends to a weak* continuous contractive homomorphism $H^\infty(\mathbb{B}_n) \to B(H)$, again denoted by Φ_T. In this case, the composition

$$r_\mu : H^\infty(\mathbb{B}_n) \overset{\Phi_T}{\longrightarrow} \mathcal{A}_T \overset{\gamma_T^{-1}}{\longrightarrow} H^\infty(\mu)$$

yields an abstract boundary-value map, that is, a contractive and weak* continuous homomorphism mapping each polynomial $p \in \mathbb{C}[z]$ to its boundary-value equivalence class $[p|\partial\mathbb{B}_n]_\mu$ in $L^\infty(\mu)$. A measure $\nu \in M^+(\partial\mathbb{B}_n)$ possessing such a weak* continuous and contractive boundary value map $r_\nu : H^\infty(\mathbb{B}_n) \to H^\infty(\nu)$ is called a Henkin measure. It is called faithful, if r_ν is isometric (and hence an isometric isomorphism and weak* homeomorphism, i.e. a dual algebra isomorphism). It is well known that the surface measure σ on $\partial\mathbb{B}_n$ is a faithful Henkin measure.

As a final remark, we state that, given two Henkin measures $\eta \ll \nu$ in $M^+(\partial\mathbb{B}_n)$, the boundary homomorphisms of η and ν are related to each other via

$$r_\eta^\nu \circ r_\nu = r_\eta$$

(to be checked on polynomials) where r_η^ν denotes the canonical weak* continuous contraction

$$r_\eta^\nu : L^\infty(\nu) \to L^\infty(\eta), \qquad [f]_\nu \mapsto [f]_\eta.$$

3.12 Lemma. *Let $T \in B(H)^n$ be spherical isometry of class $C_{\cdot 0}$. Then there exists a sequence $(J_k)_{k\geq 1}$ of isometries in $\mathcal{A}_T$ satisfying*

$$J_k^* \overset{k\to\infty}{\longrightarrow} 0 \qquad (SOT).$$

Proof. Let $\mu \in M^+(\partial\mathbb{B}_n)$ be the scalar-valued spectral measure associated with T which, by the considerations above, is a Henkin measure and thus continuous (cp. Lemma 2.2.3 in [6]). By the approximation theorem of Aleksandrov stated as Proposition 2.2 above, there exists a sequence of $(\mu+\sigma)$-inner functions

$$(\theta_k)_{k\geq 0} \quad \text{in } I_{\mu+\sigma} \subset H^\infty(\mu+\sigma) \quad \text{with} \quad \theta_k \overset{w^*}{\longrightarrow} 0 \text{ in } L^\infty(\mu+\sigma).$$

According to the estimate

$$\|f\|_{\infty,\mathbb{B}_n} = \|r_\sigma(f)\|_{\infty,\sigma} = \|r_\sigma^{\mu+\sigma}(r_{\mu+\sigma}(f))\|_{\infty,\sigma} \leq \|r_{\mu+\sigma}(f)\|_{\infty,\mu+\sigma} \quad (f \in H^\infty(\mathbb{B}_n)),$$

the measure $\mu+\sigma$ is a faithful Henkin measure and so we are able to define the desired operator-sequence as

$$J_k = \Phi_T(r_{\mu+\sigma}^{-1}(\theta_k)) \qquad (k \geq 1).$$

Indeed, $(J_k^*)_{k\geq 0}$ is a SOT-zero sequence in view of the $C_{\cdot 0}$-property of T and the weak*-continuity of $r_{\mu+\sigma}^{-1}$. In order to check that J_k is an isometry for $k \geq 1$, we write

$$J_k = \Phi_T(r_{\mu+\sigma}^{-1}(\theta_k)) = \gamma_T(r_\mu(r_{\mu+\sigma}^{-1}(\theta_k))) = \gamma_T(r_\mu^{\mu+\sigma}(\theta_k))$$

and observe that $|r_\mu^{\mu+\sigma}(\theta_k)| = 1$ ($\mu -$ a.e.) since $|\theta_k| = 1$ ($(\mu+\sigma) -$ a.e.). By Lemma 1.1, the proof is complete. □

3.13 Proposition. *A spherical isometry $T \in B(H)^n$ is of class $C_{\cdot 0}$ if and only if it is completely non-unitary.*

Proof. A result of Eschmeier (Corollary 2.4 in [8]) says that a completely non-unitary spherical isometry is of class $C_{\cdot 0}$ and this settles the difficult part.

To prove the opposite direction, fix a spherical isometry of class $C_{\cdot 0}$ and assume that $M \subset H$ is a reducing subspace for T such that $N = T|M \in B(M)^n$ is a spherical unitary tuple. In order to show that M must be the zero space, first note that the identity $\Phi_T(p)^*|M = (\Phi_T(p)|M)^* = \Phi_N(p)^*$, trivially valid for polynomials $p \in \mathbb{C}[z]$, extends by continuity to all functions $p \in A(\mathbb{B}_n)$, proving that N, viewed as a spherical isometry, is of class $C_{\cdot 0}$.

Consider the isomorphism of von Neumann algebras $\Psi_N : L^\infty(\nu) \to W^*(N)$ associated with the tuple N by the spectral theory for normal tuples (as usual, the scalar-valued spectral measure of N is regarded as a measure $\nu \in M^+(\partial\mathbb{B}_n)$). Then $\mathcal{A}_N = \Psi_N(H^\infty(\nu))$ and by Lemma 1.1 and the lemma above, there exists a sequence of isometries $J_k = \Psi_N(\theta_k)$ with inner functions $\theta_k \in H^\infty(\nu)$ such that $J_k^* \to 0$ (SOT) on M. But since in this special case the isometries $J_k = \Psi_N(\theta_k)$ are even unitaries (with inverse $\Psi_N(\overline{\theta}_k)$), we see that

$$\|x\| = \|J_k^* x\| \stackrel{k\to\infty}{\longrightarrow} 0 \qquad (x \in M).$$

This proves that M is the zero-space, as we claimed. □

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