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Abstract

We investigate the steady flow of a shear thickening generalized Newtonian fluid under homogeneous boundary conditions on a domain in $\mathbb{R}^2$. We assume that the stress tensor is generated by a potential of the form $H = h(|\varepsilon(u)|)$, $\varepsilon(u)$ denoting the symmetric part of the velocity gradient. We prove the existence of strong solutions for a large class of functions h having the property that $h'(t)/t$ increases (shear thickening case).

1 Introduction

In her monograph [La] (compare p.193) Ladyzhenskaya suggests to investigate "new equations for the description of the motion of viscous incompressible fluids", which roughly speaking means to consider viscosity coefficients, which depend on the modulus of the symmetric gradient $\varepsilon(u) = (\varepsilon(u)_{ij})$, $\varepsilon(u)_{ij} := \frac{1}{2}(\partial_i u^j + \partial_j u^i)$, of the velocity field u in a for example "monotonically increasing" way (shear thickening case). In our note we will contribute to this problem in a very special situation restricting ourselves to stationary flows through a bounded domain Ω in $\mathbb{R}^2$ with $\partial\Omega$ being of class C^1. Then we are looking for a velocity field $u : \Omega \to \mathbb{R}^2$ and a pressure function $\pi : \Omega \to \mathbb{R}$ satisfying the following set of equations

$$(1.1) \qquad \left\{ \begin{array}{l} -\operatorname{div}\left[T(\varepsilon(u))\right] + u^k \partial_k u + \nabla\pi = g \quad \text{in } \Omega\,, \\ \operatorname{div} u = 0 \quad \text{in } \Omega\,, u = 0 \quad \text{on } \partial\Omega\,. \end{array} \right.$$

We assume for simplicity that the given system of volume forces $g : \Omega \to \mathbb{R}^2$ belongs to the class $L^\infty(\Omega;\mathbb{R}^2)$ and that the tensor T is the gradient of a potential $H : \mathbb{S}^2 \to \mathbb{R}$ defined on the space $\mathbb{S}^2$ of all symmetric (2×2) matrices. Throughout this paper we adopt the summation convention, which means that the sum is taken with respect to indices repeated twice. The choice $H(\varepsilon) := \frac{\nu}{2}|\varepsilon|^2$ for some $\nu > 0$ leads to the stationary Navier–Stokes system, which is analyzed in great detail in the monographs of Ladyzhenskaya [La] and Galdi [Ga1,2]. The first extension in the spirit of Ladyzhenskaya's suggestions concerns so–called power growth potentials, for which it is required that $H(\varepsilon)$ behaves – in a sense to be made precise – like $|\varepsilon|^p$ for some exponent $p \in (1,\infty)$. Here existence and (interior) regularity results are due to Kaplický, Málek and Stará [KMS] and to Wolf [Wo] imposing lower bounds like $p > 6/5$ on p, whereas Frehse, Málek and Steinhauer [FMS] proved the existence of a weak solution to (1.1) for any $p > 1$. In connection with power law fluids we should also mention the global regularity results of Beirão da Veiga [BdV1–3] and his recent joint work [BKR] with Kaplický and Růžička. Anisotropic potentials H of (p,q)-growth are the subject of the papers [ABF] and [BFZ] exhibiting conditions like $q < p + 2$ as a sufficient hypothesis for the local regularity of a weak solution. Later on global results for the anisotropic case were established by Kaplický [Ka] under certain restrictions on the exponents p and q. Very recently the author [Fu1] discussed slow flows, i.e. $u^k \partial_k u \equiv 0$, under the assumption that

$$H(\varepsilon) = h(|\varepsilon|), \varepsilon \in \mathbb{S}^2 , \tag{1.2}$$

holds for a function $h : [0,\infty) \to [0,\infty)$ of class C^2. From (1.2) it follows that

$$DH(\varepsilon) = \frac{h'(|\varepsilon|)}{|\varepsilon|}\varepsilon ,$$

hence the viscosity coefficient is given by $\frac{h'(|\varepsilon|)}{|\varepsilon|}$. As in [Fu1] we will concentrate on shear thickening fluids, which means by definition (see, e.g. [MNRR], Definition 1.68 on p.14) that the viscosity is an increasing function of $|\varepsilon|$, and our goal is to prove the existence of rather regular solutions u and π to (1.1). The reader should note that due to the presence of the convective term in (1.1) it will not be possible to apply the variational approach used in [Fu1]. We also like to emphasize that no upper bound for D^2H is needed in order to establish the existence of a well–behaved weak solution u to (1.1). More precisely, we impose the following hypotheses on h:

$$\text{(A1)} \qquad \begin{cases} h \text{ is strictly increasing and convex together with} \\ h''(0) > 0 \text{ and } \lim\limits_{t\downarrow 0} \frac{h(t)}{t} = 0\,; \end{cases}$$

$$\text{(A2)} \qquad \begin{array}{l} h \text{ satisfies the } (\Delta 2) - \text{condition globally}\,, \\ \text{thus there is a constant } \overline{k} > 0 \text{ with the property} \\ h(2t) \leq \overline{k}\, h(t) \text{ for all } t \geq 0\,; \end{array}$$

$$\text{(A3)} \qquad \frac{h'(t)}{t} \leq h''(t) \quad \text{for any } t \geq 0\,.$$

Let us draw a few consequences:

i) We have

$$t\,h'(t) = \int_0^t \frac{d}{ds}[sh'(s)]\; ds = h(t) + \int_0^t s\,h''(s)\; ds \overset{\text{(A3)}}{\geq} 2h(t)\,,$$

hence

$$a(h) := \inf_{t>0} \frac{h'(t)t}{h(t)} \geq 2\,,$$

and together with (A1) and (A2) the latter inequality means that h is a N-function of (global) type $(\Delta 2) \cap (\nabla 2)$, we refer the reader to Corollary 4 on p.26 in the textbook [RR] of Rao and Ren. This implies that the Orlicz-Sobolev class $W^1_h(\Omega;\mathbb{R}^2)$ and its subspace $\overset{\circ}{W}{}^1_h(\Omega;\mathbb{R}^2) := \{u \in W^1_h(\Omega;\mathbb{R}^2) : u|_{\partial\Omega} = 0\}$ are reflexive (cf. [Ad] for definitions and results) and that variants of Korn's inequality are availabe, we refer to Lemma 2.3.

ii) From (A3) it easily follows that $h'(t)/t$ is increasing, thus we are in the shear thickening case, and it holds

$$h(t) \geq \frac{1}{2} h''(0) t^2 , \; t \geq 0 .$$

iii) Exactly as in [Fu1] we obtain the balancing condition

$$\text{(B)} \qquad \tfrac{1}{\overline{k}} h'(t) t \leq h(t) \leq t\, h'(t), \; t \geq 0 \; ,$$

as well as the bounds

$$\text{(1.3)} \qquad h(t) \leq c(t^m + 1) \, ,$$

$$\text{(1.4)} \qquad h'(t) \leq c(t^{m-1} + 1) \, , \; t \geq 0 \, ,$$

for a positive constant c and an exponent m being determined just through the constant $\overline{k}$ from (A2). Note that (1.4) directly follows from (1.3) and the convexity of h.

Now we can state our main result:

THEOREM 1.1. *Under the assumptions (A1-3) and (1.2) there exists at least one weak solution* $u \in \overset{\circ}{W}{}^1_h(\Omega; \mathbb{R}^2)$ *of problem (1.1), i.e.* u *satisfies* $\operatorname{div} u = 0$ *together with (*$\otimes$ *denoting the tensor product of vectors in* $\mathbb{R}^2$*)*

$$\int_\Omega DH(\varepsilon(u)) : \varepsilon(\varphi) \, dx - \int_\Omega u \otimes u : \varepsilon(\varphi) \, dx = \int_\Omega g \cdot \varphi \, dx$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^2)$ *such that* $\operatorname{div} \varphi = 0$*. Moreover,* u *belongs to the class* $W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^2)$*. If in addition we know*

$$\text{(A4)} \qquad h''(t) \leq c(t^s + 1) \, , \; t \geq 0 \, ,$$

for some constant $c > 0$ *and an arbitrary exponent* s*, then there exists a pressure function* $\pi \in \bigcap_{1 \leq r < 2} W^1_{r,\mathrm{loc}}(\Omega)$ *such that the system of partial differential equations stated in (1.1) holds almost everywhere on* Ω*.*

REMARK 1.1. *The reader should note that (A4) implies the validity of*

$$\text{(1.5)} \qquad h''(t) \leq c(1 + t^2)^{\frac{\alpha}{2}} \frac{h'(t)}{t} \, , \; t \geq 0 \, ,$$

for some suitable exponent α*. Conversely, if we have (A1-3) and (1.5), then (A4) follows from (1.4) for an appropriate choice of* s*. Suppose now that (A1-3) hold and that (1.5) is true with* $\alpha < 2$*. Then it is easy to modify the arguments from [Fu1], Step 2, and to* *prove that the solution u from Theorem 1.1 actually belongs to the space* $C^{1,\mu}(\Omega;\mathbb{R}^2)$*. If we allow values of* α *in* $[2,\infty)$*, then - with some additional work - we can deduce from [BrF] that* $u \in C^{1,\mu}(\Omega_0;\mathbb{R}^2)$ *for an open set* Ω_0 *of* Ω *such that* $\mathcal{H}-\dim(\Omega-\Omega_0)=0$*. In fact, going through the blow-up procedure presented in [BrF] taking care of the fact that now* $n=2$ *and that we already know the existence of the second weak partial derivatives in* $L^2_{\rm loc}(\Omega)$*, then it turns out that the excess function defined on discs* $B_r(x_0)$ *vanishes for all* $x_0 \in \Omega$ *as* $r \to 0$*. Therefore* $x_0 \in \Omega$ *is a regular point of u, i.e. u is* C^1 *in a neighborhood of* x_0*, if and only if* $\sup\limits_{r>0} |(\varepsilon(u))_{x_0,r}| < \infty$*, and this condition holds on a subset of* Ω *whose complement has Hausdorff dimension zero. Thus, under the assumptions (A1-4), we have constructed a field u, which solves (1.1) in the classical sense on a very large open subset of* Ω*, however we believe that interior* $C^{1,\mu}$*-regularity is true just under the hypotheses (A1-3), but we are unable to prove this conjecture.*

We finish the introduction by discussing an example of a density h satisfying (A1-3) and (1.5) exactly with a given exponent α. It turns out that for this potential the viscosity function $g(t)=\frac{h'(t)}{t}$ is bounded and that on large parts of $[0,\infty)$ the derivative $\Theta(t) := g'(t)$ of the viscosity function equals zero. The construction works like this:

- we start with a suitable function Θ;
- the viscosity function is introduced via the formula $g(t) := 1+\int_0^t \Theta(s)\,ds$;
- then we let $h(t) := \int_0^t sg(s)\,ds$.

To be precise consider a sequence $\{a_i\}$ of numbers a_i such that $0 \ll a_i < a_{i+1}$ and $\lim\limits_{i\to\infty} a_i = \infty$. We choose $\varepsilon_i > 0$ such that

$$I_i \cap I_j = \phi,\ \text{if } i \neq j,\ I_i := (a_i-\varepsilon_i, a_i+\varepsilon_i)\,,$$

and

(1.6) $$\sum_{i=1}^{\infty} \varepsilon_i a_i^{\alpha-1} < \infty\,.$$

We define the continuous function $\Theta : [0,\infty) \to [0,\infty)$ through the equation

$$\theta(t) := \begin{cases} 0 \text{ on } [0,\infty) - \bigcup\limits_{i=1}^{\infty} I_i\,, \\ \text{affine linear on } (a_i-\varepsilon_i, a_i) \quad \text{and} \\ \text{on } (a_i, a_i+\varepsilon_i) \\ \text{with value } a_i^{\alpha-1} \text{ at } t=a_i,\ i\in\mathbb{N} \end{cases}$$

and obtain

$$g(t) \le 1+\int_0^\infty g'(s)\,ds = 1+\int_0^\infty \Theta(s)\,ds = 1+\sum_{i=1}^{\infty}\varepsilon_i a_i^{\alpha_i-1}\,,$$

hence $g \in L^\infty([0,\infty))$ by (1.6). Letting h as described before, the validity of (A1) is immediate. For (A3) we observe

$$h''(t) = \frac{d}{dt}(tg(t)) = g(t) + tg'(t) \geq g(t) = \frac{1}{t}h'(t)\,.$$

Let us look at (1.5): the validity of (1.5) is equivalent to the existence of a constant c such that

$$tg'(t) \leq c\,t^\alpha g(t) \tag{1.7}$$

holds for all large t. The left-hand side of (1.7) is equal to $t\Theta(t)$, and a lower bound for the right-hand side is given by $c\,t^\alpha g(0) = c\,t^\alpha$, so that our claim (1.7) follows from the definition of Θ. The reader should note that it is not possible to replace α in (1.7) (and consequently in (1.5)) by a smaller exponent $\overline{\alpha}$. The $(\Delta 2)$ condition required in (A2) is a consequence of the estimate

$$1 = g(0) \leq g(t) \leq g_\infty,\ t \geq 0\,,$$

for a finite constant g_∞, since this inequality yields $g(2t) \leq g_\infty g(t)$ and therefore for all $t \geq 0$

$$h(2t) = \int_0^{2t} sg(s)\,ds = 4\int_0^t sg(2s)\,ds \leq 4g_\infty \int_0^t sg(s)\,ds = 4g_\infty h(t)\,.$$

Altogether we have constructed a potential $H(\varepsilon) = h(|\varepsilon|)$ being of quadratic growth in the sense that $H(\varepsilon)$ can be bounded from above and from below in terms of $|\varepsilon|^2$, but whose ellipticity behaviour (expressed through (A3) and (1.5)) at least for exponents $\alpha \geq 2$ is so bad that no regularity results for the weak solution defined in Theorem 1.1 can be deduced by applying the methods used in [BFZ] or [Fu1]. However, as discussed in Remark 1.1, it is still possible to cover the example for all values of α and to obtain a solution of class $W^2_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2)$ with very small singular set.

Our paper is organized as follows: in Section 2 we replace problem (1.1) by a suitable sequence of approximate problems, which are obtained by regularizing H from below. Moreover, we prove some basic energy estimates for the corresponding solutions. Section 3 is devoted to the study of the higher differentiability properties and the verification of local uniform bounds for the second derivatives of the solutions of the auxiliary problems. In Section 4 we will pass to the limit by the way proving Theorem 1.1.

2 Regularization and apriori energy estimates

We here replace our potential H by a suitable sequence $\{H_\ell\}_{\ell\in\mathbb{N}}$ approximating H from below. The following lemma has been established in [BF], Section 3.

Lemma 2.1. *Suppose that we have (A1-3) for the function h. For* $\ell \in \mathbb{N}$ *let* $\eta_\ell \in C^1([0,\infty))$ *such that* $0 \le \eta_\ell \le 1$, $\eta_\ell' \le 0$, $|\eta_\ell'| \le c/\ell$, $\eta_\ell \equiv 1$ *on* $[0, \frac{3}{2}\ell]$ *and* $\eta \equiv 0$ *on* $[2\ell, \infty)$. *We further define*

$$h_\ell(t) := \int_0^t s\, g_\ell(s)\; ds,\;\; t \ge 0\,,$$

where $g_\ell(t) := g(0) + \int_0^t \eta_\ell(s) g'(s)\; ds$ *and* $g(t) := h'(t)/t$, $t \ge 0$. *Then it holds:*

a) h_ℓ *satisfies (A1-3), and the constant in (A2) can be chosen uniformly in* ℓ.

b) $h_\ell(t) = h(t)$ *for all* $t \le \frac{3}{2}\ell$, $H_\ell(\sigma) := h_\ell(|\sigma|) \le h(|\sigma|) = H(\sigma)$, $\lim_{\ell\to\infty} H_\ell(\sigma) = H(\sigma)$, $\sigma \in \mathbb{S}^2$.

c) H_ℓ *is of quadratic growth, which follows from*

$$c|\tau|^2 \le D^2 H_\ell(\sigma)(\tau,\tau) \le \Lambda(\ell)|\tau|^2, \sigma, \tau \in \mathbb{S}^2\,, \tag{2.1}$$

with $c > 0$ *being independent of* ℓ, *but* $\Lambda(\ell)$ *not necessarily bounded as* $\ell \to \infty$.

REMARK 2.1. *From part a) of Lemma 2.1 it follows that we have a uniform balancing condition for the functions* h_ℓ, *i.e. (compare (B) form Section 1)*

$$c\, h_\ell'(t)t \le h_\ell(t) \le t h_\ell'(t),\;\; t \ge 0, \ell \in \mathbb{N}\,, \tag{2.2}$$

with a suitable positive constant. Here and in what follows we agree to denote all constants not depending on ℓ *by the same symbol c, but the value of c may of course change from line to line. As before (A3) implies that* $h_\ell'(t)/t$ *is increasing, hence* $\frac{h_\ell'(t)}{t} \ge h_\ell''(0) = h''(0)$ *and therefore*

$$h_\ell(t) \ge \frac{1}{2} h''(0) t^2,\;\; t \ge 0\,. \tag{2.3}$$

With h_ℓ, H_ℓ from Lemma 2.1 we consider the problem to find $u_\ell \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^2)$, $\operatorname{div} u_\ell = 0$, such that

$$\int_\Omega DH_\ell(\varepsilon(u_\ell)) : \varepsilon(\varphi)\, dx - \int_\Omega u_\ell \otimes u_\ell : \varepsilon(\varphi)\, dx = \int_\Omega g \cdot \varphi\, dx \tag{2.4}$$

holds for all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^2)$, $\operatorname{div} \varphi = 0$.

Lemma 2.2. *Let the assumptions (A1–3) together with (1.2) hold. Then (2.4) admits at least one solution* u_ℓ, *which belongs to the class* $W^2_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2) \cap C^{1,\alpha}(\Omega;\mathbb{R}^2)$ *for any* $0 < \alpha < 1$.

Proof: The existence of at least one solution $u_\ell \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^2)$, $\operatorname{div} u_\ell = 0$, of (2.4) can be obtained along standard lines as described for example in Theorem 1, p.116, of Ladyzhenskaya's monograph [La]. We also refer to Theorem 1.3 of [ABF], where one has to choose $f := H_\ell$ and (recall (2.1)) $p = q_0 = 2$. The interior regularity properties of u_ℓ can be deduced from the work of Kaplický, Málek and Stará [KMS], where besides many other results it is shown (compare Theorem 3.19 of [KMS]) that in the case of quadratic potentials any weak solution of (2.4) is regular up to the boundary provided $\partial\Omega$ is of class C^2. Alternatively we may quote Theorem 1.5 of [ABF] in order to obtain the interior regularity of the solution u_ℓ constructed as outlined in Theorem 1.3 of [ABF]. $\square$

As remarked in Section 1 it follows from (A3) that h satisfies

$$a(h) := \inf_{t>0} \frac{th'(t)}{h(t)} \geq 2$$

and therefore h is of (global) type $(\nabla 2)$. Since (A3) is also valid for h_ℓ with the consequence that again $a(h_\ell) \geq 2$, and since we have the (global) $(\Delta 2)$-condition for h_ℓ and h with uniform constant, we get from [Fu2]

Lemma 2.3. *For a finite constant c it holds*

a) $\int_\Omega h\left(|\nabla v|\right) dx \leq c \int_\Omega h\left(|\varepsilon(v)|\right) dx,\ v \in \overset{\circ}{W}{}^1_h(\Omega;\mathbb{R}^2)$,

b) $\int_\Omega h_\ell\left(|\nabla w|\right) dx \leq c \int_\Omega h_\ell\left(|\varepsilon(w)|\right) dx,\ w \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^2)$.

We wish to remark that these Korn-type inequalities are consequences of the gradient estimates for elliptic equations in Orlicz spaces obtained by Jia, Li and Wang [JLW] and by Byun, Yao and Zhou [BYZ]. As it was kindly pointed out to us by S. Zhou [Zh] the uniformity of $(\Delta 2)$ and $(\nabla 2)$ for h_ℓ implies the validity of Lemma 2.3 b) with a constant not depending on ℓ, since in this case the gradient estimates for h_ℓ hold with bounded constants c_ℓ.

Now we can state the basic energy estimate valid for the sequence of approximations.

Lemma 2.4. *Under the assumptions and with the notation from Lemma 2.2 we have*

$$\sup_\ell \int_\Omega h_\ell(|\nabla u_\ell|)\, dx < \infty\,. \tag{2.5}$$

REMARK 2.2. *From (2.3) and (2.5) it follows that*

$$\sup_\ell \|u_\ell\|_{W^1_2(\Omega;\mathbb{R}^2)} < \infty \tag{2.6}$$

is true.

Proof of Lemma 2.4: We choose $\varphi := u_\ell$ in (2.4) and observe that $\int_\Omega u_\ell \otimes u_\ell : \varepsilon(u_\ell)\,dx = 0$, therefore (2.2) together with (2.4) implies

$$\int_\Omega h_\ell(|\varepsilon(u_\ell)|)\,dx \le \|g\|_{L^\infty(\Omega)} \int_\Omega |u_\ell|\,dx\,. \tag{2.7}$$

From Corollary 1.11 of [AG] we deduce

$$\int_\Omega |u_\ell|\,dx \le c \int_\Omega |\varepsilon(u_\ell)|\,dx\,, \tag{2.8}$$

and by combining (2.3) with (2.8), it is evident how to get from (2.7) the boundedness of $\int_\Omega h_\ell(|\varepsilon(u_\ell)|)\,dx$, and a final application of Lemma 2.3 b) leads to (2.5). □

3 Local bounds for the second weak derivatives of the approximation

Suppose that we have (A1-3) together with (1.2). We then consider the solutions u_ℓ of problem (2.4) introduced in Lemma 2.2. For notational simplicity we will drop the index ℓ, but we emphasize that all constants "c" occurring in the subsequent calculations are actually independent of ℓ. We multiply the equation (π denoting a suitable pressure function)

$$-\operatorname{div}\left[DH(\varepsilon(u))\right] + u^i\partial_i u + \nabla\pi = g$$

with $\partial_k\varphi,\ \varphi \in C_0^\infty(\Omega;\mathbb{R}^2)$, integrate over Ω and obtain ($\sigma := DH(\varepsilon(u))$)

$$\int_\Omega \partial_k\sigma : \varepsilon(\varphi)\,dx - \int_\Omega u^i\partial_i u\cdot\partial_k\varphi\,dx - \int_\Omega \nabla\pi\cdot\partial_k\varphi\,dx = -\int_\Omega g\cdot\partial_k\varphi\,dx\,. \tag{3.1}$$

Choosing $\varphi := \eta^2\partial_k u$ in (3.1) for a cut-off function η (from now on we again take the sum with respect to indices repeated twice) we arrive at (using integration by parts)

$$\begin{aligned}
&\int_\Omega \partial_k\sigma : \varepsilon(\partial_k u)\eta^2\,dx \\
&= 2\int_\Omega \sigma : \partial_k\left[\eta\,\nabla\eta\odot\partial_k u\right]\,dx - 2\int_\Omega \pi\partial_k(\eta\,\nabla\eta\cdot\partial_k u)\,dx \\
&\quad + \int_\Omega u^i\partial_i u\cdot\partial_k(\eta^2\partial_k u)\,dx - \int_\Omega g\cdot\partial_k(\eta^2\partial_k u)\,dx \\
&:= T_1 + T_2 + T_3 + T_4\,.
\end{aligned} \tag{3.2}$$

Here $\odot$ denotes the symmetric product of vectors in $\mathbb{R}^2$. Note that (3.2) corresponds to (21) in [Fu1], T_1, T_2 having the same meaning as in this reference. Let us fix discs $B_r(z) \subset B_R(z)$ compactly contained in Ω, where $R < 1$, and let $\eta \in C_0^1(B_R(z))$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(z)$ and $|\nabla^k\eta| \le c/(R-r)^k$, $k = 1, 2$. Then, starting from (3.2), some modifications of the calculations leading to (25) in [Fu1] (see Appendix) show the validity of the estimate

$$
\begin{aligned}
(3.3) \int_{B_R(z)} & \eta^2 \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \, dx \\
\leq\ & c(R-r)^{-2} \left[\int_{B_R(z)} h(|\nabla u|) \, dx + \int_{B_R(z)} h'(|\varepsilon(u)|)^2 \, dx + 1 \right] + c \left[|T_3| + |T_4| \right] .
\end{aligned}
$$

Quoting inequality (2.5) we obtain a uniform bound for the quantity $\int_{B_R(z)} h(|\nabla u|)\, dx$ and this bound is also used in the discussion of T_4:

$$
\begin{aligned}
|T_4| \leq 2 & \int_{B_R(z)} |g||\nabla \eta|\eta|\nabla u| \, dx + \int_{B_R(z)} |g|\eta^2 |\nabla^2 u| \, dx \\
\leq\ & c\|g\|_{L^\infty(\Omega)} (R-r)^{-1} \int_\Omega (h(|\nabla u|) + 1) \, dx + c \int_{B_R(z)} |g|\eta^2 |\nabla \varepsilon(u)| \, dx \\
\leq\ & c(R-r)^{-2} + c\|g\|_{L^\infty(\Omega)} \int_{B_R(z)} \eta^2 |\nabla \varepsilon(u)| \, dx \, .
\end{aligned}
$$

Recalling the estimate stated before (2.3) and applying Young's inequality to the last integral, we deduce from (3.3)

$$
(3.4) \qquad \int_{B_R(z)} \eta^2 \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \, dx \leq c(R-r)^{-2} \left[1 + \int_{B_R(z)} h'(|\varepsilon(u)|)^2 \, dx \right] + c|T_3| \, .
$$

We further have

$$
\begin{aligned}
T_3 = & \int_{B_R(z)} u^i \partial_i u^j \partial_k (\eta^2 \partial_k u^j) \, dx \\
=\ & -\int_{B_R(z)} \partial_k (u^i \partial_i u^j) \eta^2 \partial_k u^j \, dx \\
=\ & -\int_{B_R(z)} \partial_k u^i \partial_i u^j \partial_k u^j \eta^2 \, dx - \int_{B_R(z)} u^i \partial_k \partial_i u^j \partial_k u^j \eta^2 \, dx \\
=\ & -\int_{B_R(z)} \partial_k u^i \partial_i u^j \partial_k u^j \eta^2 \, dx - \frac{1}{2} \int_{B_R(z)} u^i \partial_i |\nabla u|^2 \eta^2 \, dx \\
=\ & -\int_{B_R(z)} \partial_k u^i \partial_i u^j \partial_k u^j \eta^2 \, dx + \int_{B_R(z)} u \cdot \nabla \eta \, \eta |\nabla u|^2 \, dx \, ,
\end{aligned}
$$

and since we are in the $2D$-case, it is easy to see that the second last integral vanishes (cf. [MNRR]). Let $\delta > 0$ denote some arbitrary number. From Young's inequality and the discussion from above we infer

$$
\begin{aligned}
|T_3| \leq \delta & \int_{B_R(z)} |\nabla u|^4 \, dx + c(\delta)(R-r)^{-2} \int_{B_R(z)} |u|^2 \, dx \\
\leq\ & \delta \int_{B_R(z)} |\nabla u|^4 \, dx + c(\delta)(R-r)^{-2} \int_\Omega |u|^2 \, dx \\
\leq\ & \delta \int_{B_R(z)} |\nabla u|^4 \, dx + c(\delta)(R-r)^{-2} \int_\Omega |\nabla u|^2 \, dx \, ,
\end{aligned}
$$

hence (2.6) implies

$$
(3.5) \qquad |T_3| \le \delta \int_{B_R(z)} |\nabla u|^4 \, dx + c(\delta)(R-r)^{-2} \, .
$$

From Korn's inequality in L^p (see, e.g. [MM] or [Re]) we get

$$
(3.6) \qquad \int_{B_R(z)} |\nabla u|^4 \, dx \le c \left\{ \int_{B_R(z)} |\varepsilon(u)|^4 \, dx + R^{-4} \int_{B_R(z)} |u|^4 \, dx \right\} ,
$$

and with the help of Sobolev's embedding theorem and (2.6) we can bound the integral of $|u|^4$ through a constant. Finally we observe (2.3) and see after combining (3.5) and (3.6)

$$
|T_3| \le \delta c \int_{B_R(z)} h\left(|\varepsilon(u)|\right)^2 \, dx + c(\delta)(R-r)^{-4} \, .
$$

Inserting this estimate into (3.4) it is shown that

$$
(3.7) \qquad \int_{B_r(z)} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \, dx \le \delta c \int_{B_R(z)} h(|\varepsilon(u)|)^2 \, dx
$$

$$
+ c(R-r)^{-2} \left[1 + \int_{B_R(z)} h'(|\varepsilon(u)|)^2 \, dx \right] + c(\delta)(R-r)^{-4} ,
$$

and (3.7) is true for any choices of $\delta > 0$, r, $R > 0$ such that $B_r(z) \subset B_R(z) \Subset \Omega$. To the integral $\int_{B_R(z)} h'(|\varepsilon(u)|)^2 \, dx$ occurring on the right-hand side of (3.7) we can apply exactly the same arguments as used after (25) in [Fu1]. At this stage we emphasize that we use (2.2) as well as (1.3), (1.4) for h_ℓ but with exponent m (related to h), which of course is possible, and guarantees the validity of (1.3), (1.4) and (2.2) with uniform constants and uniform exponent m. As a result we end up with

$$
(3.8) \qquad \int_{B_r(z)} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \, dx \le \delta c \int_{B_R(z)} h(|\varepsilon(u)|)^2 \, dx + c(\delta)(R-r)^{-\gamma} ,
$$

where γ denotes some suitable positive exponent and where δ, r, R are as in (3.7). Note that (3.8) corresponds to (26) in [Fu1] and as outlined there we deduce from (3.8) that the functions $h(|\varepsilon(u)|)$ are in $L^2_{\text{loc}}(\Omega)$ uniformly with respect to the parameter ℓ. But then (3.8) implies

$$
\frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \in L^1_{\text{loc}}(\Omega)
$$

uniformly and recalling $\frac{h'_\ell(t)}{t} \ge h''_\ell(0) = h''(0)$ we finally arrive at

$$
(3.9) \qquad \|u_\ell\|_{W^2_2(\Omega')} \le c(\Omega') < \infty
$$

for any subdomain $\Omega' \Subset \Omega$. □

4 Passage to the limit

Let (A1-3) together with (1.2) hold. We consider the sequence $\{u_\ell\}$ from Lemma 2.2. According to (2.6) and (3.9) there exists a function

$$u \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^2) \cap W^2_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2) \tag{4.1}$$

such that after passing to a suitable subsequence it holds

$$\begin{array}{ll} \text{i)} & u_\ell \rightharpoondown u \text{ in } W^1_2(\Omega;\mathbb{R}^2)\,, \\ \text{ii)} & u_\ell \to u \text{ in } L^2(\Omega;\mathbb{R}^2) \text{ and a.e.}\,, \\ \text{iii)} & u_\ell \rightharpoondown u \text{ in } W^2_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2)\,, \\ \text{iv)} & u_\ell \to u \text{ in } W^1_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2) \text{ and } \nabla u_\ell \;\; \to \nabla u \quad \text{a.e.}\,. \end{array} \tag{4.2}$$

We claim that in addition to (4.1) we have

$$u \in \overset{\circ}{W}{}^1_h(\Omega;\mathbb{R}^2)\,. \tag{4.3}$$

In order to prove (4.3) let $\ell < m$. Then by (2.5) and the observation that $h_\ell \le h_m$ we get

$$\int_\Omega h_\ell(|\nabla u_m|)\,dx \le \int_\Omega h_m(|\nabla u_m|)\,dx \le \text{ const}\,,$$

and from (4.2)i) it follows (for all $\ell \in \mathbb{N}$) by lower semicontinuity

$$\int_\Omega h_\ell(|\nabla u|)\,dx \le \text{ const}\,.$$

Letting $\ell \to \infty$ we arrive at (4.3). Let us return to (2.4) and fix $\varphi \in C^\infty_0(\Omega;\mathbb{R}^2)$. Observing

$$|DH_\ell(\xi)| = h'_\ell(|\xi|) \le c(|\xi|^{m-1} + 1)$$

and recalling that by (4.2)iii) we have strong local convergence of ∇u_ℓ towards ∇u in any space $L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^{2\times 2})$ it is immediate that

$$\lim_{l\to\infty} \int_\Omega DH_\ell(\varepsilon(u_\ell)) : \varepsilon(\varphi)\,dx = \int_\Omega DH(\varepsilon(u)) : \varepsilon(\varphi)\,dx$$

("equi-integrability + pointwise convergence"), hence u satisfies the limit equation

$$\int_\Omega DH(\varepsilon(u)) : \varepsilon(\varphi)\,dx - \int_\Omega u \otimes u : \varepsilon(\varphi)\,dx = \int_\Omega g \cdot \varphi\,dx,\ \varphi \in C^\infty_0(\Omega;\mathbb{R}^2), \operatorname{div}\varphi = 0\,. \tag{4.4}$$

This proves the first part of Theorem 1.1. Suppose now that we have (A4) as additional hypothesis. Then by (A4) and (1.4) we see that $|D^2H(\xi)|$ grows at most as a power of $|\xi|$, so that

$$\big|\operatorname{div}[DH(\varepsilon(u))]\big| \in L^p_{\mathrm{loc}}(\Omega)$$

for any $p < 2$ on account of (4.1) and Sobolev's embedding theorem. The existence of a pressure function π with the stated properties then follows along standard lines, we refer to [Ga1], Lemma 1.1 on p.180. □

5 Appendix

Here we briefly indicate the modifications in the pressure estimate from [Fu1], which are necessary to derive (3.3). In a first step - following the arguments leading to (24) in [Fu1] - it is easy to see that in place of (3.3) we obtain

$$
\begin{aligned}
(5.1)\quad & \int_{B_R(z)} \eta^2 \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \, dx \\
&\leq \; c(R-r)^{-2} \left[\int_{B_R(z)} h(|\nabla u|) \, dx + \int_{B_R(z)} |\pi - \pi_0|^2 \, dx \right] + c \left[|T_3| + |T_4| \right] ,
\end{aligned}
$$

where T_3, T_4 denote the same quantities as in (3.3) and where π_0 is the mean value of the pressure π with respect to $B_R(z)$. According to [Ga1], Chapter III, Theorem 3.2 we can find a field $w \in \overset{\circ}{W}{}^1_2(B_R(z); \mathbb{R}^2)$ with the properties

$$
(5.2)\qquad \begin{aligned}
&\operatorname{div} w = \pi - \pi_0 \quad \text{on } B_R(z) , \\
&\|\nabla w\|_{L^2(B_R(z))} \leq c \|\pi - \pi_0\|_{L^2(B_R(z))} .
\end{aligned}
$$

Letting $\sigma := DH(\varepsilon(u))$ we obtain

$$
\begin{aligned}
&\int_{B_R(z)} (\pi - \pi_0) \operatorname{div} w \, dx = \int_{B_R(z)} \sigma : \varepsilon(w) \, dx + S_1 + S_2 , \\
&S_1 := -\int_{B_R(z)} g \cdot w \, dx , \; S_2 := \int_{B_R(z)} u^i \partial_i u \cdot w \, dx ,
\end{aligned}
$$

and Young's inequality combined with (5.2) implies

$$
(5.3)\qquad \int_{B_R(z)} |\pi - \pi_0|^2 \, dx \leq c \left[\int_{B_R(z)} |\sigma|^2 \, dx + |S_1| + |S_2| \right] .
$$

Since g is a bounded function, we have by Poincarés inequality and (5.2) (recall $R < 1$)

$$
\begin{aligned}
|S_1| &\leq \delta \int_{B_R(z)} |\nabla w|^2 \, dx + c(\delta) \|g\|^2_{L^\infty(\Omega)} \\
&\leq c \, \delta \int_{B_R} |\pi - \pi_0|^2 \, dx + c(\delta) \|g\|^2_{L^\infty(\Omega)} ,
\end{aligned}
$$

and (5.3) yields after appropriate choice of δ

$$
(5.4)\qquad \int_{B_R(z)} |\pi - \pi_0|^2 \, dx \leq c \left[\int_{B_R(z)} |\sigma|^2 \, dx + 1 + |S_2| \right]
$$

with constant c now depending on the norm of g. Observing the identity

$$
\int_{B_R(z)} u^i \partial_i u \cdot w \, dx = - \int_{B_R(z)} u^i u \cdot \partial_i w \, dx
$$

we see

$$|S_2| \le \delta \int_{B_R(z)} |\nabla w|^2 \, dx + c(\delta) \int_{B_R(z)} |u|^4 \, dx \,.$$

Recalling (2.6) and using Sobolev's embedding theorem, the last integral is bounded by a constant, thus (5.4) turns to

$$\int_{B_R(z)} |\pi - \pi_0|^2 \, dx \le c \left[\int_{B_R(z)} |\sigma|^2 \, dx + 1 \right] . \tag{5.5}$$

Since $|\sigma| = h'(|\varepsilon(u)|)$ by the definition of σ, the combination of (5.1) and (5.5) immediately leads to (3.3). □

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