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# New Korn-type inequalities and regularity of solutions to linear elliptic systems and anisotropic variational problems involving the trace-free part of the symmetric gradient

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**Abstract.** The aim of this note is to investigate a regularity theory for minimizers of energies whose density depends on the trace-free part of the symmetric gradient, where integrands of anisotropic growth are considered. An adequate coercive inequality guarantees the existence of minimizers of such energies in suitable Sobolev classes. Moreover, various other Korn-type inequalities are shown, which can be used to prove the smoothness of weak solutions to linear elliptic systems involving the trace-free part of the symmetric gradient. In particular, Campanato-type estimates for solutions to such systems are established so that all tools are available to prove the interior regularity of minimizers of energies depending on the trace-free part of the symmetric gradient.

**Keywords.** Generalized Korn inequalities · Linear elliptic systems · Campanato-type estimates · Variational problems · Nonstandard growth · Regularity

**Mathematics Subject Classification (2010).** 35E99, 35J47, 49J99, 49N60, 74B99, 83C99

## 1 Introduction

In a recent paper [7] Fuchs and the author prove a generalization of Korn's inequality, in which the symmetric gradient $\mathcal{E}v := \frac{1}{2}(\partial_i v^j + \partial_j v^i)$ of a vector field $v : \Omega \to \mathbb{R}^2$, defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, is replaced by its trace-free part $\mathcal{E}^D v := \mathcal{E}v - \frac{1}{2} \operatorname{div} v \,(\delta_{ij})$ ($\delta_{ij}$ denoting the Kronecker symbol). More precisely,

$$\|v\|_{1,p} \leqslant c\|\mathcal{E}^D v\|_p$$

holds for each vector field $v$ from the space $\overset{\circ}{W}{}^{1,p}(\Omega;\mathbb{R}^2)$, $p \in (1,\infty)$, of Sobolev functions with zero trace. Korn-type inequalities involving the trace-free part of the symmetric gradient have applications in general relativity, Cosserat elasticity, and geometry; compare [7], [9], [19], and the references therein.

As shown in [7], functionals of the type

$$\int_\Omega f(\mathcal{E}^D v)\,\mathrm{d}x$$

with an integrand $f$ of quadratic growth have a unique minimizer and are $C^{1,\alpha}$–regular under natural boundary and ellipticity conditions in the two–dimensional case. Functionals of the above type appear, for example, in general relativity and Cosserat elasticity; see [7] or [19] for some comments and further references.

One aim of this paper is to provide the tools, which are necessary to develop a regularity theory for minimizers of functionals of the above type in arbitrary dimensions and under nonstandard growth conditions. We establish Korn- and Poincaré-type inequalities involving $\mathcal{E}^D v := \mathcal{E}v - \frac{1}{n}\operatorname{div} v\,(\delta_{ij})$ for vector fields $v$ from the class $W^{1,p}(\Omega;\mathbb{R}^n)$ ($n \geqslant 2$) and discuss the regularity properties of solutions to linear elliptic systems involving the trace-free part of the symmetric gradient. Here, we show various Caccioppoli- and Campanato-type inequalities, which play a central role in regularity theory.

As an application we obtain some $C^{1,\alpha}$–regularity results for minimizers of functionals of the above type under anisotropic growth conditions. In [2] and [3] corresponding results are shown in the context of anisotropic power law fluids, where the functionals under consideration depend on the symmetric part of the gradient and are minimized in appropriate classes of solenoidal vector fields. The proofs of our regularity results follow the general line of these papers, but the arguments given there have to be adapted to our setting in a nontrivial way.

Let us give a detailed formulation of our regularity results: Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p \leqslant q < \infty$. Suppose that $f : \mathbb{M}^n \to [0,\infty)$ is a function of class $C^2$ with anisotropic growth in the following sense:

$$\lambda(1+|\sigma|^2)^{p/2-1}|\tau|^2 \leqslant D^2 f(\sigma)(\tau,\tau) \leqslant \Lambda(1+|\sigma|^2)^{q/2-1}|\tau|^2 \tag{1}$$

for all $\sigma, \tau \in \mathbb{M}^n$ with positive numbers $\lambda, \Lambda$. Here, $\mathbb{M}^n$ denotes the space of trace-free matrices of order $n$. We consider the functional

$$J[v] = J[v;\Omega] := \int_\Omega f(\mathcal{E}^D v)\,\mathrm{d}x \tag{2}$$

among vector fields $v$ from the class $\mathbb{K} := u_0 + \overset{\circ}{W}{}^{1,p}(\Omega;\mathbb{R}^n)$ with prescribed Dirichlet boundary data $u_0 \in W^{1,p}(\Omega;\mathbb{R}^n)$. Our main result is the following existence and regularity theorem, which extends the results from [7] and [19], where the case $p = q = 2$ is considered.

**Theorem 1.1.** *Let condition* (1) *hold and assume* $J[u_0] < \infty$.

*a) The minimization problem* $J \to \min$ *in* $\mathbb{K}$ *admits a unique solution* $u$.

*b) If* $n \geqslant 3$, $q \geqslant 2$ *and* $q < (1+2/n)p$, *there is an open set of full Lebesgue measure such that* $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^n)$ *for each* $\alpha \in (0,1)$.

*c) Let* $n = 2$ *and* $q < \min(2p, 2+p)$. *Then* $u \in C^{1,\alpha}(\Omega;\mathbb{R}^2)$ *for each* $\alpha \in (0,1)$.

**Corollary 1.2.** *Let* (1) *hold and suppose that* $u$ *is a local* $J$*–minimizer, that is,* $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ *fulfills for each subdomain* $\Omega' \Subset \Omega$ *the conditions*

$$J[u;\Omega'] < \infty \quad \textit{and} \quad J[u;\Omega'] \leqslant J[v;\Omega']$$

*for all* $v \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ *such that* $\operatorname{spt}(u-v) \Subset \Omega'$. *Then the statements b) and c) of Theorem 1.1 continue to hold.*

**Remark 1.3.** *It should be emphasized that in part b) of Theorem 1.1 the case* $q < 2$ *requires a different proof, which is in preparation. For some ideas concerning the subquadratic case in the framework of anisotropic power law fluids we refer to [2]. Clearly, for exponents* $1 < p \leqslant q < 2$ *the ellipticity condition* (1) *is satisfied with* $q$ *replaced by* $\overline{q} = 2$*. Therefore, we have partial regularity if* $2 < (1 + 2/n)p$*, that is,* $p > 2n/(n + 2)$ *so that values of* $p$ *and* $q$ *close to* 1 *are excluded.*

**Basic Notation.** We use the notations $L^p_{(\mathrm{loc})}$, $W^{k,p}_{(\mathrm{loc})}$, $\mathring{W}^{k,p}$, etc., for the standard Lebesgue and Sobolev spaces equipped with their standard norms $\|\cdot\|_p$, $\|\cdot\|_{k,p}$; see [1] for precise definitions and an overview. For vectors $a = (a_i)$, $b = (b_j) \in \mathbb{R}^n$ we use the notations

$$a \cdot b = a_i b_i, \quad |a| = \sqrt{a \cdot a},$$
$$a \otimes b = (a_i b_j), \quad a \odot b = \tfrac{1}{2}(a \otimes b + b \otimes a).$$

Moreover, we write

$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \quad |\sigma| = \sqrt{\sigma : \tau}, \quad \sigma^D = \sigma - \tfrac{1}{n}(\operatorname{tr}\sigma)I$$

for matrices $\sigma = (\sigma_{ij})$, $\tau = (\tau_{kl}) \in \mathbb{R}^{n\times n}$, where $\operatorname{tr}\sigma = \sigma_{ii}$ and $I = (\delta_{ij})$. Here, Einstein's convention of summation over repeated indices running from 1 to $n$ is applied. Throughout, the symbol $c$ denotes a positive constant, whose value may change from line to line.

## 2 Generalized Korn-type inequalities

In this section we collect variants of Korn's inequality involving the trace-free part of the symmetric gradient; for corresponding Korn-type inequalities in the classical setting we refer to [8] and the references given there. In the following, unless anything else is said, $c$ is a positive constant depending on $n$, $p$, and $\Omega$.

**Theorem 2.1.** *Let* $p \in (1, \infty)$ *and* $n \geqslant 3$*. Then the space*

$$D^p(\Omega) := \big\{ v \in L^p(\Omega; \mathbb{R}^n) \, : \, \mathcal{E}^D v \in L^p(\Omega; \mathbb{M}^n) \big\}$$

*(where* $\mathcal{E}^D v$ *is defined in the sense of distributions) coincides with the space* $W^{1,p}(\Omega; \mathbb{R}^n)$ *and for each* $v \in D^p(\Omega)$ *it holds*

$$\|v\|_{1,p} \leqslant c\big(\|v\|_p + \|\mathcal{E}^D v\|_p\big). \tag{3}$$

As a consequence we get the following interpolation inequality.

**Corollary 2.2.** *Let* $p \geqslant 2$ *and* $n \geqslant 3$*. Then*

$$\|v\|_p \leqslant c\big(\|v\|_2 + \|\mathcal{E}^D v\|_p\big) \tag{4}$$

*for each* $v \in W^{1,p}(\Omega; \mathbb{R}^n)$*.*

The following theorem is an extension of the Korn-type inequality shown in [7] in the two-dimensional case. For $n \geqslant 3$ this result follows by contradiction from Theorem 2.1, whereas the case $n = 2$ requires an absolutely different proof; compare Remark 2.4 a) below.

**Theorem 2.3.** *Let $p \in (1, \infty)$ and $n \geqslant 2$. Then the space*

$$D^p_{\mathrm{loc}}(\Omega) := \left\{ v \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \, : \, \mathcal{E}^D v \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^n) \right\}$$

*coincides with the space $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. Moreover,*

$$\|v\|_{1,p} \leqslant c \, \|\mathcal{E}^D v\|_p \tag{5}$$

*for each $v \in \overset{\circ}{W}{}^{1,p}(\Omega; \mathbb{R}^n)$.*

**Remark 2.4.** *a) Theorem 2.1 does not hold in the two-dimensional case. Indeed, if we assume by contradiction that (3) holds for $n = 2$, the Peetre-Tatar lemma [13] (Chapter I, Theorem 2.1) would imply that the kernel of the operator*

$$E_\Omega : W^{1,p}(\Omega; \mathbb{R}^2) \to L^p(\Omega; \mathbb{M}^2), \quad v \mapsto \mathcal{E}^D v$$

*is finite-dimensional. On the other hand, $\mathcal{E}^D v = 0$ is equivalent to the Cauchy-Riemann equations in case $n = 2$ so that $\ker E_\Omega$ coincides with the space of holomorphic functions on $\Omega$, which is a contradiction.*

*b) By scaling one easily verifies that for a ball $B_R = B_R(x_0)$ the Korn-type inequality (3) takes the form*

$$\int_{B_R} |\nabla v|^p \, \mathrm{d}x \leqslant c \left( \frac{1}{R^p} \int_{B_R} |v|^p \, \mathrm{d}x + \int_{B_R} |\mathcal{E}^D v|^p \, \mathrm{d}x \right), \tag{6}$$

*where $c = c(n,p)$ is a positive constant (being independent of $R$ and $x_0$).*

*c) From the proof of Theorem 2.3 we deduce that in case $n = 2$ the Korn-type inequality (3) holds locally in the following sense: Let $\omega' \Subset \omega \Subset \Omega$. Then there are positive numbers $c_1 = c_1(n,p,\omega',\omega)$ and $c_2 = c_2(n,p,\omega)$ such that*

$$\|v\|_{1,p;\omega'} \leqslant c_1 \|v\|_{p;\omega} + c_2 \|\mathcal{E}^D v\|_{p;\omega}$$

*for each $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. In particular, for balls $B_r = B_r(x_0)$ and $B_R = B_R(x_0)$ with $B_r \Subset B_R \Subset \Omega$,*

$$\int_{B_r} |\nabla v|^p \, \mathrm{d}x \leqslant c \left( \frac{1}{(R-r)^p} \int_{B_R} |v|^p \, \mathrm{d}x + \int_{B_R} |\mathcal{E}^D v|^p \, \mathrm{d}x \right) \tag{7}$$

*with a positive number $c = c(n,p)$ (being independent of $r$, $R$ and $x_0$).*

Before we prove the above Korn-type inequalities, we give a characterization of the kernel of the operator

$$E_\Omega : W^{1,p}(\Omega;\mathbb{R}^n) \to L^p(\Omega;\mathbb{M}^n), \quad v \mapsto \mathcal{E}^D v.$$

As already remarked, $\ker E_\Omega$ is infinite-dimensional and coincides with the space of holomorphic functions on $\Omega$ if $n = 2$. In contrast, $\ker E_\Omega$ is finite-dimensional for $n \geqslant 3$ and consists of the so-called conformal Killing vectors (Möbius transformations); compare [4] or [17]. Since we found no rigorous proof of this fact, we outline the main ideas here.

**Proposition 2.5.** *Let $n \geqslant 3$. Then $\ker E_\Omega$ coincides with the space $\mathcal{K}_\Omega$ of conformal Killing vectors $\chi : \Omega \to \mathbb{R}^n$,*

$$\chi(x) = 2(a\cdot x)x - |x|^2 a + Qx + \rho x + b$$

*with $a, b \in \mathbb{R}^n$, $\rho \in \mathbb{R}$, and a skew-symmetric matrix $Q \in \mathbb{R}^{n\times n}$.*

*Proof.* A straightforward calculation shows $\mathcal{K}_\Omega \subset \ker E_\Omega$. To prove the converse inclusion, we observe that each $\chi \in \ker E_\Omega$ satisfies the equations

$$\begin{aligned} \nabla\chi &= \overline{\mathcal{E}}\chi + \tfrac{1}{n}(\operatorname{div}\chi)I, && (8)\\ \nabla\overline{\mathcal{E}}\chi &= \tfrac{1}{n}(\partial_i \operatorname{div}\chi\,\delta_{jk} - \partial_j \operatorname{div}\chi\,\delta_{ik}), && (9)\\ \nabla^2 \operatorname{div}\chi &= 0 && (10) \end{aligned}$$

in the sense of distributions, where $\overline{\mathcal{E}}\chi := \frac{1}{2}(\partial_i\chi^j - \partial_j\chi^i)$. The first equation (8) follows from the definition of $\mathcal{E}^D\chi$, whereas (9) and (10) can be deduced by combining $\mathcal{E}^D\chi = 0$ with the relations

$$\begin{aligned} \partial_k\partial_j\chi^i &= \partial_k\mathcal{E}^D_{ij}\chi + \partial_j\mathcal{E}^D_{ik}\chi - \partial_i\mathcal{E}^D_{jk}\chi \\ &\quad + \tfrac{1}{n}(\partial_k \operatorname{div}\chi\,\delta_{ij} + \partial_j \operatorname{div}\chi\,\delta_{ik} - \partial_i \operatorname{div}\chi\,\delta_{jk}) && (11)\\ \partial_j\Delta\chi^i &= \Delta\mathcal{E}^D_{ij}\chi + \partial_j\partial_k\mathcal{E}^D_{ik}\chi - \partial_i\partial_k\mathcal{E}^D_{jk}\chi + \tfrac{1}{n-1}\partial_k\partial_l\mathcal{E}^D_{kl}\chi\,\delta_{ij} && (12)\\ \partial_j\mathcal{E}^D_{ij}\chi &= \tfrac{1}{2}\Delta\chi^i + \left(\tfrac{1}{2} - \tfrac{1}{n}\right)\partial_i \operatorname{div}\chi && (13) \end{aligned}$$

($i, j, k \in \{1, \dots, n\}$), which hold in the sense of distributions for each function $\chi \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$; compare (23), (24), and (26) in [4]. Assume without loss of generality $\chi \in C^\infty(\Omega;\mathbb{R}^n)$. Then (10) implies $\operatorname{div}\chi(x) = n(2a\cdot x + \rho)$ so that (9) gives $\overline{\mathcal{E}}\chi(x) = 2(a\otimes x - x\otimes a) + Q$. By observing $\nabla[2(a\cdot x)x - |x|^2 a] = 2[a\otimes x - x\otimes a + (a\cdot x)I]$ the claim follows from (8). □

In the proof of Theorem 2.1 we make essential use of the following less familiar lemma due to Nečas [15], [16].

**Lemma 2.6.** *Let $w \in W^{-k,p}(\Omega)$ with $k \in \mathbb{Z}$ and $p \in (1,\infty)$ such that all distributional derivatives $\partial_j w$ belong to $W^{-k,p}(\Omega)$. Then $w \in W^{-k+1,p}(\Omega)$ and*

$$\|w\|_{-k+1,p} \leqslant c\left( \|w\|_{-k,p} + \sum_{j=1}^{n} \|\partial_j w\|_{-k,p} \right).$$

*Proof of Theorem 2.1.* It is straightforward to see that $D^p(\Omega)$ is a Banach space with respect to the norm $\|v\|_{D^p(\Omega)} := \|v\|_p + \|\mathcal{E}^D v\|_p$. Therefore, it suffices to show that the mapping $\mathcal{I} : W^{1,p}(\Omega;\mathbb{R}^n) \hookrightarrow D^p(\Omega)$, $v \mapsto v$, is surjective. Let $v \in D^p(\Omega)$. Then from (12) we deduce $\partial_j \Delta v^i \in W^{-2,p}(\Omega)$, thus $\Delta v^i \in W^{-1,p}(\Omega)$ by Lemma 2.6. Moreover, we have $\partial_i \operatorname{div} v \in W^{-1,p}(\Omega)$ according to (13) (recall $n \geqslant 3$). Hence, $\partial_k \partial_j v^i \in W^{-1,p}(\Omega)$ on account of (11) so that Lemma 2.6 implies $\partial_j v^i \in L^p(\Omega)$ and

$$\|\partial_j v^i\|_p \leqslant c\left( \|\partial_j v^i\|_{-1,p} + \sum_{k=1}^{n} \|\partial_k \partial_j v^i\|_{-1,p} \right) < \infty,$$

which shows that $\mathcal{I}$ is surjective. □

*Proof of Corollary 2.2.* Since $\mathcal{K}_\Omega$ is finite-dimensional for $n \geqslant 3$ and consists of smooth functions we may introduce a basis $\{\chi_1, \dots, \chi_{s_n}\}$ in $\mathcal{K}_\Omega$, which is orthonormal with respect to the standard scalar product in $L^2(\Omega;\mathbb{R}^n)$, that is, $\int_\Omega \chi_i \cdot \chi_j \,\mathrm{d}x = \delta_{ij}$. Let

$$\mathcal{K}_\Omega^\perp := \left\{ w \in W^{1,p}(\Omega;\mathbb{R}^n) \; : \; \int_\Omega w \cdot \chi \,\mathrm{d}x = 0 \quad \text{for all } \chi \in \mathcal{K}_\Omega \right\}$$

and consider the projection operator $\pi_\Omega : W^{1,p}(\Omega;\mathbb{R}^n) \to \mathcal{K}_\Omega$,

$$\pi_\Omega w := \sum_{i=1}^{s_n} \left( \int_\Omega w \cdot \chi_i \,\mathrm{d}x \right) \chi_i.$$

Then $\pi_\Omega w = 0$ and

$$\|w\|_p \leqslant c\|\mathcal{E}^D w\|_p \tag{14}$$

for all $w \in \mathcal{K}_\Omega^\perp$. To show (14) we argue by contradiction: Assume that there is a sequence $(w_m) \subset \mathcal{K}_\Omega^\perp$ such that $\|w_m\|_p = 1$ and $\|\mathcal{E}^D w_m\|_p < 1/m$. By the Korn-type inequality (3) the sequence $(w_m)$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^n)$ so that we have $w_m \xrightarrow{m} w$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ with a function $w \in \mathcal{K}_\Omega^\perp$ (at least for a subsequence). But then $\|w\|_p = 1$ and $\|\mathcal{E}^D w\|_p = 0$, and the last statement implies $w = 0$, which is a contradiction. Now, we write $v = \pi_\Omega v + v^\perp$ with a function $v^\perp \in \mathcal{K}_\Omega^\perp$. Then (14) gives us

$$\|v\|_p \leqslant \|\pi_\Omega v\|_\infty + \|v^\perp\|_p \leqslant c\big(\|v\|_2 + \|\mathcal{E}^D v\|_p\big).$$

□

*Proof of Theorem 2.3.* For $n = 2$ the proof of the Korn-type inequality (5) is outlined in [7]. For $n \geqslant 3$ this inequality follows by contradiction using the Korn-type inequality (3). Clearly, it suffices to show

$$\|v\|_p \leqslant c\|\mathcal{E}^D v\|_p$$

for each $v \in \mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$. Assume that $(v_m) \subset \mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$ is a sequence that satisfies $\|v_m\|_p = 1$ and $\|\mathcal{E}^D v_m\|_p < 1/m$. Then, by applying (3), we see that $(v_m)$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^n)$ so that $v_m \overset{m}{\rightharpoonup} v$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as well as $v_m \overset{m}{\longrightarrow} v$ in $L^p(\Omega;\mathbb{R}^n)$ with a function $v \in \mathring{W}^{1,p}(\Omega;\mathbb{R}^n) \cap \mathcal{K}_\Omega$ satisfying $\|v\|_p = 1$ and $\mathcal{E}^D v = 0$. On the other hand (recall (13)),

$$\frac{1}{2}\int_\Omega \nabla v : \nabla\varphi \,\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{n}\right)\int_\Omega \operatorname{div} v \operatorname{div}\varphi \,\mathrm{d}x = 0$$

for each $\varphi \in \mathring{W}^{1,p/(p-1)}(\Omega;\mathbb{R}^n)$ and we may choose $\varphi := v$ since the elements of $\mathcal{K}_\Omega$ are smooth. Consequently, $v \equiv 0$, which is a contradiction. To prove the coincidence of the spaces $D^p_{\mathrm{loc}}(\Omega)$ and $W^{1,p}_{\mathrm{loc}}(\Omega)$, we fix a subdomain $\Omega' \Subset \Omega$, $v \in D^p_{\mathrm{loc}}(\Omega)$, and consider a sequence $(v_\nu)$ of mollifications of $v$. We further let $\eta \in \mathring{C}^\infty(\Omega)$ with $\eta \geqslant 0$ and $\eta \equiv 1$ in $\Omega'$. Then $\eta v_\nu \in \mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$ and we have the convergences

$$v_\nu \overset{\nu}{\to} v \quad \text{in } L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n), \quad \mathcal{E}^D v_\nu \overset{\nu}{\to} \mathcal{E}^D v \quad \text{in } L^p_{\mathrm{loc}}(\Omega;\mathbb{M}^n). \tag{15}$$

On the other hand, using (5), we find

$$\|\eta v_\nu\|_{1,p} \leqslant c\|\mathcal{E}^D(\eta v_\nu)\|_p \leqslant c\big(\|\mathcal{E}^D v_\nu\|_p + \|\nabla\eta\|_\infty\|v_\nu\|_p\big)$$

so that $(\eta v_\nu)$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^n)$. Hence, there is a function $v_0 \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $\eta v_\nu \overset{\nu}{\rightharpoonup} v_0$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ (at least for a subsequence). According to $\eta \equiv 1$ in $\Omega'$ this implies $v_\nu \overset{\nu}{\rightharpoonup} v_0$ in $W^{1,p}(\Omega';\mathbb{R}^n)$, which together with (15) proves the claim. □

As a consequence of the Korn-type inequality (3) we get Poincaré-type inequalities in dimensions $n \geqslant 3$. These inequalities do also hold in the two-dimensional case, but require a different proof [6] (Lemma A.1).

**Lemma 2.7.**
*Let $n \geqslant 2$, $p \in (1,\infty)$, and $v \in W^{1,p}(B_R;\mathbb{R}^n)$, where $B_R = B_R(x_0)$. Then there exist $\chi \in \mathcal{K}_{B_R}$ and a positive constant $c = c(n,p)$ such that*

$$\left(\fint_{B_R} |v-\chi|^p \,\mathrm{d}x\right)^{1/p} \leqslant cR\left(\fint_{B_R} |\mathcal{E}^D v|^p \,\mathrm{d}x\right)^{1/p}. \tag{16}$$

*Moreover, if $p < n$,*

$$\left(\fint_{B_R} |v-\chi|^{p^*} dx\right)^{1/p^*} \leqslant cR\left(\fint_{B_R} |\mathcal{E}^D v|^p \,\mathrm{d}x\right)^{1/p}, \tag{17}$$

*where $p^* := np/(n-p)$.*

**Remark 2.8.** *Lemma 2.7 is also valid for Sobolev functions defined on arbitrary domains with sufficiently regular boundary. In particular, if* $\Omega \subset \mathbb{R}^2$ *is a bounded domain with piecewise* $C^1$*–boundary and if* $v \in W^{1,p}(\Omega;\mathbb{R}^2)$*, it holds*

$$\|v-\chi\|_{p^{(*)}} \leqslant \left(\frac{2}{\pi}\,|\Omega|\right)^{1/2} \|\mathcal{E}^D v\|_p$$

*with a holomorphic function* $\chi : \Omega \to \mathbb{C}$*. Specifically,*

$$\chi(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{v(\zeta)}{\zeta - z}\, d\zeta,$$

*wherein* $\int_{\partial\Omega} \cdots d\zeta$ *represents the "piecewise" complex line integral, that is,* $\int_{\partial\Omega} \cdots d\zeta = \sum_{k=1}^{M} \int_{\omega_k} \cdots d\zeta$ *if* $\partial\Omega$ *can be represented as the sum of finitely many closed Jordan arcs* $\omega_1, \ldots, \omega_M$*. In the proof of the above inequalities the Cauchy-Pompeiu formula [18] (§ IV.4)*

$$v(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{v(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \int_{\Omega} \frac{\partial_{\overline{z}} v(\zeta)}{\zeta - z}\, d\mathcal{L}^2(\zeta) \quad (z \in \Omega),$$

*valid for functions* $v \in C^\infty(\overline{\Omega};\mathbb{C})$*, is combined with a well-known estimate for the Riesz potential [12] (Lemma 7.12); see [6] for the details.*

*Proof of Lemma 2.7.* In case $n \geqslant 3$ the first inequality (16) is a simple consequence of (3) and the Peetre-Tatar lemma [13] (Chapter I, Theorem 2.1) applied to the operator $E_{B_R}$; compare [19] (Lemma 5.5). For $n \geqslant 3$ a different proof for (16) based on an integral representation for $\mathcal{E}^D v$ is provided by Reshetnyak [17]. The second inequality (17) can be obtained by combining (16) with (3), whereby one can argue exactly as in the proof of the classical Sobolev-Poincaré inequality [5] (Section 4.5.2). □

## 3 Existence of minimizers: proof of Theorem 1.1 a)

Since (1) implies $f(\sigma) \geqslant a|\sigma|^p - b$ for all $\sigma \in \mathbb{M}^n$ with suitable constants $a > 0, b \geqslant 0$, using (5), for each function $w \in \mathbb{K}$ we get

$$\begin{aligned} J[w] &\geqslant c\left( \int_\Omega |\mathcal{E}^D w|^p \,\mathrm{d}x - |\Omega| \right) \\ &\geqslant c\left( \int_\Omega |\mathcal{E}^D (w-u_0)|^p \,\mathrm{d}x - \int_\Omega |\nabla u_0|^p \,\mathrm{d}x - |\Omega| \right) \\ &\geqslant c\left( \int_\Omega |\nabla (w-u_0)|^p \,\mathrm{d}x - \int_\Omega |\nabla u_0|^p \,\mathrm{d}x - |\Omega| \right). \end{aligned} \tag{18}$$

Now, if $(u_m) \subset \mathbb{K}$ is a $J$–minimizing sequence, (18) together with the assumption $J[u_0] < \infty$ implies $\sup_m \|u_m\|_{1,p} < \infty$ so that

$$u_m \overset{m}{\rightharpoondown} u \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^n) \quad \text{and} \quad u_m \overset{m}{\longrightarrow} u \quad \text{in } L^p(\Omega;\mathbb{R}^n)$$

with a function $u \in \mathbb{K}$ (at least for a subsequence). By the lower semicontinuity of $J$ we deduce that $u$ is a solution to the minimization problem $J \to \min$ in $\mathbb{K}$. To show the uniqueness, suppose that $v \in \mathbb{K}$ is a second solution such that $\mathcal{E}^D v \neq \mathcal{E}^D u$ holds on a subset $U$ of $\Omega$ with positive Lebesgue measure. But then

$$f\bigg(\mathcal{E}^D\Big(\frac{u+v}{2}\Big)\bigg) < \frac{1}{2} f(\mathcal{E}^D u) + \frac{1}{2} f(\mathcal{E}^D v) \quad \text{a.e. in } U$$

by the strict convexity of $f$, which leads to the contradiction $J[(u+v)/2] < \inf_{\mathbb{K}} J$. Hence, $\mathcal{E}^D(u-v) = 0$ a.e. in $\Omega$ and (5) implies $u = v$.

## 4 Linear elliptic systems

Based on the Korn-type inequalities shown in Section 2 we establish in this section the smoothness of solutions $v$ to linear elliptic systems involving $\mathcal{E}^D v$, where we adjust the arguments from the classical setting [10] to our situation. By the way we obtain various Caccioppoli- and Campanato-type estimates. Corresponding results in the context of the linear Stokes problem are shown in [8] (Lemma 3.0.5).

**Lemma 4.1.** *Let $A$ be a symmetric bilinear form on the space $\mathbb{M}^n$ that satisfies $\lambda_0|\tau|^2 \leqslant A(\tau,\tau) \leqslant \Lambda_0|\tau|^2$ for each $\tau \in \mathbb{M}^n$ with positive numbers $\lambda_0, \Lambda_0$. Then, if $v \in W^{1,2}(\Omega;\mathbb{R}^n)$ satisfies for each $\varphi \in \overset{\circ}{W}{}^{1,2}(\Omega;\mathbb{R}^n)$*

$$\int_\Omega A(\mathcal{E}^D v, \mathcal{E}^D \varphi)\,\mathrm{d}x = 0, \tag{19}$$

*$v \in C^\infty(\Omega;\mathbb{R}^n)$. Moreover, there is a positive constant $c = c(n,\lambda_0,\Lambda_0)$ such that for balls $B_r = B_r(x_0) \Subset B_R = B_R(x_0) \subset \Omega$ we have:*

*i)* $\displaystyle\int_{B_r} |\mathcal{E}^D v|^2\,\mathrm{d}x \leqslant \frac{c}{(R-r)^2}\int_{B_R} |v-\chi|^2\,\mathrm{d}x \quad (\chi \in \mathcal{K}_{B_R})$

*ii)* $\displaystyle\int_{B_r} |\nabla v|^2\,\mathrm{d}x \leqslant \frac{c}{(R-r)^2}\int_{B_R} |v-\xi|^2\,\mathrm{d}x \quad (\xi \in \mathbb{R}^n)$

*iii)* $\displaystyle\int_{B_r} |\nabla v|^2\,\mathrm{d}x \leqslant c\Big(\frac{r}{R}\Big)^n \int_{B_R} |\nabla v|^2\,\mathrm{d}x$

*iv)* $\displaystyle\int_{B_r} |v-(v)_{x_0,r}|^2\,\mathrm{d}x \leqslant c\Big(\frac{r}{R}\Big)^{n+2} \int_{B_R} |v-(v)_{x_0,R}|^2\,\mathrm{d}x$

v) $\displaystyle\int_{B_r} |\nabla v - (\nabla v)_{x_0,r}|^2 \,\mathrm{d}x \leqslant c\left(\frac{r}{R}\right)^{n+2} \int_{B_R} |\nabla v - (\nabla v)_{x_0,R}|^2 \,\mathrm{d}x$

vi) $\displaystyle\int_{B_r} |\mathcal{E}^D v - (\mathcal{E}^D v)_{x_0,r}|^2 \,\mathrm{d}x \leqslant c\left(\frac{r}{R}\right)^{n+2} \int_{B_R} |\mathcal{E}^D v - (\mathcal{E}^D v)_{x_0,R}|^2 \,\mathrm{d}x$

Here, we used the symbol $(v)_{x_0,r}$ to denote the mean value $⨍_{B_r} v \,\mathrm{d}x$.

By combining the result of Lemma 4.1 with the well-known freezing technique [10] (§ III.3) and the Korn-type inequality (5), we obtain the following regularity result for systems with continuous coefficients; compare [19].

**Corollary 4.2.** *Let $A$ be a symmetric bilinear form on the space $\mathbb{M}^n$ with coefficients $A_{ij}^{kl} = A_{ij}^{kl}(x) \in C^0(\Omega)$ such that*

$$\lambda_0 |\tau|^2 \leqslant A(x)(\tau,\tau) \leqslant \Lambda_0 |\tau|^2$$

*for all $x \in \Omega$, $\tau \in \mathbb{M}^n$ with positive numbers $\lambda_0, \Lambda_0$. Then, if $v \in W^{1,2}(\Omega;\mathbb{R}^n)$ satisfies for each $\varphi \in \mathring{W}^{1,2}(\Omega;\mathbb{R}^n)$*

$$\int_\Omega A(\mathcal{E}^D v, \mathcal{E}^D \varphi) \,\mathrm{d}x = 0,$$

*$v \in C^{0,\alpha}(\Omega;\mathbb{R}^n)$ for each $\alpha \in (0,1)$.*

**Remark 4.3.** *Clearly, it is possible to show regularity results for solutions to inhomogeneous systems. If $v \in W^{1,2}(\Omega;\mathbb{R}^n)$ satisfies*

$$\int_\Omega A(\mathcal{E}^D v, \mathcal{E}^D \varphi) \,\mathrm{d}x + \int_\Omega g : \mathcal{E}^D \varphi \,\mathrm{d}x = \int_\Omega h \cdot \varphi \,\mathrm{d}x$$

*with data $g$, $h$ belonging to suitable Lebesgue and Morrey spaces, then $v \in C^{0,\alpha}(\Omega;\mathbb{R}^n)$. Moreover, if the data $A$, $g$ and $h$ are of class $C^{k,\alpha}$ for some $k \in \mathbb{N}$, $v$ belongs to $C^{k+1,\alpha}$.*

*Proof of Lemma 4.1.* The regularity of $v$ follows by combining the well-known difference quotient technique with Theorem 2.3. We write

$$\Delta_h w(x) := \frac{w(x + he_k) - w(x)}{h} \quad (h \neq 0)$$

for the difference quotient of a function $w$ in direction $k \in \{1, \dots, n\}$. Next, we fix a ball $B_R = B_R(x_0) \Subset \Omega$, radii $r < s < R$, and consider $\varphi := \eta^2 \Delta_h v$, where $\eta \in \mathring{C}^\infty(B_R)$, $\eta \geqslant 0$, is assumed to satisfy $\eta \equiv 1$ in $B_r$, $\eta \equiv 0$ outside $B_s$, and $|\nabla\eta| \leqslant c/(s-r)$ in $B_R$. From (19) we infer for sufficiently small $h$

$$\int_{B_s} \eta^2 A(\Delta_h \mathcal{E}^D v, \Delta_h \mathcal{E}^D v) \,\mathrm{d}x = -2 \int_{B_s} \eta A(\Delta_h \mathcal{E}^D v, (\nabla\eta \odot \Delta_h v)^D) \,\mathrm{d}x.$$

Using the Cauchy-Schwarz and Young's inequality with some $\delta \in (0,1)$, we find

$$\begin{aligned}\int_{B_s} \eta^2 A(\Delta_h \mathcal{E}^D v, \Delta_h \mathcal{E}^D v)\,\mathrm{d}x \leqslant \delta \int_{B_s} \eta^2 A(\Delta_h \mathcal{E}^D v, \Delta_h \mathcal{E}^D v)\,\mathrm{d}x \\ + c(\delta) \int_{B_s} A((\nabla\eta \odot \Delta_h v)^D, (\nabla\eta \odot \Delta_h v)^D)\,\mathrm{d}x,\end{aligned}$$

which in turn implies (choose $\delta = 1/2$)

$$\int_{B_r} |\Delta_h \mathcal{E}^D v|^2\,\mathrm{d}x \leqslant \frac{c}{(s-r)^2} \int_{B_s} |\Delta_h v|^2\,\mathrm{d}x \leqslant \frac{c}{(s-r)^2} \int_{B_{R+h}} |\nabla v|^2\,\mathrm{d}x.$$

Consequently, the weak derivatives $\partial_k \mathcal{E}^D v$ exist in $L^2_{\mathrm{loc}}(\Omega; \mathbb{M}^n)$ and Theorem 2.3 implies $v \in W^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. But then, $w \coloneqq \partial_k v$ also satisfies (19), and we can repeat the above steps to get $w \in W^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. Hence, $v \in W^{3,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, and, by iterating this procedure, we obtain $v \in W^{\ell,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ for each $\ell \in \mathbb{N}$. Now, we enter into the proof of the various inequalities stated in the lemma. The first inequality can be easily obtained by inserting $\varphi \coloneqq \eta^2(v - \chi) \in \mathring{W}^{1,2}(B_R; \mathbb{R}^n)$ in (19). To show ii), we combine i) with the local Korn-type inequality (7): For each $\rho \in (r, R)$ we get

$$\int_{B_r} |\nabla v|^2\,\mathrm{d}x \leqslant c\left(\frac{1}{(\rho - r)^2} \int_{B_\rho} |v - \xi|^2\,\mathrm{d}x + \int_{B_\rho} |\mathcal{E}^D v|^2\,\mathrm{d}x\right),$$

from which ii) follows by choosing $\rho \coloneqq (r + R)/2$ and applying i) (with $v$ replaced by $v - \xi$ and $\chi = 0$) to the second integral on the right-hand side. Using the $W^{\ell,2}_{\mathrm{loc}}$–regularity of $v$, iii) can be deduced along the lines of [11] (Proposition 1.9): Let $r < R/2$ (otherwise iii) is trivial). Let us consider the case $R = 1$ and $x_0 = 0$. Then, for $\ell > n/2$, we have

$$\int_{B_s} |\nabla v|^2\,\mathrm{d}x \leqslant c s^n \|\nabla v\|^2_{\infty; \overline{B}_{1/2}} \leqslant c s^n \|\nabla v\|^2_{\ell,2; B_{1/2}} \tag{20}$$

for each $s < 1/2$. We write

$$\|\nabla v\|^2_{\ell,2;B_{1/2}} = \int_{B_{1/2}} |\nabla v|^2\,\mathrm{d}x + \sum_{\nu=2}^{\ell} \left( \int_{B_{1/2}} \sum_{|\gamma|=\nu} |\partial^\gamma v|^2\,\mathrm{d}x \right)$$

and define $\rho_\nu \coloneqq 1/(2\nu - 2)$ for $\nu \in \{2, \ldots, \ell\}$. Then $B_{1/2} \Subset B_{1/2 + j\rho_\nu} \subset B_1$ for all $j \in \{1, \ldots, \nu - 1\}$, $B_{1/2 + (\nu-1)\rho_\nu} = B_1$, and repeated application of ii) leads to

$$\begin{aligned}\int_{B_{1/2}} \sum_{|\gamma|=\nu} |\partial^\gamma v|^2\,\mathrm{d}x &\leqslant \frac{c}{\rho_\nu^2} \int_{B_{1/2+\rho_\nu}} \sum_{|\gamma|=\nu-1} |\partial^\gamma v|^2\,\mathrm{d}x \\ &\leqslant \left(\frac{c}{\rho_\nu^2}\right)^2 \int_{B_{1/2+2\rho_\nu}} \sum_{|\gamma|=\nu-2} |\partial^\gamma v|^2\,\mathrm{d}x \leqslant \cdots \leqslant \left(\frac{c}{\rho_\nu^2}\right)^{\nu-1} \int_{B_1} |\nabla v|^2\,\mathrm{d}x.\end{aligned}$$

Returning to (20), we have established

$$\int_{B_s} |\nabla v|^2 \,\mathrm{d}x \leqslant c s^n \int_{B_1} |\nabla v|^2 \,\mathrm{d}x,$$

from which iii) follows by rescaling. Inequality iv) can be proved by applying Poincaré's inequality on the left-hand side of iii) and ii) on the right-hand side. Inequality v) is a direct consequence of iv) (replace $v$ by $\nabla v$). It remains to prove vi): Let $r < R/2$ (otherwise vi) is obvious). From v) we deduce

$$\begin{aligned}\int_{B_r} |\mathcal{E}^D v - (\mathcal{E}^D v)_{x_0,r}|^2 \,\mathrm{d}x &\leqslant c\Big(\frac{r}{R}\Big)^{n+2} \int_{B_{R/2}} |\nabla v - (\nabla v)_{x_0,R/2}|^2 \,\mathrm{d}x \\ &\leqslant c\Big(\frac{r}{R}\Big)^{n+2} \int_{B_{R/2}} |\nabla v - Q|^2 \,\mathrm{d}x,\end{aligned}$$

where $Q \in \mathbb{R}^{n\times n}$ is an arbitrary matrix. Let $w(x) \coloneqq v(x) - (\nabla v)_{x_0,R}\, x$ and choose $\chi \in \mathcal{K}_{B_R}$ according to Lemma 2.7 such that

$$\int_{B_R} |w - \chi|^2 \,\mathrm{d}x \leqslant c R^2 \int_{B_R} |\mathcal{E}^D w|^2 \,\mathrm{d}x.$$

Now, if we set $Q \coloneqq (\nabla v)_{x_0,R} + \nabla\chi$ and $\overline{w} \coloneqq w - \chi$, we have $\nabla\overline{w} = \nabla v - Q$ as well as $\mathcal{E}^D\overline{w} = \mathcal{E}^D w = \mathcal{E}^D v - (\mathcal{E}^D v)_{x_0,R}$ so that, using ii) with $v$ replaced by $\overline{w}$ and $\xi = 0$, we find

$$\int_{B_{R/2}} |\nabla v - Q|^2 \,\mathrm{d}x \leqslant c \int_{B_R} |\mathcal{E}^D w|^2 \,\mathrm{d}x = c \int_{B_R} |\mathcal{E}^D v - (\mathcal{E}^D v)_{x_0,R}|^2 \,\mathrm{d}x.$$

□

## 5 Regularization, higher integrability and a Caccioppoli-type inequality

Assume that we are in the situation of Theorem 1.1. Since the functional $J$ (defined in (2)) is anisotropic according to (1), we have to consider a more regular functional. To this purpose we fix a ball $B_R = B_R(x_0) \Subset \Omega$ and consider a sequence $(u_\nu)$ of mollifications of $u$. We define

$$\begin{aligned}\delta_\nu &\coloneqq \big(1 + \nu + \|\mathcal{E}^D u_\nu\|_{q;B_R}^{2q}\big)^{-1}, \\ f_\nu(\sigma) &\coloneqq f(\sigma) + \delta_\nu (1 + |\sigma|^2)^{q/2} \quad (\sigma \in \mathbb{M}^n)\end{aligned}$$

and let $v_\nu$ denote the unique minimizer of the functional

$$J_\nu[w] \coloneqq \int_{B_R} f_\nu(\mathcal{E}^D w) \,\mathrm{d}x$$

in the class $u_\nu + \mathring{W}^{1,q}(B_R;\mathbb{R}^n)$.

**Lemma 5.1.** *Let* (1) *hold with* $q \geqslant 2$*, and let* $\Gamma_\nu \coloneqq 1 + |\mathcal{E}^D v_\nu|^2$*.*

*a)* $v_\nu \in W^{2,2}_{\mathrm{loc}}(B_R;\mathbb{R}^n)$ *and* $\tau_\nu \coloneqq Df_\nu(\mathcal{E}^D v_\nu) \in W^{1,q/(q-1)}_{\mathrm{loc}}(B_R;\mathbb{M}^n)$

*b)* $\Gamma_\nu^{p/4},\ \Gamma_\nu^{q/4} \in W^{1,2}_{\mathrm{loc}}(B_R)$

*c)* $v_\nu \overset{\nu}{\rightharpoonup} u$ *in* $W^{1,p}(B_R;\mathbb{R}^n)$

*d)* $\displaystyle\int_{B_R} f_{(\nu)}(\mathcal{E}^D v_\nu)\,\mathrm{d}x \overset{\nu}{\to} \int_{B_R} f(\mathcal{E}^D u)\,\mathrm{d}x$ *and* $\displaystyle\delta_\nu \int_{B_R} \Gamma_\nu^{q/2}\,\mathrm{d}x \overset{\nu}{\to} 0$

**Remark 5.2.** *Note that if* (1) *is satisfied with exponents* $1 < p \leqslant q < 2$*, it also holds with* $q$ *replaced by* $\overline{q} = 2$*. Therefore, if we replace* $q$ *by* $\overline{q} = 2$ *in the definition of* $f_\nu$*, the above lemma is also available in the situation of part c) of Theorem 1.1.*

*Proof of Lemma 5.1.* To prove the statements a) and b) we can argue as in [2]. Let us give a sketch of the proof of the $W^{2,2}_{\mathrm{loc}}$–regularity of $v_\nu$. From the growth of $Df_\nu$ we deduce $\tau_\nu \in L^{q/(q-1)}(B_R;\mathbb{M}^n)$ and since $v_\nu$ minimizes $J_\nu$ we have the Euler-Lagrange equation

$$\int_{B_R} \tau_\nu : \mathcal{E}^D \varphi\,\mathrm{d}x = 0$$

for all $\varphi \in \overset{\circ}{W}{}^{1,q}(B_R;\mathbb{R}^n)$, from which we deduce

$$\int_{B_R} \Delta_h \tau_\nu : \mathcal{E}^D \varphi\,\mathrm{d}x = 0 \tag{21}$$

for all $\varphi \in W^{1,q}(B_R;\mathbb{R}^n)$ with $\operatorname{spt}\varphi \subset B_R$ and sufficiently small $h \neq 0$. Let $B_r = B_r(\overline{x}) \Subset B_R$, $0 < \rho < \rho' < r$, and $\varphi \coloneqq \eta^2 \Delta_h u$, where $\eta \in \overset{\circ}{C}{}^\infty(B_r(\overline{x}))$, $\eta \geqslant 0$, is chosen such that $\eta \equiv 1$ in $B_\rho(\overline{x})$, $\eta \equiv 0$ outside $B_{\rho'}(\overline{x})$ as well as $|\nabla \eta| \leqslant c/(\rho' - \rho)$. Then from (21) we infer

$$\int_{B_{\rho'}(\overline{x})} \eta^2 \Delta_h \tau_\nu : \Delta_h \mathcal{E}^D v_\nu\,\mathrm{d}x = -2 \int_{B_{\rho'}(\overline{x})} \eta \Delta_h \tau_\nu : (\nabla\eta \odot \Delta_h v_\nu)^D\,\mathrm{d}x \tag{22}$$

Now, we use the relation

$$\Delta_h \tau_\nu = \int_0^1 D^2 f_\nu(\mathcal{E}^D v_\nu + th\,\Delta_h \mathcal{E}^D v_\nu)(\Delta_h \mathcal{E}^D v_\nu, \cdot\,)\,\mathrm{d}t$$

and consider the positive bilinear form

$$\mathcal{B}_x \coloneqq \int_0^1 D^2 f(\mathcal{E}^D v_\nu(x) + th\,\Delta_h \mathcal{E}^D v_\nu(x))\,\mathrm{d}t$$

acting on trace-free matrices. Then, if we write $\mathcal{Q}_x$ for the associated quadratic form, (22) reads as

$$\int_{B_{\rho'}(\overline{x})} \eta^2 \mathcal{Q}_x(\Delta_h \mathcal{E}^D v_\nu)\,\mathrm{d}x = -2 \int_{B_{\rho'}(\overline{x})} \eta \mathcal{B}_x(\Delta_h \mathcal{E}^D v_\nu, (\nabla\eta \odot \Delta_h v_\nu)^D)\,\mathrm{d}x. \tag{23}$$

To estimate the right-hand side, we apply the Cauchy-Schwarz inequality for $\mathcal{B}_x$ as well as Young's inequality with some $\delta > 0$:

$$\begin{aligned}
&\int_{B_{\rho'}(\overline{x})} \eta |\mathcal{B}_x(\Delta_h \mathcal{E}^D v_\nu, (\nabla\eta \odot \Delta_h v_\nu)^D)| \,\mathrm{d}x \\
&\leqslant \int_{B_{\rho'}(\overline{x})} \eta \mathcal{Q}_x(\Delta_h \mathcal{E}^D v_\nu)^{1/2} \mathcal{Q}_x((\nabla\eta \odot \Delta_h v_\nu)^D)^{1/2} \,\mathrm{d}x \\
&\leqslant \delta \int_{B_{\rho'}(\overline{x})} \eta^2 \mathcal{Q}_x(\Delta_h \mathcal{E}^D v_\nu) \,\mathrm{d}x + c(\delta) \int_{B_{\rho'}(\overline{x})} \mathcal{Q}_x((\nabla\eta \odot \Delta_h v_\nu)^D) \,\mathrm{d}x,
\end{aligned}$$

where an upper bound for the second integral on the right-hand side is given by (using (1) and Hölder's inequality)

$$\frac{c}{(\rho' - \rho)^2} \left( \int_{B_{\rho'}(\overline{x})} (1 + |\mathcal{E}^D v_\nu|^2 + |h \Delta_h \mathcal{E}^D v_\nu|^2)^{q/2} \,\mathrm{d}x \right)^{1-2/q} \left( \int_{B_{\rho'}(\overline{x})} |\Delta_h v_\nu|^q \,\mathrm{d}x \right)^{2/q}.$$

Therefore, by combining these estimates with (23) we finally arrive at (compare [2], proof of Lemma 3.1):

$$\omega(\rho) \leqslant \frac{c}{(\rho' - \rho)^2} \left( 1 + \int_{B_{r+h}(\overline{x})} |\nabla v_\nu|^q \,\mathrm{d}x \right), \tag{24}$$

where we have abbreviated

$$\omega(\rho) := \int_{B_\rho(\overline{x})} \mathcal{Q}_x(\Delta_h \mathcal{E}^D v_\nu) \,\mathrm{d}x.$$

On the other hand, since $q \geqslant 2$, the definition of $f_\nu$ implies

$$\omega(\rho) \geqslant c\delta_\nu \int_{B_\rho(\overline{x})} |\Delta_h \mathcal{E}^D v_\nu|^2 \,\mathrm{d}x,$$

which together with (24) shows that $\partial_k \mathcal{E}^D v_\nu$ exists in $L^2_{\mathrm{loc}}(\Omega; \mathbb{M}^n)$. Hence, $v_\nu \in W^{2,2}_{\mathrm{loc}}(B_R; \mathbb{R}^n)$ according to Theorem 2.3. Now, to justify c), we observe that by the growth of $f$ and the minimality of $v_\nu$ we have

$$\|\mathcal{E}^D v_\nu\|_p^p \leqslant c \left( \int_{B_R} f(\mathcal{E}^D u_\nu) \,\mathrm{d}x + \delta_\nu \int_{B_R} (1 + |\mathcal{E}^D u_\nu|^2)^{q/2} \,\mathrm{d}x + 1 \right),$$

wherein the second term on the right-hand side vanishes as $\nu \to \infty$ by definition of $\delta_\nu$. On the other hand, the minimality of $v_\nu$ and Jensen's inequality gives

$$\int_{B_R} f_{(\nu)}(\mathcal{E}^D v_\nu) \,\mathrm{d}x \leqslant \int_{B_R} f_{(\nu)}(\mathcal{E}^D u_\nu) \,\mathrm{d}x \leqslant \int_{B_R} f(\mathcal{E}^D u) \,\mathrm{d}x + O(\nu), \tag{25}$$

where $O(\nu) \xrightarrow{\nu} 0$. Therefore, $\sup_\nu \|\mathcal{E}^D v_\nu\|_p \leqslant c$ and (5) implies

$$\|v_\nu\|_{1,p} \leqslant \|v_\nu - u_\nu\|_p + \|u_\nu\|_p \leqslant c\big(\|\mathcal{E}^D v_\nu\|_p + \|u_\nu\|_{1,p}\big).$$

Consequently, $v_\nu \xrightharpoonup{\nu} \widetilde{u}$ in $W^{1,p}(B_R;\mathbb{R}^n)$ (at least for a subsequence) with a function $\widetilde{u} \in u + \mathring{W}^{1,p}(B_R;\mathbb{R}^n)$. Hence, c) and d) follow from (25) as in [2] (proof of Lemma 4.1). □

**Lemma 5.3.** *Let* $H_\nu := D^2 f_\nu(\mathcal{E}^D v_\nu)(\partial_k \mathcal{E}^D v_\nu, \partial_k \mathcal{E}^D v_\nu)^{1/2}$ *(with summation with respect to* $k \in \{1,\dots,n\}$*). Then for all* $\eta \in \mathring{C}^\infty(B_R)$ *and* $\chi \in \mathcal{K}_{B_R}$ *we have the estimate*

$$\int_{B_R} \eta^2 H_\nu^2 \,\mathrm{d}x \leqslant c \int_{B_R} |\nabla \eta|^2 \Gamma_\nu^{q/2-1} |\nabla v_\nu - \nabla \chi|^2 \,\mathrm{d}x,$$

*where* $c = c(\Lambda)$ *is a positive number (being independent of* $\nu$ *and* $R$*).*

*Proof.* By inserting $\varphi := \eta^2 \Delta_h(v_\nu - \chi)$ in (21) we get

$$\begin{aligned}\int_{B_R} & \eta^2 \Delta_h \tau_\nu : \Delta_h \mathcal{E}^D v_\nu \,\mathrm{d}x \\ &= -2 \int_{B_R} \eta \Delta_h \tau_\nu : \big(\nabla\eta \odot \Delta_h(v_\nu - \chi)\big)^D \,\mathrm{d}x.\end{aligned} \tag{26}$$

Since $\Delta_h \tau_\nu : \Delta_h \mathcal{E}^D v_\nu \geqslant 0$ and

$$\Delta_h \tau_\nu : \Delta_h \mathcal{E}^D v_\nu \xrightarrow{(h\to 0)} D^2 f_\nu(\mathcal{E}^D v_\nu)(\partial_k \mathcal{E}^D v_\nu, \partial_k \mathcal{E}^D v_\nu)$$

a.e. in $B_R$, Fatou's lemma implies

$$\begin{aligned}\int_{B_R} & \eta^2 D^2 f_\nu(\mathcal{E}^D v_\nu)(\partial_k \mathcal{E}^D v_\nu, \partial_k \mathcal{E}^D v_\nu) \,\mathrm{d}x \\ &\leqslant \liminf_{h\to 0} \int_{B_R} \eta^2 \Delta_h \tau_\nu : \Delta_h \mathcal{E}^D v_\nu \,\mathrm{d}x.\end{aligned}$$

For the right-hand side of (26) we observe $|\mathcal{E}^D v_\nu|^{q/2} \in W^{1,2}_{\mathrm{loc}}(B_R)$ (by Lemma 5.1 b)) so that by Sobolev's imbedding theorem $|\mathcal{E}^D v_\nu|^{q/2} \in L^t_{\mathrm{loc}}(B_R)$ for some $t > q$. Since from $v_\nu \in W^{1,q}(B_R;\mathbb{R}^n)$ we also get $|v_\nu| \in L^t(B_R)$, Theorem 2.3 gives $v_\nu \in W^{1,t}_{\mathrm{loc}}(B_R;\mathbb{R}^n)$. Hence, using Young's inequality, we get

$$\eta |\Delta_h \tau_\nu| |\nabla \eta| |\Delta_h(v_\nu - \chi)| \leqslant c(\eta, t_1, t_2)\big(|\Delta_h \tau_\nu|^{t_1} + |\Delta_h(v_\nu - \chi)|^{t_2}\big)$$

with suitable exponents $t_1 < q/(q-1)$ and $t_2 \in (q,t)$ so that we have equi-integrability. Therefore, by Vitali's theorem (26) (with summation with respect to $k$) turns into

$$\int_{B_R} \eta^2 H_\nu^2 \,\mathrm{d}x \leqslant -2 \int_{B_R} \eta \partial_k \tau_\nu : \big(\nabla \eta \odot \partial_k(v_\nu - \chi)\big)^D \,\mathrm{d}x. \tag{27}$$

Now, by combining the Cauchy-Schwarz inequality with (1), we see

$$\Gamma_\nu^{1-q/2}|\nabla\tau_\nu|^2 \leqslant c(\Lambda)\, H_\nu^2 \tag{28}$$

so that, using Hölder's and Young's inequality with some $\delta \in (0,1)$, we arrive at

$$\text{r.h.s. of (27)} \leqslant \delta \int_{B_R} \eta^2 H_\nu^2 \,\mathrm{d}x + c(\delta) \int_{B_R} |\nabla\eta|^2 \Gamma_\nu^{q/2-1}|\nabla v_\nu - \nabla\chi|^2 \,\mathrm{d}x,$$

from which the desired estimate follows by choosing $\delta = 1/2$. □

**Lemma 5.4.** *Let $0 < \rho < r$ such that $r - \rho \leqslant 1$ and $B_r(\overline{x}) \Subset B_R(x_0)$. Then*

$$\int_{B_\rho(\overline{x})} H_\nu^2 \,\mathrm{d}x \leqslant \frac{c}{(r-\rho)^4} \int_{B_r(\overline{x})} \Gamma_\nu^{q/2} \,\mathrm{d}x,$$

*where $c = c(n, q, \Omega)$ is independent of $m$ and the balls.*

*Proof.* From Lemma 5.3 we infer

$$\int_{B_R} \eta^2 H_\nu^2 \,\mathrm{d}x \leqslant c\|\nabla\eta\|_\infty^2 \bigg( \int_{\operatorname{spt}\nabla\eta} \Gamma_\nu^{q/2} \,\mathrm{d}x \bigg)^{1-2/q} \bigg( \int_{\operatorname{spt}\nabla\eta} |\nabla v_\nu - \nabla\chi|^q \,\mathrm{d}x \bigg)^{2/q}.$$

Let $\eta \in \overset{\circ}{C}{}^\infty(B_{(\rho+r)/2}(\overline{x}))$, $\eta \geqslant 0$, such that $\eta \equiv 1$ in $B_\rho(\overline{x})$ and $|\nabla\eta| \leqslant c/(r-\rho)$. By applying the Korn-type inequality (7) we get

$$\begin{aligned}\bigg( \int_{B_{(\rho+r)/2}(\overline{x})} |\nabla v_\nu - \nabla\chi|^q \,\mathrm{d}x \bigg)^{1/q} &\leqslant \frac{c}{r-\rho} \bigg( \int_{B_r(\overline{x})} |v_\nu - \chi|^q \,\mathrm{d}x \bigg)^{1/q} \\ &\quad + c\bigg( \int_{B_r(\overline{x})} |\mathcal{E}^D v_\nu|^q \,\mathrm{d}x \bigg)^{1/q}\end{aligned}$$

and if we choose $\chi$ according to Lemma 2.7, we find

$$\bigg( \int_{\operatorname{spt}\nabla\eta} |\nabla v_\nu - \nabla\chi|^q \,\mathrm{d}x \bigg)^{2/q} \leqslant c\Big(1 + \frac{r}{r-\rho}\Big)^2 \bigg( \int_{B_r(\overline{x})} |\mathcal{E}^D v_\nu|^q \,\mathrm{d}x \bigg)^{2/q}$$

and the desired estimate follows at once. □

Using Lemma 5.4 we can show uniform higher integrability of $\mathcal{E}^D v_\nu$ now, which in turn gives us uniform higher integrability of the gradient.

**Lemma 5.5.** *There exists an exponent $\widetilde{q} > q$ such that*

$$\int_{B_\rho(\overline{x})} \Gamma_\nu^{\widetilde{q}/2} \,\mathrm{d}x \leqslant c\big(n, p, q, \rho, R, \Omega, \overline{x}, J[u; B_R]\big) < \infty$$

*for each ball $B_\rho(\overline{x}) \Subset B_R$.*

*Proof.* Let $\widetilde{q} \coloneqq p\kappa > q$ with

$$\kappa \coloneqq \begin{cases} \frac{n}{n-2} & ; \quad n \geqslant 3 \\ \text{any number} > \frac{p}{2p-q} & ; \quad n = 2 \end{cases}$$

(recall $q < (1 + 2/n)p$ in case $n \geqslant 3$ and $q < 2p$ in case $n = 2$). From Lemma 5.1 we know $\phi_\nu \coloneqq \mathit{\Gamma}_\nu^{p/4} \in W^{1,2}_{\mathrm{loc}}(B_R; \mathbb{R}^n)$ so that

$$\begin{aligned} \int_{B_\rho(\overline{x})} \mathit{\Gamma}_\nu^{\widetilde{q}/2} \, \mathrm{d}x &\leqslant \int_{B_R} (\eta\phi_\nu)^{2\kappa} \, \mathrm{d}x \leqslant c \bigg( \int_{B_R} |\nabla(\eta\phi_\nu)|^2 \, \mathrm{d}x \bigg)^\kappa \\ &\leqslant c \bigg( \int_{B_R} |\nabla\eta|^2 \phi_\nu^2 \, \mathrm{d}x + \int_{B_R} \eta^2 |\nabla\phi_\nu|^2 \, \mathrm{d}x \bigg)^\kappa, \end{aligned}$$

where $\eta \in \overset{\circ}{C}{}^\infty(B_R)$, $\eta \geqslant 0$, is chosen such that $\eta \equiv 1$ in $B_\rho(\overline{x})$, $\eta \equiv 0$ outside $B_{(\rho+r)/2}(\overline{x})$ for some $r \in (\rho, R)$, and $|\nabla\eta| \leqslant c/(r - \rho)$. Now, by combining the formula for $\nabla\phi_\nu$ with the lower bound for $D^2 f_\nu$ we get

$$|\nabla\phi_\nu|^2 \leqslant c\,\mathit{\Gamma}_\nu^{p/2-1} |\nabla\mathcal{E}^D v_\nu|^2 \leqslant c H_\nu^2 \tag{29}$$

so that, using the estimate from Lemma 5.4, we find

$$\bigg( \int_{B_\rho(\overline{x})} \mathit{\Gamma}_\nu^{\widetilde{q}/2} \, \mathrm{d}x \bigg)^{1/\kappa} \leqslant \frac{c}{(r-\rho)^4} \bigg( \int_{B_R} \mathit{\Gamma}_\nu^{p/2} \, \mathrm{d}x + \int_{B_r(\overline{x})} \mathit{\Gamma}_\nu^{q/2} \, \mathrm{d}x \bigg). \tag{30}$$

Owing to $q < \widetilde{q}$ and since we may assume $q > p$ (otherwise replace $p$ by a slightly smaller number $p_0$ such that $q < (1 + 2/n)p_0$ in case $n \geqslant 3$ or $q < 2p_0$ in case $n = 2$, respectively), there is a number $\theta = \theta(p, \kappa) \in (0, 1)$ such that

$$\frac{1}{q} = \frac{1-\theta}{\widetilde{q}} + \frac{\theta}{p}.$$

Since the definition of $\kappa$ together with the condition for $q$ implies

$$(1-\theta)\,\frac{q}{p} = \frac{\kappa}{\kappa-1}\bigg(\frac{q}{p} - 1\bigg) < 1, \tag{31}$$

we may apply the interpolation inequality for Lebesgue spaces [12] ((7.9), p. 146) to $\mathit{\Gamma}_\nu^{1/2}$ with the result:

$$\begin{aligned} \int_{B_r(\overline{x})} \mathit{\Gamma}_\nu^{q/2} \, \mathrm{d}x &\leqslant \bigg( \int_{B_r(\overline{x})} \mathit{\Gamma}_\nu^{\widetilde{q}/2} \, \mathrm{d}x \bigg)^{(1-\theta)q/\widetilde{q}} \bigg( \int_{B_r(\overline{x})} \mathit{\Gamma}_\nu^{p/2} \, \mathrm{d}x \bigg)^{\theta q/p} \\ &\leqslant \delta \bigg( \int_{B_r(\overline{x})} \mathit{\Gamma}_\nu^{\widetilde{q}/2} \, \mathrm{d}x \bigg)^{1/\kappa} + \delta^{-\gamma} \bigg( \int_{B_r(\overline{x})} \mathit{\Gamma}_\nu^{p/2} \, \mathrm{d}x \bigg)^\beta \end{aligned}$$

with $\delta \in (0,1)$ and suitable exponents $\beta > 1$ and $\gamma > 0$. Here, we used Young's inequality in the second step, which is possible on account of (31). If we choose $\delta = (2c)^{-1}(r-\rho)^4$ and use the latter inequality on the right-hand side of (30), we arrive at

$$\left(\int_{B_\rho(\overline{x})} \Gamma_\nu^{\widetilde{q}/2}\,\mathrm{d}x\right)^{1/\kappa} \leqslant \frac{1}{2}\left(\int_{B_r(\overline{x})} \Gamma_\nu^{\widetilde{q}/2}\,\mathrm{d}x\right)^{1/\kappa} + \frac{c}{(r-\rho)^{\widetilde{\gamma}}}\left(\int_{B_R} \Gamma_\nu^{p/2}\,\mathrm{d}x\right)^{\beta},$$

where $\widetilde{\gamma} = 4(1+\gamma)$. But now we are in the same situation as in (4.19) of [2] and can argue in the same way as in [2] to complete the proof. □

**Corollary 5.6.** *For each ball* $B_r(\overline{x}) \Subset B_R$ *we have* $\sup_\nu \|v_\nu\|_{1,\widetilde{q};B_r(\overline{x})} < \infty$. *Moreover,* $u \in W^{1,\widetilde{q}}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$.

*Proof.* From Lemma 5.1 c) we know $\sup_\nu \|v_\nu\|_{1,p} < \infty$ so that by Sobolev's imbedding theorem we get $\sup_\nu \|v_\nu\|_{p_1} < \infty$, where $p_1 > p$ is given by

$$p_1 := \begin{cases} \widetilde{q} & ; \quad p \geqslant n \\ p^* & ; \quad p < n. \end{cases}$$

In case $n \geqslant 3$ we can argue as in the proof of Corollary 4.2 in [2] by using the Korn-type inequality (3), which does not hold in the two-dimensional case. For $n = 2$ we have to use the local variant (7) of (3). For this purpose we choose a small radius $\rho'$ such that $B_\rho(\overline{x}) \Subset B_{\rho'}(\overline{x}) \Subset B_R$. If $\widetilde{q} \leqslant p_1$, we get

$$\|v_\nu\|_{1,\widetilde{q};B_\rho(\overline{x})} \leqslant c\big(\|v_\nu\|_{p_1;B_{\rho'}(\overline{x})} + \|\mathcal{E}^D v_\nu\|_{\widetilde{q};B_{\rho'}(\overline{x})}\big),$$

wherein the right-hand side is uniformly bounded with respect to $\nu$ (by Lemma 5.5), thus, $\sup_\nu \|v_\nu\|_{1,\widetilde{q};B_\rho(\overline{x})} < \infty$. In the same way we get $\sup_\nu \|v_\nu\|_{1,p_1;B_{\rho'}(\overline{x})} < \infty$ in case $\widetilde{q} > p_1$. But then Sobolev's imbedding theorem implies $\sup_\nu \|v_\nu\|_{\widetilde{q};B_{\rho'}(\overline{x})} < \infty$ (observe $p_1 = p^* \geqslant 2 = n$), which leads to $\sup_\nu \|v_\nu\|_{1,\widetilde{q};B_\rho(\overline{x})} < \infty$ by using (7) once more. □

**Lemma 5.7.** *Let* $\phi := (1+|\mathcal{E}^D u|^2)^{p/4}$. *Then* $\phi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *and* $\phi_\nu \overset{\nu}{\rightharpoonup} \phi$ *in* $W^{1,2}_{\mathrm{loc}}(B_R)$.

*Proof.* By combining (29) with Lemmas 5.4 and 5.5 the sequence $(\nabla\phi_\nu)$ is seen to be bounded in $L^2_{\mathrm{loc}}(B_R;\mathbb{R}^n)$ so that there exists a function $\widetilde{\phi} \in W^{1,2}_{\mathrm{loc}}(B_R)$ such that $\phi_\nu \overset{\nu}{\rightharpoonup} \widetilde{\phi}$ in $W^{1,2}_{\mathrm{loc}}(B_R)$ as well as $\phi_\nu \overset{\nu}{\to} \widetilde{\phi}$ a.e. in $B_R$ (at least for a subsequence). To get the desired convergence, it suffices to show

$$\mathcal{E}^D v_\nu \overset{\nu}{\to} \mathcal{E}u \quad \text{a.e. in } B_R. \tag{32}$$

For this purpose we consider the decomposition

$$\begin{aligned}\int_{B_R} f(\mathcal{E}^D v_\nu) - f(\mathcal{E}^D u)\,\mathrm{d}x &= \int_{B_R} Df(\mathcal{E}^D u) : \mathcal{E}^D w_\nu\,\mathrm{d}x \\ &+ \int_{B_R}\int_0^1 (1-t)D^2 f(\mathcal{E}^D u + t\mathcal{E}^D w_\nu)(\mathcal{E}^D w_\nu, \mathcal{E}^D w_\nu)\,\mathrm{d}t\,\mathrm{d}x =: I_1 + I_2,\end{aligned} \tag{33}$$

where $w_\nu := v_\nu - u$. (Note that both integrals on the right-hand side of (33) are well defined on account of the growth properties of $Df$ and $D^2 f$ and since $u \in W^{1,q}(B_R;\mathbb{R}^n)$ by Corollary 5.6.) Since $v_\nu \in u_\nu + \overset{\circ}{W}{}^{1,q}(B_R;\mathbb{R}^n)$ and $u_\nu \overset{\nu}{\to} u$ in $W^{1,q}(B_R;\mathbb{R}^n)$ (recall that $u_\nu$ is a mollification of $u$) we get

$$\begin{aligned} I_1 &= \int_{B_R} Df(\mathcal{E}^D u) : (\mathcal{E}^D v_\nu - \mathcal{E}^D u_\nu)\,\mathrm{d}x + \int_{B_R} Df(\mathcal{E}^D u) : (\mathcal{E}^D u_\nu - \mathcal{E}^D u)\,\mathrm{d}x \\ &= \int_{B_R} Df(\mathcal{E}^D u) : (\mathcal{E}^D u_\nu - \mathcal{E}^D u)\,\mathrm{d}x \overset{\nu}{\to} 0, \end{aligned}$$

where we used the Euler-Lagrange equation for $u$ as well as the growth of $Df$ in the last step. Now, by Lemma 5.1 d) the left-hand side of (33) vanishes as $\nu \to \infty$ so that from (33) we infer $I_2 \overset{\nu}{\to} 0$. Since on the other hand, (1) implies

$$\int_{B_R} \phi_\nu^{2-4/p} |\mathcal{E}^D w_\nu|^2\,\mathrm{d}x \leqslant \int_{B_R} (1 + |\mathcal{E}^D v_\nu|^2 + |\mathcal{E}^D u|^2)^{p/2-1} |\mathcal{E}^D w_\nu|^2\,\mathrm{d}x \leqslant c I_2,$$

(32) follows from the convergence $\phi_\nu \overset{\nu}{\to} \widetilde{\phi}$ a.e. in $B_R$. □

## 6 Partial regularity: proof of Theorem 1.1 b)

Let $n \geqslant 3$, and let (1) be fulfilled with $q \geqslant 2$ and $p \leqslant q$ such that $q < (1 + 2/n)p$. To prove part b) of Theorem 1.1, we adjust the well-known blowup technique (compare [2] or [8]) to our setting. We define the excess of $u$ with respect to a ball $B_r(x_0) \Subset \Omega$ by

$$(Eu)_{x_0,r} := \fint_{B_r(x_0)} |\mathcal{E}^D u - (\mathcal{E}^D u)_{x_0,r}|^2\,\mathrm{d}x + \fint_{B_r(x_0)} |\mathcal{E}^D u - (\mathcal{E}^D u)_{x_0,r}|^q\,\mathrm{d}x$$

Note that according to Corollary 5.6 $(Eu)_{x_0,r}$ is well defined.

**Lemma 6.1.** *Let $\ell > 0$ be given. Then there is a positive constant $c_*$ with the property: To each $\tau \in (0, 1/4)$ there exists a positive number $\varepsilon = \varepsilon(\ell, \tau)$ such that for every ball $B_r(x_0) \Subset \Omega$ for which*

$$|(\mathcal{E}^D u)_{x_0,r}| < \ell \quad \text{and} \quad (Eu)_{x_0,r} < \varepsilon$$

*hold, we have*

$$(Eu)_{x_0,\tau r} \leqslant c_* \tau^2 (Eu)_{x_0,r}.$$

From the above lemma we deduce by a standard iteration procedure (compare [8] or [19]) that $\mathcal{E}^D u$ is of class $C^{0,\alpha}$ on the set

$$\Omega_0 := \Big\{ x \in \Omega \,:\, \sup_{r>0} |(\mathcal{E}^D u)_{x,r}| < \infty \text{ and } \liminf_{r \searrow 0} (Eu)_{x,r} = 0 \Big\}.$$

Moreover, $\Omega_0$ is an open set of full Lebesgue measure.

Now, let $\omega \Subset \Omega_0$. Then $\mathcal{E}^D u \in C^{0,\alpha}(\overline{\omega}; \mathbb{M}^n)$ and with the arguments from [2] (p. 386) we get $u \in W^{2,2}_{\mathrm{loc}}(\omega; \mathbb{R}^n)$ as well as

$$\int_\omega D^2 f(\mathcal{E}^D u)(\mathcal{E}^D(\partial_k u), \mathcal{E}^D \varphi)\,\mathrm{d}x = 0 \tag{34}$$

for each $\varphi \in \mathring{C}^1(\omega; \mathbb{R}^n)$ and $k \in \{1, \dots, n\}$. Therefore, $w := \partial_k u \in W^{1,2}(\omega; \mathbb{R}^n)$ solves the elliptic system (34) with continuous matrix $D^2 f(\mathcal{E}^D u)$. Hence, $w \in C^{0,\alpha}(\overline{\omega}; \mathbb{R}^n)$ by Corollary 4.2, which proves Theorem 1.1 b).

Consequently, it remains to prove the blowup lemma.

*Proof of Lemma 6.1.* We argue by contradiction and assume that there exist $\tau \in (0, 1/4)$ and a sequence of balls $B_{r_m}(x_m) \Subset \Omega$ such that

$$\begin{gathered} |(\mathcal{E}^D u)_{x_m, r_m}| < \ell, \quad (Eu)_{x_m, r_m} =: \lambda_m^2 \xrightarrow{\nu} 0, \\ (Eu)_{x_m, \tau r_m} > c_* \tau^2 \lambda_m^2. \end{gathered} \tag{35}$$

We define

$$u_m(z) := \frac{u(x_m + r_m z) - r_m A_m z - \chi_m(z)}{\lambda_m r_m} \quad (z \in B_1),$$

where $A_m := (\mathcal{E}^D u)_{x_m, r_m}$ and $\chi_m \in \mathcal{K}_{B_1}$ is chosen according to Lemma 2.7 such that

$$\fint_{B_1} |u_m|^2\,\mathrm{d}z \leqslant c \fint_{B_1} |\mathcal{E}^D u_m|^2\,\mathrm{d}z. \tag{36}$$

Observing $\mathcal{E}^D u_m = \lambda_m^{-1}[\mathcal{E}^D u(x_m + r_m z) - A_m]$, the definition of $\lambda_m$ implies

$$\fint_{B_1} |\mathcal{E}^D u_m|^2\,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_1} |\mathcal{E}^D u_m|^q\,\mathrm{d}z = 1, \tag{37}$$

which together with (36) and the Korn-type inequality (3) gives boundedness of $(u_m)$ in $W^{1,2}(B_1; \mathbb{R}^n)$. Hence, we have (at least for a subsequence)

$$\begin{aligned} u_m &\xrightarrow{m} : v \quad \text{in } W^{1,2}(B_1; \mathbb{R}^n) \\ \lambda_m \mathcal{E}^D u_m &\xrightarrow{m} 0 \quad \text{in } L^2(B_1; \mathbb{M}^n) \text{ and a.e. in } B_1. \end{aligned} \tag{38}$$

Moreover, $A_m \xrightarrow{m}: A$ (for a subsequence) with a matrix $A \in \mathbb{M}^n$, $|A| \leqslant \ell$, and $v$ fulfills

$$\int_{B_1} D^2 f(A)(\mathcal{E}^D v, \mathcal{E}^D \varphi)\,\mathrm{d}z = 0 \tag{39}$$

for all $\varphi \in \mathring{C}^1(B_1; \mathbb{R}^n)$, which can be shown as in [2] (Proposition 5.1).

By virtue of (39) and Lemma 4.1 $v$ belongs to $C^\infty(B_1; \mathbb{R}^n)$ and satisfies

$$\fint_{B_\tau} |\mathcal{E}^D v - (\mathcal{E}^D v)_{0,\tau}|^2\,\mathrm{d}z \leqslant c^* \tau^2 \fint_{B_1} |\mathcal{E}^D v - (\mathcal{E}^D v)_{0,1}|^2\,\mathrm{d}z \leqslant c^* \tau^2 \tag{40}$$

with a constant $c^* = c^*(n,p,q,\ell)$, where in the last step we used $(\mathcal{E}^D v)_{0,1} = 0$ (which follows from $(\mathcal{E}^D u_m)_{0,1} = 0$ and (38)) as well as (37). Suppose that we can show $\mathcal{E}^D u_m \xrightarrow{m} \mathcal{E}^D v$ in $L^2_{\mathrm{loc}}(B_1;\mathbb{M}^n)$ and $\lambda_m^{1-2/q}\mathcal{E}^D u_m \xrightarrow{m} 0$ in $L^q_{\mathrm{loc}}(B_1;\mathbb{M}^n)$ in case $q > 2$. Then (40) turns into

$$\lim_m \fint_{B_\tau} |\mathcal{E}^D u_m - (\mathcal{E}^D u_m)_{0,\tau}|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} |\mathcal{E}^D u_m - (\mathcal{E}^D u_m)_{0,\tau}|^q \,\mathrm{d}z \leqslant c^* \tau^2.$$

But then, choosing $c_* = 2c^*$, we get a contradiction to our assumption (35) since the third condition in (35) is equivalent to

$$\fint_{B_\tau} |\mathcal{E}^D u_m - (\mathcal{E}^D u_m)_{0,\tau}|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} |\mathcal{E}^D u_m - (\mathcal{E}^D u_m)_{0,\tau}|^q \,\mathrm{d}z > c_* \tau^2.$$

Therefore, we have to show: □

**Lemma 6.2.** *a)* $\mathcal{E}^D u_m \xrightarrow{m} \mathcal{E}^D v$ *in* $L^2_{\mathrm{loc}}(B_1;\mathbb{M}^n)$

*b)* $\lambda_m^{1-2/q}\mathcal{E}^D u_m \xrightarrow{m} 0$ *in* $L^q_{\mathrm{loc}}(B_1;\mathbb{M}^n)$ *if* $q > 2$

In the proof of Lemma 6.2 we need the convergences

$$\lambda_m^{1-2/q} u_m \xrightarrow{m} 0 \text{ in } L^q(B_1;\mathbb{R}^n), \quad \lambda_m^{1-2/q} \nabla u_m \xrightarrow{m} 0 \text{ in } L^q(B_1;\mathbb{R}^{n\times n}) \tag{41}$$

in case $q > 2$. To see this, we observe that by the interpolation inequality (4) we have

$$\|u_m\|_q \leqslant c\big(\|u_m\|_2 + \|\mathcal{E}^D u_m\|_q\big)$$

so that the Korn-type inequality (3) together with (38) implies

$$\|u_m\|_{1,q} \leqslant c\big(\|u_m\|_2 + \|\mathcal{E}^D u_m\|_q\big) \leqslant c\big(1 + \|\mathcal{E}^D u_m\|_q\big).$$

Thus,

$$\lambda_m^{q-2} \int_{B_1} |u_m|^q + |\nabla u_m|^q \,\mathrm{d}z \leqslant c\left(1 + \lambda_m^{q-2} \int_{B_1} |\mathcal{E}^D u_m|^q \,\mathrm{d}z\right),$$

wherein the right-hand side is uniformly bounded on account of (37). Therefore, we have $\lambda_m^{1-2/q} u_m \xrightarrow{m} : \overline{v}$ in $W^{1,q}(B_1;\mathbb{R}^n)$ as well as $\lambda_m^{1-2/q} u_m \xrightarrow{m} \overline{v}$ in $L^2(B_1;\mathbb{R}^n)$ (at least for a subsequence), which together with (35) and (38) implies (41). Moreover, we need the estimate contained in the following lemma in the proof of Lemma 6.2.

**Lemma 6.3.** *Let* $\rho \in (0,1)$ *and* $\psi_m := \lambda_m^{-1}[\Theta(A_m + \lambda_m \mathcal{E}^D u_m) - \Theta(A_m)]$*, where* $\Theta(\sigma) := (1 + |\sigma|^2)^{p/4}$ $(\sigma \in \mathbb{M}^n)$*. Then* $\psi_m \in W^{1,2}_{\mathrm{loc}}(B_1;\mathbb{R}^n)$ *and*

$$\int_{B_\rho} |\nabla \psi_m|^2 \,\mathrm{d}z \leqslant c(\rho) \int_{B_1} (1 + |A_m + \lambda_m \mathcal{E}^D u_m|^2)^{q/2-1} |\nabla u_m|^2 \,\mathrm{d}z,$$

*where the constant* $c(\rho)$ *is independent of* $m$*.*

*Proof.* By Lemma 5.7 and the definition of $u_m$ we have $\psi_m \in W^{1,2}_{\mathrm{loc}}(B_1;\mathbb{R}^n)$ with $\nabla\psi_m(z) = r_m\lambda_m^{-1}\nabla\Theta(\mathcal{E}^D u(x_m + r_m z))\nabla\mathcal{E}^D u(x_m + r_m z)$ so that after rescaling we obtain

$$\int_{B_\rho} |\nabla\psi_m|^2 \,\mathrm{d}z = \lambda_m^{-2} r_m^{2-n} \int_{B_{\rho r_m(x_m)}} |\nabla\phi|^2 \,\mathrm{d}x, \tag{42}$$

where $\phi \coloneqq \Theta(\mathcal{E}^D u) = (1 + |\mathcal{E}^D u|^2)^{p/4}$. To estimate the right-hand side, we recall $\nabla\phi_\nu \xrightarrow{\nu} \nabla\phi$ in $L^2_{\mathrm{loc}}(B_R)$ and show a limit version of the estimate

$$\int_{B_R} \eta^2 |\nabla\phi_\nu|^2 \,\mathrm{d}x \leqslant c \int_{B_R} |\nabla\eta|^2 \Gamma_\nu^{q/2-1} |\nabla v_\nu - Q|^2 \,\mathrm{d}x, \tag{43}$$

which is valid for each $\eta \in \mathring{C}^\infty(B_R)$ and $Q \in \mathbb{R}^{n\times n}$. Note that (43) can be obtained by analogous calculations as in the proof of Lemma 5.3 (recall (29)). The Korn-type inequality (3) implies

$$\|\nabla v_\nu - \nabla u\|_{p;B_r(\overline{x})} \leqslant c\big(\|v_\nu - u\|_{p;B_r(\overline{x})} + \|\mathcal{E}^D v_\nu - \mathcal{E}^D u\|_{p;B_r(\overline{x})}\big)$$

for each ball $B_r(\overline{x}) \Subset B_R$ so that the convergences stated in Lemma 5.1 and (32) together with the higher integrability results stated in Corollary 5.6 imply $\nabla v_\nu \xrightarrow{\nu} \nabla u$ in $L^p_{\mathrm{loc}}(B_R;\mathbb{R}^{n\times n})$ and a.e. in $B_R$ (at least for a subsequence). But then we can argue as in the proof of Lemma 4.6 in [2] to get from (43) the estimate

$$\int_{B_R} \eta^2 |\nabla\phi|^2 \,\mathrm{d}x \leqslant c \int_{B_R} |\nabla\eta|^2 \Gamma^{q/2-1} |\nabla u - Q|^2 \,\mathrm{d}x, \tag{44}$$

where $\Gamma \coloneqq 1 + |\mathcal{E}^D u|^2$. Now, if we choose $\eta \geqslant 0$ such that $\eta \equiv 1$ in $B_{\rho r_m}(x_m)$ and $|\nabla\eta| \leqslant c/(r_m - \rho r_m)$, (6.3) together with (44) implies (by scaling)

$$\int_{B_\rho} |\nabla\psi_m|^2 \,\mathrm{d}z = c(\rho)\lambda_m^{-2} \int_{B_1} (1 + |\mathcal{E}^D u(x_m + r_m z)|^2)^{q/2-1} |\nabla u(x_m + r_m z) - Q|^2 \,\mathrm{d}z.$$

Finally, choosing $Q \coloneqq A_m + r_m^{-1}\nabla\chi_m(z)$, the desired estimate follows. □

*Proof of Lemma 6.2.* Following the lines of [2] we show

$$\lim_m \int_{B_\rho} \int_0^1 (1-t)(1 + |A_m + \lambda_m \mathcal{E}^D v + t\lambda_m \mathcal{E}^D w_m|^2)^{p/2-1} |\mathcal{E}^D w_m|^2 \,\mathrm{d}t \,\mathrm{d}z = 0 \tag{45}$$

for each $\rho \in (0,1)$, where $w_m \coloneqq u_m - v$. From the minimality of $u$ we get for $r \in (0,1)$ (by scaling)

$$\int_{B_r} f(A_m + \lambda_m \mathcal{E}^D u_m) \,\mathrm{d}z \leqslant \int_{B_r} f(A_m + \lambda_m \mathcal{E}^D \varphi) \,\mathrm{d}z \tag{46}$$

for each $\varphi \in u_m + \mathring{W}^{1,2}(B_r;\mathbb{R}^n)$. Let us specify $\varphi$: let $\eta \in \mathring{C}^1(B_1)$, $\eta \geqslant 0$, with $\eta \equiv 1$ in $B_\rho$ and $\eta \equiv 0$ outside $B_r$ for some $r \in (\rho, 1)$. Then $\varphi \coloneqq u_m - \eta w_m$ is admissible in

(46). Next, we consider the relation

$$\begin{aligned}
&\int_{B_r}\int_0^1 (1-t)\eta D^2 f(A_m+\lambda_m\mathcal{E}^D v+t\lambda_m\mathcal{E}^D w_m)(\mathcal{E}^D w_m,\mathcal{E}^D w_m)\,\mathrm{d}t\,\mathrm{d}z\\
&=\lambda_m^{-2}\int_{B_r}\eta[f(A_m+\lambda_m\mathcal{E}^D u_m)-f(A_m+\lambda_m\mathcal{E}^D v)]\,\mathrm{d}z\\
&\quad-\lambda_m^{-1}\int_{B_r}\eta Df(A_m+\lambda_m\mathcal{E}^D v):\mathcal{E}^D w_m\,\mathrm{d}z.
\end{aligned}\tag{47}$$

Owing to (1) and the properties of $\eta$ the left-hand side of (47) is bounded from below by the left-hand side of (45). Therefore, we have to show that the right-hand side of (47) vanishes as $m\to\infty$. Using (46) and the convexity of $f$ we get

$$\begin{aligned}
\text{r.h.s. of (47)}&=\lambda_m^{-2}\Bigg(\int_{B_r} f(A_m+\lambda_m\mathcal{E}^D v+t\lambda_m\mathcal{E}^D u_m)\,\mathrm{d}z\\
&\quad-\int_{B_r}[(1-\eta)f(A_m+\lambda_m\mathcal{E}^D v+t\lambda_m\mathcal{E}^D u_m)\\
&\qquad+\eta f(A_m+\lambda_m\mathcal{E}^D v+t\lambda_m\mathcal{E}^D v)]\,\mathrm{d}z\Bigg)\\
&\quad-\lambda_m^{-1}\int_{B_r}\eta Df(A_m+\lambda_m\mathcal{E}^D v):\mathcal{E}^D w_m\,\mathrm{d}z\\
&\leqslant\lambda_m^{-2}\Bigg(\int_{B_r} f(A_m+\lambda_m[\mathcal{E}^D u_m-\mathcal{E}^D(\eta w_m)])\,\mathrm{d}z\\
&\quad-\int_{B_r} f(A_m+\lambda_m[(1-\eta)\,\mathcal{E}^D u_m+\eta\,\mathcal{E}^D v])\,\mathrm{d}z\Bigg)\\
&\quad-\lambda_m^{-1}\int_{B_r}\eta Df(A_m+\lambda_m\mathcal{E}^D v):\mathcal{E}^D w_m\,\mathrm{d}z\\
&=:\lambda_m^{-2}T_1-\lambda_m^{-1}T_2.
\end{aligned}\tag{48}$$

Let $\sigma_m\coloneqq A_m+\lambda_m[(1-\eta)\,\mathcal{E}^D u_m+\eta\,\mathcal{E}^D v]$. Then

$$\begin{aligned}
\lambda_m^{-2}T_1&=\lambda_m^{-2}\Bigg(\int_{B_r} f(\sigma_m-\lambda_m(\nabla\eta\odot w_m)^D)-f(\sigma_m)\,\mathrm{d}z\Bigg)\\
&=\int_{B_r}\int_0^1(1-t)D^2 f(\sigma_m-t\lambda_m(\nabla\eta\odot w_m)^D)\\
&\qquad((\nabla\eta\odot w_m)^D,(\nabla\eta\odot w_m)^D)\,\mathrm{d}t\,\mathrm{d}z\\
&\quad-\lambda_m^{-1}\int_{B_r} Df(\sigma_m):(\nabla\eta\odot w_m)^D\,\mathrm{d}z\\
&=:I_1-\lambda_m^{-1}I_2.
\end{aligned}$$

For the first integral on the right-hand side we obtain (using (1) and (35))

$$
\begin{aligned}
I_1 &\leqslant c \int_{B_r} (1 + |\sigma_m|^2 + \lambda_m^{q-2} |\nabla\eta|^{q-2} |w_m|^2)^{q/2-1} |\nabla\eta|^2 |w_m|^2 \,\mathrm{d}z \\
&\leqslant c \bigg( \int_{B_r} |\nabla\eta|^2 |w_m|^2 \,\mathrm{d}z + \lambda_m^{q-2} \int_{B_r} |\nabla\eta|^q |w_m|^q \,\mathrm{d}z \\
&\quad + \int_{B_r} |\nabla\eta|^2 |w_m|^2 |\sigma_m|^{q-2} \,\mathrm{d}z \bigg) \\
&\leqslant c \bigg( \int_{B_r} |\nabla\eta|^2 |w_m|^2 \,\mathrm{d}z + \lambda_m^{q-2} \int_{B_r} |\nabla\eta|^q |w_m|^q \,\mathrm{d}z \\
&\quad + \lambda_m^{q-2} \int_{B_r} |\nabla\eta|^2 |w_m|^2 \big( |\mathcal{E}^D u_m|^{q-2} + |\mathcal{E}^D v|^{q-2} \big) \,\mathrm{d}z \bigg),
\end{aligned}
$$

wherein the right-hand side vanishes as $m \to \infty$ on account of (37), (38), (41), and $v \in W^{1,\infty}_{\mathrm{loc}}(B_1;\mathbb{R}^n)$. Returning to (48), we have shown

$$
\text{r.h.s. of (47)} \leqslant O(m) + \lambda_m^{-1} |I_2 + T_2|, \tag{49}
$$

where $O(m) \overset{m}{\longrightarrow} 0$. For the remaining term we observe that $I_2 + T_2$ can be rewritten as follows:

$$
\begin{aligned}
I_2 + T_2 &= \int_{B_r} [Df(\sigma_m) - Df(A_m + \lambda_m \mathcal{E}^D v)] : (\nabla\eta \odot w_m)^D \,\mathrm{d}z \\
&\quad + \int_{B_r} Df(A_m + \lambda_m \mathcal{E}^D v) : \mathcal{E}^D(\eta w_m) \,\mathrm{d}z =: J_1 + J_2.
\end{aligned}
$$

Since $\sigma_m = A_m + \lambda_m \mathcal{E}^D v + \lambda_m (1-\eta)\, \mathcal{E}^D w_m$, we have

$$
\begin{aligned}
|J_1| = \lambda_m \bigg| \int_{B_r} \int_0^1 &(1-\varphi) D^2 f(A_m + \lambda_m \mathcal{E}^D v + t\lambda_m (1-\eta)\, \mathcal{E}^D w_m) \\
&(\mathcal{E}^D w_m, (\nabla\eta \odot w_m)^D) \,\mathrm{d}t \,\mathrm{d}z \bigg|,
\end{aligned}
$$

which can be estimated similar to $I_1$ with the result that $\lambda_m^{-1} J_1 \overset{m}{\longrightarrow} 0$. Similarly, we get $\lambda_m^{-1} J_2 \overset{m}{\longrightarrow} 0$ by observing

$$
\begin{aligned}
\lambda_m^{-1} |J_2| &= \lambda_m^{-1} \bigg| \int_{B_r} [Df(A_m + \lambda_m \mathcal{E}^D v) - Df(A_m)] : \mathcal{E}^D(\eta w_m) \,\mathrm{d}z \bigg| \\
&= \bigg| \int_{B_r} \int_0^1 D^2 f(A_m + t\lambda_m \mathcal{E}^D v)(\mathcal{E}^D v, \mathcal{E}^D(\eta w_m)) \,\mathrm{d}t \,\mathrm{d}z \bigg|.
\end{aligned}
$$

Returning to (49), we have established that the right-hand side of (47) vanishes as $m \to \infty$ so that (45) follows from (1) and the choice of $\eta$. Now, we can argue as in [2]. First, we note that (45) immediately implies part a) as well as part b) in the case $p = q$. In case $p \geqslant 2$, $q > p$ we use the estimate

$$\lambda_m^{q-2} |\mathcal{E}^D u_m|^q \leqslant \lambda_m^{q-2+(2/p-1)q} \psi_m^{2q/p}$$

on the set $B_\rho - U_m$, where $U_m := \{z \in B_\rho : \lambda_m |\mathcal{E}^D u_m| \leqslant M\}$ with a sufficiently large number $M$. Since $q < (1 + 2/n)p$, we have $2q/p < 2n/(n-2)$ so that Sobolev's imbedding theorem together with the estimate from Lemma 6.3 and the convergences (38), (41) shows

$$\int_{B_\rho} \psi_m^{2q/p} \, \mathrm{d}z \leqslant c(\rho).$$

On the other hand, it is straightforward to show $\int_{U_m} \lambda_m^{q-2} |\mathcal{E} u_m|^q \, \mathrm{d}z \overset{m}{\longrightarrow} 0$ and we conclude b). The remaining case $p < 2$ can be handled by similar arguments; compare [2] (p. 397). □

## 7 The two-dimensional case: proof of Theorem 1.1 c)

Let us assume in the following that we are in the situation of part c) of Theorem 1.1, that is, we have $n = 2$ and $q < \min(2p, p+2)$. To show full regularity of the minimizer $u$, we use a technique described in [3], whose main ingredient is a higher integrability lemma.

First, we note that according to Corollary 5.6 we have

$$v_\nu \overset{\nu}{\rightharpoonup} u \quad \text{in } W^{1,t}(B_r(\overline{x}); \mathbb{R}^n) \tag{50}$$

for each $t \in (1, \infty)$ and each ball $B_r(\overline{x}) \Subset B_R$. (Note that $\widetilde{q}$ may be replaced by each exponent $t$ in case $n = 2$.)

From the proof of Lemma 5.3 (compare (27)) we infer

$$\int_{B_R} \eta^2 H_\nu^2 \, \mathrm{d}x \leqslant c \int_{B_R} \eta |\nabla \eta| |\nabla \tau_\nu| |\partial_k v_\nu - \chi^k| \, \mathrm{d}x$$

for each $\eta \in \overset{\circ}{C}{}^\infty(B_R)$ and $\chi^k \in \mathcal{K}_{B_R}$. We choose $\eta \geqslant 0$ such that $\eta \equiv 1$ in $B_{r/2}(\overline{x})$, $\eta \equiv 0$ outside $B_r(\overline{x})$, and $|\nabla \eta| \leqslant c/r$ and get the starting inequality (recall (28))

$$\fint_{B_{r/2}(\overline{x})} H_\nu^2 \, \mathrm{d}x \leqslant \frac{c}{r} \fint_{B_r(\overline{x})} \Gamma_\nu^{(q-2)/4} H_\nu |\partial_k v_\nu - \chi^k| \, \mathrm{d}x. \tag{51}$$

Let $\gamma := 4/3$. Then $v_\nu \in W^{2,\gamma}_{\mathrm{loc}}(B_R; \mathbb{R}^2)$ according to Lemma 5.1. Hence, using the Hölder and the Poincaré-type inequality (17) with suitable functions $\chi^k$ on the right-hand side of (51), we find

$$\fint_{B_{r/2}(\overline{x})} H_\nu^2\,\mathrm{d}x \leqslant c\bigg(\fint_{B_r(\overline{x})} \big(\Gamma_\nu^{(q-2)/4} H_\nu\big)^\gamma\,\mathrm{d}x\bigg)^{1/\gamma} \frac{1}{r}\bigg(\fint_{B_r(\overline{x})} |\partial_k v_\nu - \chi^k|^4\,\mathrm{d}x\bigg)^{1/4}$$

$$\leqslant c\bigg(\fint_{B_r(\overline{x})} \big(\Gamma_\nu^{(q-2)/4} H_\nu\big)^\gamma\,\mathrm{d}x\bigg)^{1/\gamma} \bigg(\fint_{B_r(\overline{x})} |\nabla \mathcal{E}^D v_\nu|^\gamma\,\mathrm{d}x\bigg)^{1/\gamma}.$$

Applying (29) on the right-hand side we arrive at

$$\bigg(\fint_{B_{r/2}(\overline{x})} H_\nu^2\,\mathrm{d}x\bigg)^{1/2} \leqslant c\bigg(\fint_{B_r(\overline{x})} \big(h_\nu H_\nu\big)^\gamma\,\mathrm{d}x\bigg)^{1/\gamma}, \tag{52}$$

where $h_\nu \coloneqq \max\big\{\Gamma_\nu^{(2-p)/4}, \Gamma_\nu^{(q-2)/4}\big\}$. This is exactly the situation of Lemma 1.2 in [3] with the choices

$$d = 2/\gamma = 3/2, \quad f = H_\nu^\gamma, \quad g = h_\nu^\gamma, \quad h = 0.$$

Note that from Lemmas 5.4 and 5.5 we know that $H_\nu$ is uniformly bounded in $L^2_{\mathrm{loc}}(B_R)$. Thus, we can apply Lemma 1.2 from [3] if $\exp(\beta h_\nu^2) \in L^1_{\mathrm{loc}}(B_R)$ holds for some $\beta > 0$. In fact, since $\phi_\nu$ is uniformly bounded in $W^{1,2}_{\mathrm{loc}}(B_R)$ (according to Lemma 5.7) one can show

$$\int_{B_\rho(\overline{x})} \exp(\beta \phi_\nu^{2-\kappa})\,\mathrm{d}x \leqslant c(\beta,\kappa,\rho) < \infty$$

for all $\beta > 0$, $\kappa \in (0,1)$ and for each ball $B_\rho(\overline{x}) \Subset B_R$, from which it follows that

$$\int_{B_\rho(\overline{x})} \exp(\beta h_\nu^2)\,\mathrm{d}x \leqslant c(\beta,\rho) < \infty;$$

see [3] (p. 141) for similar arguments. Lemma 1.2 from [3] implies

$$\int_{B_\rho(\overline{x})} H_\nu^2 \log^{c_0\beta}(e + H_\nu)\,\mathrm{d}x \leqslant c(\beta,\rho) < \infty$$

and, using the estimate $|\nabla \tau_\nu| \leqslant c h_\nu H_\nu$, we end up with

$$\int_{B_\rho(\overline{x})} |\nabla \tau_\nu|^2 \log^{\beta'}(e + H_\nu)\,\mathrm{d}x \leqslant c(\beta',\rho) < \infty$$

for each $\beta' > 1$, which gives us continuity of $\tau_\nu = Df_\nu(\mathcal{E}^D v_\nu)$ uniformly with respect to $\nu$; compare [3]. Together with the convergence (32) this implies continuity of $\tau = Df(\mathcal{E}^D u)$ so that $\mathcal{E}^D u$ is continuous as well. Therefore, we can argue as in the previous section to get that $u$ belongs to $W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ and that each partial derivative of $u$ solves the system (34) with continuous matrix $D^2 f(\mathcal{E}^D u)$, but now on each subdomain $\omega \Subset \Omega$. Hence, $\nabla u \in C^{0,\alpha}(\Omega;\mathbb{R}^{2\times 2})$ follows from Corollary 4.2.

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