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Abstract

Harmonic weak Maaß forms have recently been shown to have quite a few interesting arithmetic applications, including their connection to Ramanujan's work on (mock) theta functions [2, 3, 5, 12].

The peculiar behaviour of these functions under differential operators suggests that the $(\mathfrak{g}, K)$-modules generated by these functions, viewed as functions on the group $\mathrm{SL}_2(\mathbb{R})$, have a simple structure, in spite of being more complicated than the well-known picture in the case of cusp forms. In this note we show that this is indeed the case.

1. Preliminaries.

We denote by $\mathbb{H}$ the upper half plane in $\mathbb{C}$ and by G the group $\mathrm{GL}_2^+(\mathbb{R})$. The matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ acts on $\mathbb{H}$ by fractional linear transformations $z \longmapsto gz = \frac{az+b}{cz+d}$ and on functions on $\mathbb{H}$ by $(f|_k g)(z) = \det(g)^{\frac{k}{2}}(cz+d)^{-k} f(gz)$ resp. $(f[g]_k)(z) = (\dfrac{c\overline{z}+d}{|cz+d|})^k f(gz)$ with $z = \{\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right) \mid u > 0\}$ acting trivially.

We recall that $g \in G$ has a decomposition $g = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \kappa_\theta$ with $\kappa_\theta = r(\theta) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in K = \mathrm{SO}_2(\mathbb{R})$, where $x, y, u, \theta \in \mathbb{R}$, $y, u > 0$, with x, y, u uniquely determined and θ unique modulo $2\pi\mathbb{Z}$.

With x, y, u, θ as above as coordinates on G we have the differential operators

$$\begin{aligned} R^{(G)} &= e^{2i\theta}(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}) \\ L^{(G)} &= e^{-2i\theta}(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}) \\ \Delta^{(G)} &= -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial^2}{\partial x\partial\theta} \end{aligned}$$

on C^∞-functions on G; the operator $\Delta^{(G)}$ is the Laplace-Beltrami operator for G.

We note that sometimes the parametrization of G is considered with θ replaced by $-\theta$, i.e., κ_θ above is written as $\kappa_{(-\theta)}$; this leads to sign changes in the definitions of the differential operators above.

For $k \in \mathbb{Z}$, we have the corresponding differential operators on functions on $\mathbb{H}$ as in [9, 6] given by

$$\begin{aligned} R_k^{(R)} &= iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{k}{2} \\ L_k^{(R)} &= -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{k}{2} \\ \Delta_k^{(R)} &= -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + iky\frac{\partial}{\partial x} \end{aligned}$$

(the superscript (R) standing for Roelcke). They satisfy

$(R_k f)[g]_{k+2} = R_k(f[g]_k), \quad (L_k f)[g]_{k-2} = L_k(f[g]_k), \quad (\Delta_k f)[g]_k = \Delta_k(f[g]_k).$

One has moreover: $\Delta_k^{(R)} = -L_{k+2}^{(R)} R_k^{(R)} - \frac{k}{2}(\frac{k}{2}+1) = -R_{k-2}^{(R)} L_k^{(R)} + \frac{k}{2}(1-\frac{k}{2})$,
$R_k^{(R)} \circ \Delta_k^{(R)} = \Delta_{k+2}^{(R)} \circ R_k^{(R)}, \quad L_k^{(R)} \circ \Delta_k^{(R)} = \Delta_{k-2}^{(R)} \circ L_k^{(R)}$. The operators $R^{(G)}$, $R_k^{(R)}$ are called raising operators, the operators $L^{(G)}$, $L_k^{(R)}$ are called lowering operators.

We consider a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ with a character $\chi : \Gamma \longrightarrow \mathbb{C}^\times$ and for $k \in \mathbb{Z}$ functions f on $\mathbb{H}$ satisfying $f[\gamma]_k = \chi(\gamma) f \quad$ for all $\gamma \in \Gamma$.
Such an f induces a function $F = \sigma_k(f)$ on G given by $F(g) = (f[g]_k)(i)$; the function $F = \sigma_k(f)$ satisfies then $F(g\kappa_\theta) = e^{ik\theta}F(g) \quad (\theta \in R)$ and $F(\gamma g \left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right)) = \chi(\gamma)F(g) \quad (\gamma \in \Gamma,\ u > 0)$.
The map $f \longmapsto \sigma_k(f)$ is a bijection between the respective types of functions on $\mathbb{H}$ and G, with inverse given by

$$F \longmapsto f,\ f(x+iy) = F\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

On C^2-functions the bijection commutes with the actions of $L_k^{(R)},\ R_k^{(R)},\ \Delta_k^{(R)}$ resp. $L^{(G)},\ R^{(G)},\ \Delta^{(G)}$.

In the theory of modular forms it is more usual to consider functions $f : \mathbb{H} \longrightarrow \mathbb{C}$ with $f|_k\gamma = \chi(\gamma) f \quad (\gamma \in \Gamma)$ instead of the transformation equation $f[\gamma]_k = \chi(\gamma) f$ treated above; for such an f the function $f^{(R)}$ given by $f^{(R)}(x+iy) = y^{\frac{k}{2}} f(x+iy)$ satisfies the latter transformation equation.
If f as above is an eigenfunction of the weight k hyperbolic Laplacian $\Delta_k = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + iky(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}) = \Delta_k^{(R)} - ky\frac{\partial}{\partial y}$, the function $f^{(R)}$ is an eigenfunction of $\Delta_k^{(R)}$ with eigenvalue $\lambda - \frac{k}{2}(\frac{k}{2}-1)$.
The corresponding lowering and raising operators $L_k = y(L_k^{(R)} + k/2)$, $R_k = y^{-1}(R_k^{(R)} + k/2)$ have the disadvantage of changing the eigenvalue of a

Laplace eigenfunction and the advantage that the kernel of L_k consists of the holomorphic functions on $\mathbb{H}$.

2. Weak Maass forms of integral weight.

We recall from [4]:

Definition 1. *A C^∞-function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called a weak Maaß form of weight k and character (or nebentypus) χ for the congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ if it satisfies*

a) *$f|_k\gamma = \chi(\gamma)f$ for all $\gamma \in \Gamma$*

b) *For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there is a polynomial*

$$P_{f,\gamma} = \sum_{n\geq 0} c^+_{f,\gamma}(-n)X^n \in \mathbb{C}[X]$$

and an $\epsilon = \epsilon_{f,\gamma} > 0$ such that $(f|_k\gamma)(z) - P_{f,\gamma}(e^{-2\pi iz/h_\gamma}) = \mathrm{O}(e^{-\epsilon y})$ (where h_γ is the width of the cusp $\gamma(\infty)$) as $y = \mathrm{Im}(z) \longrightarrow \infty$.

$P_{f,\gamma}(q^{-1/h_\gamma}) = \sum_{n\leq 0} c^+_{f,\gamma}(n)q^{n/h_\gamma}$ is called the principal part of f at the cusp $\gamma(\infty)$.

c) *f is an eigenfunction of Δ_k.*

If the Δ_k-eigenvalue in c) is 0, the function f is called harmonic.

Definition and Lemma 2. *If $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a weak Maaß form as above with Δ_k-eigenvalue λ, the corresponding function $f^{(R)}$ (satisfying $f^{(R)}[\gamma]_k = \chi(\gamma)f$ for all $f \in \Gamma$) is called a weak Maaß-Roelcke form of weight k (MR-form for short).*

With $P^{(R)}_{f,\gamma}(z) = y^{\frac{k}{2}}P_{f,\gamma}(e^{-2\pi iz/h_\gamma})$ the function $f^{(R)}$ satisfies a cusp condition analogous to the one given in the previous definition, it is an eigenfunction of $\Delta^{(R)}_k$ with eigenvalue $\lambda^{(R)} = \lambda - \frac{k}{2}(\frac{k}{2} - 1)$.

The functions $L^{(R)}_k f^{(R)}$, $R^{(R)}_R f^{(R)}$ are then weak Maaß-Roelcke forms of weights $k-2$, $k+2$ respectively and with the same $\Delta^{(R)}_{k-2}$ resp. $\Delta^{(R)}_{k+2}$-eigenvalue $\lambda^{(R)}$.

Proof. It is easily checked (or deduced from the results of [4]) that $L^{(R)}_k f^{(R)}$ and $R^{(R)}_k f^{(R)}$ satisfy the required cusp conditions.

The rest of the assertion is obvious from the facts stated in Section 1. □

It is well-known that one has compatible actions of the complexified Lie algebra $\mathfrak{g}$ of $\mathrm{SL}_2(\mathbb{R})$ and of the maximal compact subgroup $K = \mathrm{SO}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ on analytic functions on G: The action of K is by right translation, the generators

$$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \; L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \; H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of $\mathfrak{g}$ act by $R^{(G)}$, $L^{(G)}$, $H^G = -i\frac{\partial}{\partial\theta}$ respectively, and after extension of this action to the universal eveloping algebra $U(\mathfrak{g})$ the Casimir element $H^2+2RL+2LR$ acts by $-4\Delta^{(G)}$. It is also well-known that under this action of $\mathfrak{g}$ and K a function $F \in C^\infty(G)$ that is both K-finite and $Z(G)$-finite generates an admissible $(\mathfrak{g}, K)$-module V, i.e., V is the direct sum of finite dimensional K-invariant subspaces and each finite dimensional irreducible representation of K occurs only with finite multiplicity.

The structure of this $(\mathfrak{g}, K)$-module is also well-known, if F is the function on G corresponding to a holomorphic cusp form of weight $k > 0$ in the way described in Section 1: One obtains then an irreducible $(\mathfrak{g}, K)$-module which is a (limit of) discrete series representation $\mathcal{D}^+(k)$ and in which F is a vector of lowest K-type k. The K-types occurring in this representation are the set $\Sigma^+(k) = \{\ell \in Z \mid \ell \equiv k \bmod 2,\ \ell \geq k\}$ (notations as in [6]), the Laplace eigenvalue is $\lambda = \frac{k}{2}(1 - \frac{k}{2})$. The raising and lowering operators push the K-types up and down, acting injectively with the exception that $L^{(G)}$ annihilates the lowest K-type k.

Similarly, an antiholomorphic cusp form (i.e., a complex conjugate of a holomorphic cusp form) of weight k gives rise to a Maaß-Roelcke form of weight $-k$ and a function on G of K-type $-k$ which generates an irreducible $(\mathfrak{g}, K)$-module of type $\mathcal{D}^-(k)$ with set of K-types $\Sigma^-(k) = \{\ell \in \mathbb{Z} \mid \ell \equiv k \bmod 2,\ \ell \leq -k\}$; again $L^{(G)}$ and $R^{(G)}$ act injectively with the exception that $R^{(G)}$ annihilates the highest K-type $-k$.

We denote (see [6]) for $k \geq 1$ by $V(k)$ the $(\mathfrak{g}, K)$-module of K-finite vectors in the induced representation $H(s, -s, k)$ (with $s = \frac{k-1}{2}$) of $\mathrm{GL}_2^+(\mathbb{R})$ which is given by normalized induction from the character

$$\chi \begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} = (\mathrm{sgn}(y_1))^k |y_1|^s |y_2|^{-s}$$

of the upper triangular subgroup. It is well known (see [6]) that $V(k)$ is indecomposable with each K-type $l \equiv k \bmod 2$ appearing with multiplicity 1 and contains precisely two irreducible submodules which are of types $\mathcal{D}^+(k)$, $\mathcal{D}^-(k)$.

In contrast to the case of holomorphic cusp forms, the functions on G corresponding to weak Maaß forms do not embed into $L^2(\Gamma \backslash G)$ so that one can not expect complete reducibility. Instead, the known results about the action of the raising and lowering operators on weak harmonic Maaß forms, see [4], suggest that the $(\mathfrak{g}, K)$-module generated by the function on G corresponding to a weak harmonic Maaß form is indecomposable, but not irreducible. We have in fact

Proposition 3. *Let f be a harmonic weak Maaß form of weight $2-k$ with $k \geq 1$ for the congruence subgroup Γ and denote by F the corresponding function on $G = \mathrm{GL}_2^+(\mathbb{R})$ (which is an eigenfunction of the Laplace-Bertrami operator $\Delta^{(G)}$ with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$).*

Then $L^{(G)}F$ generates (if it is non-zero) a $(\mathfrak{g}, K)$-module of type $\mathcal{D}^-(k)$ and corresponds to an antiholomorphic modular form of weight k.

Similarly, $(R^{(G)})^{k-1}F$ generates, if it is non-zero, a $(\mathfrak{g}, K)$-module of type $\mathcal{D}^+(k)$ and corresponds to a weakly holomorphic (i.e., holomorphic with possible poles at ∞) modular form of weight k. The $(\mathfrak{g}, K)$-module generated by F is indecomposable but not irreducible; it is isomorphic to $V(k)$ if both $L^{(G)}(F)$, $(R^{(G)})^{k-1}(F)$ are non-zero, and to $V(k)/\mathcal{D}^+(k)$ resp. $V(k)/\mathcal{D}^-(k)$ if one (and only one) of them is zero.

Proof. The $(\mathfrak{g}, K)$-module structure is completely determined by the actions of $L^{(G)}$, $R^{(G)}$, $\Delta^{(G)}$ and the occurring K-types and their multiplicities. The latter are contained in $\{\ell \in \mathbb{Z} \mid \ell \equiv k \bmod 2\}$ and occur with multiplicity 1. The $\Delta^{(G)}$-action is given as multiplication with $\lambda = \frac{k}{2}(1-\frac{k}{2})$, and $L^{(G)}$, $R^{(G)}$ push the K-type up or down by 2, with $R^{(G)}$ possibly annihilating a vector of type $\ell = -k$ or $\ell = k-2$, and $L^{(G)}$ possibly annihilating a vector of type $\ell = k$ or $\ell = 2-k$.

Since we are starting at K-type $2-k$, all K-types $\leq k-2$ appear if $L^{(G)}F \neq 0$, and all K-types $\geq 2-k$ appear if $(R^{(G)})^{k-1}F \neq 0$. Moreover, since the Laplace eigenvalue is $\frac{k}{2}(1-\frac{k}{2})$, we see by translating the identities

$$\begin{aligned}\Delta_\ell^{(R)} + L_{\ell-1}^{(R)} \circ R_\ell^{(R)} &= -\frac{\ell}{2}(\frac{\ell}{2}+1)\\ \Delta_\ell^{(R)} + R_{\ell-2}^{(R)} \circ R_\ell^{(R)} &= \frac{\ell}{2}(1-\frac{\ell}{2})\end{aligned}$$

into identities for the action of $\Delta^{(G)}$, $R^{(G)}$, $L^{(G)}$ on the respective K-types that $L^{(G)} \circ R^{(G)}$ annihilates the K-type $\ell = k-2$ and $R^{(G)} \circ L^{(G)}$ annihilates the K-type $\ell = 2-k$.

This proves all the assertions. □

Remark 4. a) *A similar phenomenon occurs for the non holomorphic Eisenstein series of weight 2 for the full modular group whose $(\mathfrak{g}, K)$-module is $V(2)/\mathcal{D}^-(2)$.*

b) *The representation theoretic point of view to interpret the (anti)-holomorphic forms related to a harmonic weak Maaß form as vectors in the same $(\mathfrak{g}, K)$-module should facilitate an adelic treatment and generalizations to the number field case as well as generalizations to other (e.g. symplectic or orthogonal) groups and to vector valued modular forms for these. The fact that these modules are indecomposable but not irreducible causes some phenomena which do not occur in the representation theoretic investigation of cusp forms.*

c) *Using the Fourier expansion of f given in* [4] *one sees that it cannot happen that both $L^{(G)}F$ and $(R^{(G)})^{k-1}F$ are zero for non constant F, essentially because these two differential equations do not have a non trivial periodic solution.*

3. Half integral weight.

In order to deal with the case of half integral weight we have to modify the group theoretic setup, shifting from $\mathrm{GL}_2^+(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R})$ to the metaplectic group $\widetilde{\mathrm{SL}_2}(\mathbb{R}) =: \widetilde{G}$.
We will see that the situation is almost completely parallel to the case of integral weight, but since the necessary ingredients are somewhat scattered through the literature we prefer to go through this setup in some detail.
We recall that the metaplectic group $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ is as a set $\mathrm{SL}_2(\mathbb{R}) \times \{\pm 1\}$, with multiplication given by $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, c(g_1, g_2)\epsilon_1\epsilon_2)$, where $c(\cdot, \cdot)$ is the cocycle given by

$$c(g_1, g_2) = (x(g_1), x(g_2))_{\mathbb{R}}(-x(g_1)x(g_2), x(g_1 g_2)_{\mathbb{R}},$$

where $(\cdot, \cdot)_{\mathbb{R}}$ denotes the Hilbert symbol for $\mathbb{R}$ and

$$x(g) = \begin{cases} c & c \neq 0 \\ d & c = 0 \end{cases} \quad \text{for } g = \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

$\widetilde{G}$, although not linear, is a real Lie group with Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and maximal compact subgroup $\widetilde{K} \cong \mathbb{R}/4\pi\mathbb{Z}$, an isomorphism being given by

$$\theta + 4\pi\mathbb{Z} \longmapsto \widetilde{r}(\theta) = (r(\theta), \epsilon_\theta)$$

with

$$\epsilon_\theta = \begin{cases} 1 & -\pi < \theta \leq \pi \\ -1 & \pi < \theta \leq 3\pi \end{cases} ;$$

the map

$$i\,\theta\, H \longmapsto \widetilde{r}(\theta)$$

coincides with the exponential map

$$i\,\theta\, H = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \longmapsto \exp(i\,\theta\, H) \in \widetilde{G}$$

(see [1, Sect. 5].
We denote the image of this map by $\widetilde{K}$. The elements of $\widetilde{G}$ can be written uniquely as $g = b(z)\widetilde{r}(\theta)$ with $z = x + iy \in \mathbb{H}$, $b(z) = \left(\begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{\frac{1}{2}} \end{pmatrix}, 1\right) \in \widetilde{G}$.
With this parametrization we have again the differential operators $R^{(\widetilde{G})}, L^{(\widetilde{G})}, \Delta^{(\widetilde{G})}$ as in Section 1. We also have the same relations to the action of the Lie algebra $\mathfrak{sl}_2$ as before, i.e., R acts by $R^{(\widetilde{G})}$, L by $L^{(\widetilde{G})}$, H by $-i\frac{\partial}{\partial\theta}$ and $H^2 + 2RL + 2LR$ by $-4\Delta^{(\widetilde{G})}$.

We have a correspondence between functions F on $\widetilde{G}$ satisfying

$$F(g\widetilde{r}(\theta)) = e^{ik\theta}F(g) \quad (g \in \widetilde{G},\ \theta \in \mathbb{R})$$

and functions on $\mathbb{H}$, given by $f \longmapsto F = \widetilde{\sigma}_k(f)$ with

$$\begin{aligned} F(b(z)\widetilde{r}(\theta)) &= e^{ik\theta}f(z) \\ f(z) &= F(b(z)); \end{aligned}$$

this correspondence commutes with the actions of $L^{(\widetilde{G})}, R^{(\widetilde{G})}, \Delta^{(\widetilde{G})}$ on functions on H on the one side and of $L_k^{(R)}, R_k^{(R)}, \Delta_k^{(R)}$ on functions on $\mathbb{H}$ on the other side.

The fact that $\widetilde{r}(\theta) = \exp(i\,\theta\, H)$ is true implies that the $\mathfrak{sl}_2$-action on functions on G is compatible with the action of $\widetilde{K}$ by right translation. In particular, functions F satisfying

$$F(g\widetilde{r}(\theta)) = e^{ik\theta}F(g)$$

for $g \in \widetilde{G}$, $\theta \in \mathbb{R}$ are eigenfunctions of $H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$ with eigenvalue k, so that the study of $(\mathfrak{g}, \widetilde{K})$-modules of functions on $\widetilde{G}$ becomes completely parallel to that of $(\mathfrak{g}, K)$-modules of functions on G discussed in Section 2.

Although one can not embed $\mathrm{SL}_2(\mathbb{Z})$ or $\Gamma_0(N)$ for some $N \in \mathbb{N}$ homomorphically into $\widetilde{G}$, the group $\Gamma_1(4)$ can be embedded by the map

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (\gamma, \chi(\gamma)) =: \widetilde{\gamma},$$

where $\chi(\gamma) = (\frac{c}{d})$ with $(\frac{c}{d})$ denoting the extended Jacobi symbol as in [10] satisfying

$$(\frac{c}{d}) = \begin{cases} (\frac{c}{|d|}) & \text{if } c \neq 0,\ c > 0 \text{ or } d > 0 \\ -(\frac{c}{|d|}) & \text{if } c < 0,\ d < 0 \\ 1 & \text{if } c = 0, \end{cases}$$

see [7, Sect. 2.2]; we write $\widetilde{\Gamma_1(4)}$ for the image of this map. For $\gamma \in \Gamma_1(4)$ we write $j(\gamma, z) = (\frac{c}{d})(cz+d)^{\frac{1}{2}}$ with $-\frac{\pi}{2} < \arg(cz+d)^{\frac{1}{2}} \leq \frac{\pi}{2}$. For half integral $k = \kappa + \frac{1}{2}$ ($\kappa \in \mathbb{Z}$) we write $f[\gamma]_k(z) = (\frac{\overline{j(\gamma, z)}}{|j(\gamma, z)|})^{2k} f(\gamma z)$ and $f|_k\gamma(z) = j(\gamma, z)^{-2k}f(\gamma z)$. As before, if $f : \mathbb{H} \longrightarrow \mathbb{C}$ transforms like a classical modular form of weight k for $\Gamma_1(4N)$, i.e., satisfies $f|_k\gamma = f$ for all $\gamma \in \Gamma_1(4N)$, the function $f^{(R)}$ given by $f^{(R)}(z) = \mathrm{Im}(z)^{\frac{k}{2}} f(z)$ satisfies $f^{(R)}[\gamma]_k = f^{(R)}$ for all $\gamma \in \Gamma_1(4N)$ and $F = \widetilde{\sigma}_k(f^{(R)})$ satisfies $F(\widetilde{\gamma}g) = F(g)$ for all $\widetilde{\gamma} \in \widetilde{\Gamma_1(4N)}$, $g \in \widetilde{G}$.

Moreover, defining L_k, R_k, Δ_k as usual (see Section 1) we obtain as before that for an eigenfunction f of Δ_k with eigenvalue λ the corresponding $f^{(R)}$ is a $\Delta_k^{(R)}$-eigenfunction of eigenvalue $\lambda - \frac{k}{2}(\frac{k}{2} - 1)$. The weight raising and lowering properties and the commutation relations for $L_k^{(R)}$, $R_k^{(R)}$ have been discussed in [8]; since

$$\begin{aligned} L^{(\widetilde{G})}\widetilde{\sigma}_k(f^{(R)}) &= \widetilde{\sigma}_{k-2}(L_k^{(R)}f^{(R)}), \\ R^{(\widetilde{G})}\widetilde{\sigma}_k(f^{(R)}) &= \widetilde{\sigma}_{k+2}(R_k^{(R)}f^{(R)}), \\ \Delta^{(\widetilde{G})}\widetilde{\sigma}_k(f^{(R)}) &= \widetilde{\sigma}_k(\Delta_k^{(R)}f^{(R)}) \end{aligned}$$

hold, this discussion carries over to $L^{(\widetilde{G})}, R^{(\widetilde{G})}, \Delta^{(\widetilde{G})}$.
Taken together we have

Lemma 5. *The assertions of Lemma 2 are true for the case of half integral weight $k = \frac{\kappa}{2} + 1$ as well.*

Lemma 6. *Let $k = \frac{1}{2} + \kappa \in \frac{1}{2} + \mathbb{Z}$ and denote for $\nu = \pm\frac{1}{2}$ by $\mathcal{B}(k-1, \nu)$ the (induced) representation of $\widetilde{G} = \widetilde{\mathrm{SL}_2}(\mathbb{R})$ (acting by right translation) on the space of square integrable functions $\varphi : \widetilde{G} \longrightarrow \mathbb{C}$ satisfying*

$$\begin{aligned} \varphi((b(z), \epsilon)\widetilde{g}) &= \epsilon y^k \varphi(\widetilde{g}) \quad (z = x + iy \in \mathbb{H}) \\ \varphi(\widetilde{r}(\pi)\widetilde{g}) &= e^{i\nu\pi}\varphi(\widetilde{g}). \end{aligned}$$

Then the $(\mathfrak{g}, \widetilde{K})$-module $\widetilde{V}(k, \nu)$ of $\widetilde{K}$-finite functions in the space of $\mathcal{B}(k-1, \nu)$ is spanned by the functions $\widetilde{\varphi}_n$ for $n \in \nu + 2\mathbb{Z}$ with

$$\widetilde{\varphi}_n(\widetilde{g}\widetilde{r}(\theta)) = e^{in\theta}\widetilde{\varphi}_n(\widetilde{g}) \quad (\widetilde{g} \in \widetilde{G},\ \theta \in \mathbb{R})$$

(which condition determines $\widetilde{\varphi}_n$ up to scalar multiples).
The $(\mathfrak{g}, \widetilde{K})$-module of the discrete series representation $\widetilde{\pi}_k^+$ of $\widetilde{G}$ is for $\nu \equiv k \bmod 2\mathbb{Z}$ realized as the unique nontrivial $(\mathfrak{g}, \widetilde{K})$-submodule of $\widetilde{V}(k, \nu)$ or equivalently as the unique nontrivial $(\mathfrak{g}, \widetilde{K})$-quotient of $\widetilde{V}(2-k, \nu)$, its $\widetilde{K}$-types are indexed by the $n \in \nu + 2\mathbb{Z}$ with $n \geq k$.
If $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a holomorphic cusp form of weight k (for some congruence subgroup) and $f^{(\widetilde{G})}$ the corresponding function on $\widetilde{G}$, the function $f^{(\widetilde{G})}$ generates a $(\mathfrak{g}, \widetilde{K})$-module isomorphic to $\widetilde{\pi}_k^+$ and is a vector of the lowest $\widetilde{K}$-type k in this module.
Analogous statements hold for antiholomorphic cusp forms and discrete series respesentations $\widetilde{\pi}_k^-$.

Proof. This is a reformulation of Proposition 6 in [11]. □

The situation for harmonic weak Maaß forms is now similar to that in Section 2, except that due to the fact that $2k$ is odd we cannot get both holomorphic and antiholomorphic forms by applying raising and lowering operators.

Proposition 7. *Let f be a harmonic weak Maaß form of half integral weight $2-k$ with $k \geq 1$, $k \in \nu + 2\mathbb{Z}$ for the congruence subgroup $\Gamma \subseteq \Gamma_1(4)$ and denote by F the corresponding function on $\widetilde{G} = \widetilde{\mathrm{SL}_2}(\mathbb{R})$*

(which is an eigenfunction of the Laplace-Beltrami operator $\Delta^{(\widetilde{G})}$ *with eigenvalue* $\lambda = \frac{k}{2}(1 - \frac{k}{2})$*).*

Then $L^{(\widetilde{G})}F$ *generates (if it is non zero) a* $(\mathfrak{g}, \widetilde{K})$*-module of type* $\widetilde{\pi}_k^-$ *and corresponds to an antiholomorphic modular form of weight* k*. The* $(\mathfrak{g}, \widetilde{K})$*-module generated by* F *is isomorphic to* $\widetilde{V}(k, -\nu)$ *if* $L^{(G)}F \neq 0$*, it is isomorphic to the discrete series representation* $\widetilde{\pi}_{2-k}^+$ *if* $L^{(G)}F = 0$*. In both cases the* $(\mathfrak{g}, \widetilde{K})$*-module is indecomposable, but it is irreducible only in the second case.*

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