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Abstract

We consider entire solutions of the equations for stationary flows of shear thickening fluids in $2D$ and prove Liouville results under conditions like global boundedness of the velocity field or finiteness of the energy.

1 Introduction

In our paper we study entire solutions $u : \mathbb{R}^2 \to \mathbb{R}^2$, $\pi : \mathbb{R}^2 \to \mathbb{R}$ of the following set of equations

(1.1) $$\begin{cases} -\operatorname{div}\left[T(\varepsilon(u))\right] + u^k \partial_k u + \nabla \pi = 0\,, \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^2 \end{cases}$$

and derive Liouville-type results under rather natural assumptions to be made precise below. In physical terms (1.1) describes the stationary flow of an incompressible generalized Newtonian fluid, u denoting the velocity field, π the pressure function, and T represents the stress tensor. As usual $\varepsilon(u)$ stands for the symmetric derivative of u, i.e.

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right) = \frac{1}{2}(\partial_i u^k + \partial_k u^i)_{1 \le i,k \le 2}\,,$$

and $u^k \partial_k u$ (summation w.r.t. $k = 1, 2$) is the so-called convective term. We assume that the stress tensor T comes from a potential $H : \mathbb{S}^2 \to [0, \infty)$ defined on the space $\mathbb{S}^2$ of all symmetric (2×2)-matrices and that H satisfies the structural condition

(1.2) $$H(\varepsilon) = h(|\varepsilon|)$$

with $h : [0, \infty) \to [0, \infty)$ at least of class C^2. Note that (1.2) implies

$$DH(\varepsilon) = \mu(|\varepsilon|)\varepsilon,\ \mu(t) := \frac{h'(t)}{t},\ t = |\varepsilon|\,,$$

which means that the viscosity coefficient may depend on the modulus of ε as proposed by Ladyzhenskaya on p.193 of her book [La]. For further mathematical and physical explanations the reader is referred to the monographs of Galdi [Ga1,2] and of Málek, Necǎs, Rokyta, Růžička [MNRR] (compare also [FS]). Here we concentrate on shear thickening fluids, which means by definition (see [MNRR], Def. 1.68 on p.14) that $\mu(|\varepsilon|)$ is an increasing function. Of course the case of the stationary Navier-Stokes system falls into this category but we can also cover the (nondegenerate) p-case with $p > 2$, in which the function h grows like t^p generating a strongly nonlinear behaviour of the leading part in the first equation in (1.1).

Let us recall what is known about Liouville theorems for entire solutions of the Navier-Stokes system in $2D$: from the work of Giaquinta and Modica (see Remark 1.6 in [GM]) it follows that in case

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx < \infty \tag{1.3}$$

the velocity field is a constant vector, provided the convective term is neglected in (1.1). This restriction was removed by Galdi (see [Ga2], Chapter X, Theorem 3.1) so that the constants are the only entire solutions having finite energy of the stationary Navier-Stokes system in the plane.

Recently Koch [Ko] and Koch, Nadirashvili, Seregin, Sverǎk [KNSS] investigated the situation for the instationary Navier-Stokes equation in two spatial variables replacing (1.3) by the condition

$$|u(x,t)| \leq \text{ const} \tag{1.4}$$

and showing that (1.4) implies

$$u(x,t) = b(t) \quad \text{on } \mathbb{R}^2 \times (-\infty, 0)$$

for a bounded function $b : (-\infty, 0) \to \mathbb{R}^2$.

In order to describe our results we now give a precise formulation of the properties of the potential h. Henceforth we assume:

$$\text{(A1)} \qquad \begin{cases} h \text{ is strictly increasing and convex} \\ \text{together with } h''(0) > 0 \text{ and } \lim\limits_{t \to 0} \frac{h(t)}{t} = 0\,. \end{cases}$$

(A2) (doubling property) there exists a constant such that $h(2t) \leq a h(t) \quad$ for all $t \geq 0$.

$$\text{(A3)} \qquad \text{we have } \frac{h'(t)}{t} \leq h''(t) \quad \text{for any } t > 0\,.$$

From (A1) - (A3) it immediately follows:

i) $\mu(t) = \frac{h'(t)}{t}$ is an increasing function, thus we are in the shear thickening case. (For the proof we just quote (A3).)

ii) We have $h(0) = h'(0) = 0$ and

$$h(t) \geq \frac{1}{2} h''(0) t^2 \,. \tag{1.5}$$

For (1.5) we observe that by i) for all $t > 0$

$$\frac{h'(t)}{t} \geq \lim_{s \to 0} \frac{h'(s)}{s} = h''(0) \,,$$

hence $h(t) = \int_0^t h'(s)\, ds \geq h''(0) \int_0^t s\, ds$.

iii) The function h satisfies the balancing condition, i.e.

$$\frac{1}{a}\, h'(t) t \leq h(t) \leq h(t) \leq t\, h'(t),\; t \geq 0 \,. \tag{1.6}$$

In fact, the second inequality is a consequence of the convexity of h. We further have by (A2)

$$h(t) \geq \frac{1}{a}\, h(2t) = \frac{1}{a} \int_0^{2t} h'(s)\, ds \geq \frac{1}{a} \int_t^{2t} h'(s)\, ds \geq \frac{1}{a} t\, h'(t),\;\; t \geq 0 \,,$$

and (1.6) follows.

iv) For an exponent $m \geq 2$ and a constant $c \geq 0$ it holds

$$h(t) \leq c(t^m + 1) \,,\;\; t \geq 0 \,, \tag{1.7}$$

which is an immediate consequence of (A2).

In order to formulate our results we assume from now on that $u \in C^2(\mathbb{R}^2; \mathbb{R}^2)$ and $\pi \in C^1(\mathbb{R}^2)$ are entire solutions of (1.1) with $T = DH$ and H satisfying (1.2), h being defined according to (A1) - (A3). Note that this degree of smoothness is motivated by the results in [Fu1,2] and the non-degeneracy of $D^2 H$, however it will become clear from the proofs that we could also consider weak solutions with (second) derivatives having a sufficient degree of local integrability. Our first theorem is in the spirit of Giaquinta and Modica [GM] and of Galdi [Ga2].

THEOREM 1.1. *Suppose that we have a finite energy solution in the sense that*

$$\int_{\mathbb{R}^2} h\left(|\varepsilon(u)|\right)\, dx < \infty \tag{1.8}$$

is true.

- *a) If the convective term vanishes, then u must be a rigid motion, and reduces to a constant vector, if (1.8) is replaced by the stronger assumption that $\int_{\mathbb{R}^2} h(|\nabla u|)\, dx$ is finite.*

b) If we allow the convective term to be non-zero, but require in addition to (1.8) the validity of

$$\int_{\mathbb{R}^2} |u|^2 \, dx < \infty \,, \tag{1.9}$$

then u is identically zero.

Next we consider bounded solutions. We have

THEOREM 1.2. *Suppose that u is in the space $L^\infty(\mathbb{R}^2;\mathbb{R}^2)$. Then u is a constant vector, if*

i) the convective term vanishes

or

ii) $\sup\limits_{\mathbb{R}^2 - B_R(0)} |u - u_\infty| \to 0$ as $R \to \infty$ for some vector $u_\infty \in \mathbb{R}^2$.

REMARK 1.1. *We conjecture that any bounded solution u must be a constant vector, but we are unable to prove this. From (4.19) it follows that*

$$\int_{B_R(0)} h(|\varepsilon(u)|) \, dx \leq c\, R$$

for any $R \geq 1$, and the choice $\gamma = r^{-1}$ in (5.24) implies

$$\int_{\mathbb{R}^2} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \, dx < \infty \,,$$

in particular $\int_{\mathbb{R}^2} |\nabla^2 u|^2 \, dx < \infty$, and a more careful analysis might yield $\nabla u = 0$.

REMARK 1.2. *From the proof of Theorem 1.2 it will become evident that the condition required in ii) of the theorem can be replaced by the hypothesis that $\int_{\mathbb{R}^2} |u - u_\infty|^2 \, dx < \infty$.*

Let us finally say a few words concerning our notation: throughout this paper the convention of summation with respect to indices repeated twice is used. All constants are denoted by the symbol "c", and the value of c may change from line to line. Whenever it is necessary we will indicate the dependence of c on parameters. As usual $B_R(x_0)$ denotes the open disc with center x_0 and radius $R > 0$, and the symbols "$:$", "$\cdot$" will be used for the scalar products of matrices and vectors, respectively, $|\cdot|$ denoting the associated Euclidean norms.

Our paper is organized as follows: in Section 2 we present a measure theoretic result originating in the work of Giaquinta and Modica [GM] and being of crucial importance for proving Theorem 1.2. Moreover, we collect in Section 2 various technical tools. Section 3 presents the proof of Theorem 1.1. In Section 4 we derive an energy estimate for bounded solutions, which is used during the proof of Theorem 1.2 to be presented in Section 5.

2 Auxiliary results

Our first and most important tool originates in the work of Giaquinta and Modica and formulates the "ε"-lemma 0.5 of [GM] for the situation at hand.

Lemma 2.1. *Suppose that we are given a function $f \geq 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and some number $s > 0$. Then we can find $\beta_0 := \beta_0(s) > 0$ as follows: if for some $\beta \in (0, \beta_0)$ it is possible to calculate a constant $c(\beta) > 0$ such that the inequality*

$$\int_{Q_R(x_0)} f\,dx \leq \beta \int_{Q_{2R}(x_0)} f\,dx + c(\beta)\left[\int_{Q_{2R}(x_0)} 1\,dx + R^{-s}\int_{Q_{2R}(x_0)} 1\,dx\right]$$

holds for all squares $Q_R(x_0) \subset \mathbb{R}^2$, then we obtain the inequality

$$\int_{Q_R(x_0)} f\,dx \leq c\left[\int_{Q_{2R}(x_0)} 1\,dx + R^{-s}\int_{Q_{2R}(x_0)} 1\,dx\right]$$

again for all squares.

REMARK 2.1. *For $z \in \mathbb{R}^2$ and $R > 0$ we have by definition $Q_R(z) = \{x \in \mathbb{R}^2 : |x_i - z_i| < R, i = 1, 2\}$. In Sections 4 and 5 Lemma 2.1 will be applied on discs in place of squares, but this modification can be justified by some elementary considerations.*

REMARK 2.2. *In Lemma 0.5 of [GM] it is formally required that f is in $L^1(Q_0)$ for some cube Q_0. But going through the calculations it is easy to see that actually Lemma 2.1 will follow. Of course we could give a more general form of Lemma 2.1, but this simple variant is sufficient for our purposes.*

The next result can be traced in [Ga1], Chapter III, Section 3 (see also [FS], Lemma 3.0.4, for further references).

Lemma 2.2. *Suppose that we are given numbers $1 < p_1 \leq p \leq p_2 < \infty$. Then there exists a constant $c = c(p_1, p_2)$ with the following property: if $f \in L^p(B_R(x_0))$ satisfies $⨍_{B_R(x_0)} f\,dx = 0$, then there exists a field v in the Sobolev space $\overset{\circ}{W}{}^1_p(B_R(x_0); \mathbb{R}^2)$ satisfying*

$$\operatorname{div} v = f$$

together with the estimate

$$\int_{B_R(x_0)} |\nabla v|^s\,dx \leq c \int_{B_R(x_0)} |f|^s\,dx$$

for any exponent $s \in [p_1, p]$. The same is true if the disc $B_R(x_0)$ is replaced by the annulus $T_R(x_0) := B_{2R}(x_0) - B_R(x_0)$.

We also need the following inequalities, which for simplicity we take from Acerbi and Mingione (see Proposition 2.7 in [AM]), who callected these estimates in a form being suitable for our applications. Moreover, in [AM] the reader will find more on the history of these results.

Lemma 2.3. *a) (Korn type inequality) Let $p \in (1, \infty)$.*
Then for fields $v \in \overset{\circ}{W}{}^1_p(B_R(x_0); \mathbb{R}^2)$ it holds

$$\|\nabla v\|_{L^p(B_R(x_0))} \leq c \|\varepsilon(v)\|_{L^p(B_R(x_0))}$$

with c independent of R.

b) Let $w \in W^1_2(B_R(x_0); \mathbb{R}^2)$ and $q \in (1,2)$. Then there is a rigid motion γ such that ($q^* := \frac{2q}{2-q}$)

$$\begin{cases} \|w - \gamma\|_{L^2(B_R(x_0))} & \leq & cR\|\varepsilon(w)\|_{L^2(B_R(x_0))}\,, \\ \|w - \gamma\|_{L^{q^*}(B_R(x_0))} & \leq & c\|\varepsilon(w)\|_{L^q(B_R(x_0))} \end{cases}$$

with c being indepent of R. The same statements hold if we replace $B_R(x_0)$ by $T_R(x_0) := B_{2R}(x_0) - B_R(x_0)$.

The next lemma goes back to Ladyzhenskaya (see [La], Lemma 1 on p.8)

Lemma 2.4. *For smooth functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with compact support we have*

$$\int_{\mathbb{R}^2} \varphi^4 \, dx \leq 2 \int_{\mathbb{R}^2} \varphi^2 \, dx \int_{\mathbb{R}^2} |\nabla \varphi|^2 \, dx\,.$$

We finish this section with an elementary result concerning the growth of h and h'.

Lemma 2.5. *There is a number $\tau \in (1,2)$ such that*

$$h'(t) \leq c\left(h(t)^{1/\tau} + 1\right)$$

or equivalently

$$|DH(\varepsilon)| \leq c\left(H(\varepsilon)^{1/\tau} + 1\right)$$

holds for all $t \geq 0$ and $\varepsilon \in \mathbb{S}^2$. Moreover we even have the sharper estimate

$$h'(t) \leq c\left[h(t)^{1/\tau} + t\right],\ t \geq 0\,.$$

Proof: For $t \geq 1$ it follows from (1.6) and (1.7) that

$$h'(t) \leq c\frac{h(t)}{t} = c\,h(t)^{1-\delta}\ \frac{h(t)^\delta}{t} \leq c\,h(t)^{1-\delta}\ \frac{t^{\delta m}}{t} = c\,h(t)^{1-\delta}\ t^{\delta m - 1} \leq c\,h(t)^{1-\delta}\,,$$

provided δ is sufficiently small. Letting $\tau := \frac{1}{1-\delta}$ and recalling that $h'(t) \leq c\,t$ for $t \in [0,1]$, all our claims follow. □

3 Finite energy solutions: proof of Theorem 1.1

Suppose that our entire solution u satisfies (1.8). Fix discs $B_R \subset B_{2R}$ centered at the origin, let $T_R := B_{2R} - B_R$ and choose $\eta \in C_0^\infty(B_{2R})$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_R, $|\nabla \eta| \le c/R$. We let $p := \tau/(\tau-1) > 2$ with τ from Lemma 2.5 and use Lemma 2.3b) to find a rigid motion γ such that

$$\int_{T_R} |u-\gamma|^2\,dx \le c\,R^2 \int_{T_R} |\varepsilon(u)|^2\,dx \tag{3.1}$$

and

$$\Big(\int_{T_R} |u-\gamma|^p\,dx\Big)^{1/p} \le c\Big(\int_{T_R} |\varepsilon(u)|^q\,dx\Big)^{1/q}, \tag{3.2}$$

where $q := \frac{2p}{p+2} \in (1,2)$. Quoting Lemma 2.2 with $f := \operatorname{div}[\eta^2(u-\gamma)]$ we find $w \in \overset{\circ}{W}{}^1_p(T_R;\mathbb{R}^2)$ such that

$$\left\{\begin{array}{l} \operatorname{div} w = \operatorname{div}[\eta^2(u-\gamma)] = \nabla\eta^2\cdot(u-\gamma) \text{ on } T_R\,,\\ \|\nabla w\|_{L^2(T_R)} \le c\|\nabla\eta^2\cdot(u-\gamma)\|_{L^2(T_R)}\,,\\ \|\nabla w\|_{L^p(T_R)} \le c\|\nabla\eta^2\cdot(u-\gamma)\|_{L^p(T_R)}\,. \end{array}\right. \tag{3.3}$$

In order to justify the application of Lemma 2.2 we have to check that $\int_{T_R} f\,dx = 0$: if ν denotes the exterior unit normal to ∂T_R, then (since $\eta = 0$ on ∂B_{2R} and $\eta = 1$ on ∂B_R)

$$\begin{aligned} \int_{T_R} f\,dx &= \int_{T_R} \operatorname{div}\left[\eta^2(u-\gamma)\right]\,dx = \int_{\partial T_R} \eta^2(u-\gamma)\cdot\nu\;d\mathcal{H}^1\\ &= \int_{\partial B_R} \eta^2(u-\gamma)\cdot\nu\;d\mathcal{H}^1 = -\int_{\partial B_R} (u-\gamma)\cdot\frac{x}{R}\,d\mathcal{H}^1\\ &= \int_{B_R} \operatorname{div}(u-\gamma)\,dx = 0\,. \end{aligned}$$

We now let

$$\varphi := \left\{\begin{array}{ll} u-\gamma & \text{in } B_R\\ \eta^2(u-\gamma)-w & \text{in } T_R\,, \end{array}\right.$$

thus $\varphi = 0$ outside of B_{2R} and $\operatorname{div}\varphi = 0$. Let us assume for the moment that $u^k\partial_k u = 0$. Then the multiplication of (1.1) with φ and integration by parts yields

$$0 = \int_{B_R} DH(\varepsilon(u)):\varepsilon(u)\,dx + \int_{T_R} DH(\varepsilon(u)):\varepsilon(\eta^2[u-\gamma])\,dx + \int_{T_R} DH(\varepsilon(u)):\varepsilon(-w)\,dx\,,$$

hence

$$\begin{aligned} &\int_{B_{2R}} \eta^2 DH(\varepsilon(u)):\varepsilon(u)\,dx\\ \le\;& 2\int_{T_R} \eta h'(|\varepsilon(u)|)\,|\nabla\eta||u-\gamma|\,dx + \int_{T_R} h'(|\varepsilon(u)|)\,|\varepsilon(w)|\,dx\\ =:\;& U_1 + U_2\,. \end{aligned} \tag{3.4}$$

Clearly we have by (1.6)

$$\text{(3.5)} \qquad \text{l.h.s. of (3.4) } \geq c \int_{B_R} h\left(|\varepsilon(u)|\right)\, dx\,.$$

From the last inequality in Lemma 2.5 we infer

$$U_1 \leq c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)^{1/\tau} |\nabla\eta||u-\gamma|\, dx + \int_{T_R} |\varepsilon(u)||\nabla\eta||u-\gamma|\, dx\right] =: c\,[U_3 + U_4]$$

with

$$U_3 \leq c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)\, dx + R^{-p}\int_{T_R} |u-\gamma|^p\, dx\right]$$

and

$$U_4 \leq c\left[\int_{T_R} |\varepsilon(u)|^2\, dx + \frac{1}{R^2}\int_{T_R} |u-\gamma|^2\, dx\right]\,.$$

Using (3.1) and recalling (1.5) we find by (3.4), (3.5) and the above estimates

$$\text{(3.6)} \qquad \int_{B_R} h\left(|\varepsilon(u)|\right)\, dx \leq c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)\, dx + R^{-p}\int_{T_R} |u-\gamma|^p\, dx + |U_2|\right]\,.$$

Similar to the discussion of U_1 we have

$$\begin{aligned} U_2 &\leq c\left[\int_{T_R} h\left(\varepsilon(u)\right)^{1/\tau} |\varepsilon(w)|\, dx + \int_{T_R} |\varepsilon(u)||\varepsilon(w)|\, dx\right] \\ &\leq c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)\, dx + \int_{T_R} |\varepsilon(w)|^p\, dx + \int_{T_R} |\varepsilon(u)|^2\, dx + \int_{T_R} |\varepsilon(w)|^2\, dx\right] \\ &\overset{(3.3),(1.5)}{\leq} c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)\, dx + R^{-2}\int_{T_R} |u-\gamma|^2\, dx + R^{-p}\int_{T_R} |u-\gamma|^p\, dx\right] \end{aligned}$$

and by quoting (3.1) one more time (3.6) implies

$$\text{(3.7)} \qquad \int_{B_R} h\left(|\varepsilon(u)|\right)\, dx \leq c\left[\int_{T_R} h\left(|\varepsilon(u)|\right)\, dx + R^{-p}\int_{T_R} |u-\gamma|^p\, dx\right]\,.$$

By (3.2) it holds

$$\begin{aligned} R^{-p}\int_{T_R} |u-\gamma|^p\, dx &\leq c\, R^{-p}\left(\int_{T_R} |\varepsilon(u)|^q\, dx\right)^{p/q} \\ &\leq c\, R^{-p}\left[\left(\int_{T_R} |\varepsilon(u)|^2\, dx\right)^{q/2} \mathcal{L}^2(T_R)^{1-q/2}\right]^{p/q} \\ &= c\, R^{-p} R^{(2-q)\frac{p}{q}}\left(\int_{T_R} |\varepsilon(u)|^2\, dx\right)^{p/2} \leq c\, R^{2\frac{p}{q}-2p}\left(\int_{T_R} h\left(|\varepsilon(u)|\right)\, dx\right)^{p/2}\,, \end{aligned}$$

and with (3.7) it is shown

$$\int_{B_R} h\left(|\varepsilon(u)|\right) \, dx \le c \left[\int_{T_R} h\left(|\varepsilon(u)|\right) \, dx + R^{2\frac{p}{q}-2p} \left(\int_{T_R} h\left(|\varepsilon(u)|\right) \, dx \right)^{p/2} \right] . \tag{3.8}$$

Now on account of (1.8) the r.h.s. of (3.8) vanishes as $R \to \infty$, thus $\varepsilon(u) \equiv 0$ and therefore u is a rigid motion.

Next we drop our hypothesis $u^k \partial_k u \equiv 0$ and assume in addition to (1.8) the validity of (1.9). From Lemma 2.3a) it follows

$$\int_{B_t} |\nabla u|^2 \, dx \le c \left[\int_{B_{2t}} |\varepsilon(u)|^2 \, dx + \frac{1}{t^2} \int_{B_{2t}} |u|^2 \, dx \right] ,$$

hence by (1.8) and (1.9)

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx < \infty . \tag{3.9}$$

Therefore u is in the space $W^1_2(\mathbb{R}^2;\mathbb{R}^2) = \overset{\circ}{W}{}^1_2(\mathbb{R}^2;\mathbb{R}^2)$ and Lemma 2.4 yields

$$\int_{\mathbb{R}^2} |u|^4 \, dx \le c \int_{\mathbb{R}^2} |u|^2 \, dx \int_{\mathbb{R}^2} |\nabla u|^2 \, dx . \tag{3.10}$$

In the presence of the convective term on the r.h.s. of (3.4) the additional quantity $\int_{B_{2R}} u^k \partial_k u \cdot \varphi \, dx$ occurs. It holds

$$\begin{aligned} \int_{B_{2R}} u^k \partial_k u^i \varphi^i \, dx &= - \int_{B_{2R}} u^k u^i \partial_k \varphi^i \, dx \\ = \; & - \int_{B_{2R}} u^k u^i \varepsilon(\varphi)_{ik} \, dx = - \int_{B_{2R}} u^k u^i \varepsilon(\eta^2(u-\gamma))_{ik} \, dx \\ & + \int_{T_R} u^k u^i \varepsilon(w)_{ik} \, dx =: -V_1 + V_2 , \end{aligned}$$

where

$$\begin{aligned} |V_2| &\le \int_{T_R} |u|^2 |\varepsilon(w)| \, dx \le c \left[\int_{T_R} |u|^4 \, dx + \int_{T_R} |\varepsilon(w)|^2 \, dx \right] \\ &\overset{(3.1),(3.3)}{\le} \quad c \left[\int_{T_R} |u|^4 \, dx + \int_{T_R} |\varepsilon(u)|^2 \, dx \right] \end{aligned}$$

and by (3.10) and (1.8) we see

$$\lim_{R \to \infty} V_2 = 0 . \tag{3.11}$$

Next we observe (recall $\eta \equiv 1$ on B_R)

$$V_1 = \int_{B_R} u^i u^k \varepsilon(u)_{ik} \, dx + \int_{T_R} u^i u^k \varepsilon(\eta^2(u-\gamma))_{ik} \, dx =: V_3 + V_4 .$$

V_4 is estimated as follows:

$$\begin{aligned} V_4 = \int_{T_R} u^i u^k \eta^2 \varepsilon(u)_{ik}\, dx + 2\int_{T_R} u^i u^k \eta\; \partial_i \eta (u^k - \gamma^k)\, dx \\ \leq \quad c \left[\|u\|^2_{L^4(T_R)} \|\varepsilon(u)\|_{L^2(T_R)} + \frac{1}{R}\int_{T_R} |u|^2 |u-\gamma|\, dx \right] \\ \leq \quad c\|u\|^2_{L^4(T_R)} \left[\|\varepsilon(u)\|_{L^2(T_R)} + \frac{1}{R}\|u-\gamma\|_{L^2(T_R)} \right] \\ \overset{(3.1)}{\leq} \quad c\|u\|^2_{L^4(T_R)} \|\varepsilon(u)\|_{L^2(T_R)}\,, \end{aligned}$$

thus

(3.12) $$\lim_{R\to\infty} V_4 = 0\,.$$

Finally we look at V_3: it holds

$$V_3 = \int_{B_R} u^i u^k \partial_k u^i\, dx = \int_{B_R} \partial_k \left[u^i u^k u^i \right]\, dx - \int_{B_R} u^k \partial_k u^i u^i\, dx\,,$$

thus (recall the choice of η)

$$\begin{aligned} V_3 = \frac{1}{2}\int_{B_R} \partial_k \left[u^k |u|^2 \right]\, dx = \frac{1}{2}\int_{\partial B_R} |u|^2 u^k \,\frac{x^k}{R}\, d\mathcal{H}^1 \\ = \quad -\frac{1}{2}\int_{\partial T_R} \eta^2 |u|^2 u\cdot\nu\, d\mathcal{H}^1 = -\frac{1}{2}\int_{T_R} \operatorname{div}\left(\eta^2|u|^2 u\right)\, dx\,. \end{aligned}$$

This yields

$$\begin{aligned} |V_3| \leq c\left[\frac{1}{R}\int_{T_R} |u|^3\, dx + \int_{T_R} |u|^2 |\nabla u|\, dx \right] \\ \leq \quad c\left[\frac{1}{R}\left(\int_{T_R} |u|^4\, dx \right)^{3/4} R^{2(1-\frac{3}{4})} + \|u\|^2_{L^4(T_R)} \|\nabla u\|_{L^2(T_R)} \right] \end{aligned}$$

and we may apply (3.9) and (3.10) to get

(3.13) $$\lim_{R\to\infty} V_3 = 0\,.$$

Summing up it follows from (3.11) - (3.13) that $\int_{B_{2R}} u^k \partial_k u \cdot \varphi\, dx$ vanishes as $R \to \infty$, and we again arrive at $\varepsilon(u) = 0$. But (3.9) implies that u is constant, and from (1.9) we finally deduce that $u = 0$. This completes the proof of Theorem 1.1. □

4 Energy estimates for bounded solutions

We start with the following result concerning the growth of the energy.

Lemma 4.1. *Suppose that u is a bounded (smooth) solution of problem (1.1) under the conditions (A1)-(A3) concerning h. Then it holds*

$$\int_{B_t(x_0)} H\left(\varepsilon(u)\right)\, dx \le c[t+1] \tag{4.1}$$

for all discs $B_t(x_0) \subset \mathbb{R}^2$.

Proof: Consider an arbitrary disc $B_R(x_0)$ and a cut-off function $\eta \in C_0^\infty(B_R(x_0))$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{R/2}(x_0)$ and $|\nabla \eta| \le c/R$. From (1.1) we deduce as usual

$$\int_{B_R(x_0)} DH(\varepsilon(u)) : \varepsilon(\varphi)\, dx + \int_{B_R(x_0)} u^k \partial_k u \cdot \varphi\, dx = 0 \tag{4.2}$$

for any φ vanishing on $\partial B_R(x_0)$ and satisfying $\operatorname{div} \varphi = 0$. For $\ell \in \mathbb{N}$ to be specified later we let $\varphi := \eta^{2\ell} u - w$, where $w \in \overset{\circ}{W}{}^1_p(B_R(x_0); \mathbb{R}^2)$ is defined in Lemma 2.2 with the choices $p := \tau/(\tau-1)$, τ from Lemma 2.5, and $f := \operatorname{div}(\eta^{2\ell} u) = \nabla \eta^{2\ell} \cdot u$, thus we have the estimates

$$\begin{cases} \|\nabla w\|_{L^p(B_R(x_0))} \le c \|\nabla \eta^{2\ell} \cdot u\|_{L^p(B_R(x_0))}\,, \\ \|\nabla w\|_{L^2(B_R(x_0))} \le c \|\nabla \eta^{2\ell} \cdot u\|_{L^2(B_R(x_0))}\,. \end{cases} \tag{4.3}$$

From (4.2) we get

$$\begin{aligned} &\int_{B_R(x_0)} DH(\varepsilon(u)) : \varepsilon(u) \eta^{2\ell}\, dx \\ &= -\int_{B_R(x_0)} DH(\varepsilon(u)) : (u \otimes \nabla \eta^{2\ell})\, dx + \int_{B_R(x_0)} DH(\varepsilon(u)) : \varepsilon(w)\, dx \\ &\quad - \int_{B_R(x_0)} u^k \partial_k u \cdot u \eta^{2\ell}\, dx + \int_{B_R(x_0)} u^k \partial_k u \cdot w\, dx \\ &=: T_1 + T_2 + T_3 + T_4\,, \end{aligned} \tag{4.4}$$

and the balancing property (1.6) implies

$$\text{l.h.s. of (4.4) } \ge c \int_{B_R(x_0)} H(\varepsilon(u)) \eta^{2\ell}\, dx\,. \tag{4.5}$$

We further have on account of our assumption that the field u is bounded (with c depending on ℓ and on $\|u\|_{L^\infty(\mathbb{R}^2)}$)

$$\begin{aligned} |T_1| &\le c \int_{B_R(x_0)} \eta^{2\ell-1} |\nabla \eta|\, |DH(\varepsilon(u))|\, dx \\ &\le c \int_{B_R(x_0)} \eta^{2\ell-1} |\nabla \eta| \left[H(\varepsilon(u))^{1/\tau} + |\varepsilon(u)| \right] dx\,, \end{aligned}$$

where we have used Lemma 2.5. Young's inequality yields for any $\delta > 0$

$$\begin{aligned} T_1 \le \delta \int_{B_R(x_0)} \eta^{(2\ell-1)\tau} H(\varepsilon(u))\, dx + c(\delta) \int_{B_R(x_0)} |\nabla \eta|^p \, dx \\ + \delta \int_{B_R(x_0)} \eta^{2(2\ell-1)} |\varepsilon(u)|^2 \, dx + c(\delta) \int_{B_R(x_0)} |\nabla \eta|^2 \, dx \, . \end{aligned}$$

Let us choose ℓ so large that $(2\ell - 1)\tau \ge 2\ell$. Observing that by (1.5)

$$\int_{B_R(x_0)} \eta^{2(2\ell-1)} |\varepsilon(u)|^2 \, dx \le c \int_{B_R(x_0)} \eta^{2\ell} H(\varepsilon(u))\, dx$$

we can absorb the δ-terms occurring in the estimate for T_1 into the r.h.s. of (4.5), hence we deduce from (4.4) after δ being fixed

$$\int_{B_R(x_0)} \eta^{2\ell} H(\varepsilon(u))\, dx \le c \left[1 + R^{2-p} + |T_2| + |T_3| + |T_4| \right] . \tag{4.6}$$

Next we use (4.3) and Young's inequality:

$$\begin{aligned} |T_2| &\le c \left[\int_{B_R(x_0)} |\varepsilon(u)| |\varepsilon(w)| \, dx + \int_{B_R(x_0)} H(\varepsilon(u))^{1/\tau} |\varepsilon(w)| \, dx \right] \\ &\le \delta \int_{B_R(x_0)} H(\varepsilon(u))\, dx + c(\delta) \left[\int_{B_R(x_0)} |\nabla w|^2 \, dx + \int_{B_R(x_0)} |\nabla w|^p \, dx \right] \\ &\le \delta \int_{B_R(x_0)} H(\varepsilon(u))\, dx + c(\delta) \left[1 + R^{2-p} \right] , \end{aligned}$$

where δ is an arbitrary parameter. Inserting this bound for T_2 into (4.6), we find

$$\int_{B_R(x_0)} \eta^{2\ell} H(\varepsilon(u))\, dx \le \delta \int_{B_R(x_0)} H(\varepsilon(u))\, dx + c(\delta) \left[1 + R^{2-p} \right] + c \left[|T_3| + |T_4| \right] . \tag{4.7}$$

For discussing T_3 we observe

$$\begin{aligned} \int_{B_R(x_0)} u^k \partial_k u^i u^i \eta^{2\ell} \, dx &= - \int_{B_R(x_0)} u^i \partial_k \left[u^k u^i \eta^{2\ell} \right] \, dx \\ &= \; - \int_{B_R(x_0)} u^k u^i \partial_k u^i \eta^{2\ell} \, dx - \int_{B_R(x_0)} u^k |u|^2 \partial_k \eta^{2\ell} \, dx \, , \end{aligned}$$

hence

$$|T_3| = \frac{1}{2} \left| \int_{B_R(x_0)} u^k |u|^2 \partial_k \eta^{2\ell} \, dx \right| \le cR \, ,$$

and for T_4 we finally get

$$T_4 = \int_{B_R(x_0)} u^k \partial_k u^i w^i \, dx = - \int_{B_R(x_0)} u^i \partial_k (u^k w^i) \, dx = - \int_{B_R(x_0)} u^i u^k \partial_k w^i \, dx \, ,$$

thus

$$|T_4| \le c \int_{B_R(x_0)} |\nabla w|\, dx \le cR \|\nabla w\|_{L^2(B_R(x_0))} \overset{(4.3)}{\le} cR\,.$$

Returning to (4.7) it is shown that

$$\int_{B_{R/2}(x_0)} H(\varepsilon(u))\, dx \le \delta \int_{B_R(x_0)} H(\varepsilon(u))\, dx + c(\delta) \left[1 + R + R^{2-p}\right] \tag{4.8}$$

valid for discs $B_R(x_0)$ and any $\delta > 0$. In case $R \le 1$ it holds

$$1 + R + R^{2-p} \le cR^{-p} \int_{B_R(x_0)} 1\, dx\,,$$

whereas for $R > 1$ we have

$$1 + R + R^{2-p} \le cR \le c \int_{B_R(x_0)} 1\, dx\,,$$

thus in both cases we obtain

$$1 + R + R^{2-p} \le c \left[\int_{B_R(x_0)} 1\, dx + R^{-p} \int_{B_R(x_0)} 1\, dx\right]\,.$$

Therefore (4.8) implies

$$\begin{aligned} \int_{B_{R/2}(x_0)} H\left(\varepsilon(u)\right)\, dx &\le \delta \int_{B_R(x_0)} H(\varepsilon(u))\, dx \\ &+ c(\delta) \left[R^{-p} \int_{B_R(x_0)} 1\, dx + \int_{B_R(x_0)} 1\, dx\right]\,. \end{aligned} \tag{4.9}$$

If we apply Lemma 2.1 to inequality (4.9), we find

$$\int_{B_{r/2}(x_0)} H\left(\varepsilon(u)\right)\, dx \le c \left[r^{-p} \int_{B_r(x_0)} 1\, dx + \int_{B_r(x_0)} 1\, dx\right]\,, \tag{4.10}$$

and (4.10) holds for all discs $B_r(x_0)$. Clearly (4.10) implies the growth estimate

$$\int_{B_t(x_0)} H(\varepsilon(u))\, dx \le c\, t^2 \tag{4.11}$$

for all radii $t \ge 1$. Going through our calculations again (cf. (4.6)), we can restate our result in the form $(0 < R < \infty)$

$$\int_{B_{R/2}(x_0)} H(\varepsilon(u))\, dx \le c \left[1 + R^{2-p} + |T_2| + |T_3| + |T_4|\right]\,,$$

where the term $1+R^{2-p}$ comes from the discussion of T_1, and the bounds derived for T_3, T_4 yield

$$\int_{B_{R/2}(x_0)} H(\varepsilon(u))\,dx \le c\left[1+R+R^{2-p}+|T_2|\right]. \tag{4.12}$$

Now we estimate T_2 as follows:

$$\begin{aligned}
|T_2| &\le c\Bigg[\|\varepsilon(u)\|_{L^2(B_R(x_0))}\|\varepsilon(w)\|_{L^2(B_R(x_0))}\\
&\qquad + \left(\int_{B_R(x_0)} H(\varepsilon(u))\,dx\right)^{1/\tau}\|\varepsilon(w)\|_{L^p(B_R(x_0))}\Bigg]\\
&\overset{(4.3)}{\le} c\left[\left(\int_{B_R(x_0)} H(\varepsilon(u))\,dx\right)^{1/2} + \left(\int_{B_R(x_0)} H(\varepsilon(u))\,dx\right)^{1/\tau}\frac{1}{R}R^{2/p}\right],
\end{aligned}$$

and if we assume $R \ge 1$, then the application of (4.11) yields

$$|T_2| \le c\left[R + R^{2/\tau}R^{-1}R^{2/p}\right] = c\,R\,.$$

In combination with (4.12) it is therefore shown that in place of (4.11) we have

$$\int_{B_t(x_0)} H(\varepsilon(u))\,dx \le c\,t \tag{4.13}$$

for all $t \ge 1$. If t is in $(0,1)$, then by (4.13)

$$\int_{B_t(x_0)} H(\varepsilon(u))\,dx \le \int_{B_1(x_0)} H(\varepsilon(u))\,dx \le c\,,$$

hence we have established (4.1) and Lemma 4.1 is proved. □

In the following we will use (4.13) to derive an estimate (see (4.19)) for $\int_{B_R} H(\varepsilon(u))\,dx$, $B_R = B_R(0)$, $R \ge 1$, which incorporates the quantity $\sup\limits_{\mathbb{R}^2 - B_R} |u-u_\infty|$. At this stage u_∞ denotes some arbitrary vector and we just assume u to be a bounded function without requirering $\sup \ldots \to 0$ as $R \to \infty$. We return to (4.2) choosing now

$$\varphi := \eta^2(u-u_\infty) - w\,,$$

where η is as before, but w is an element of the space $\overset{\circ}{W}{}^1_m(B_R;\mathbb{R}^2)$ with $f := \nabla\eta^2\cdot(u-u_\infty)$ and exponent p in (4.3) replaced by m, where m is defined according to (1.7). Note that (1.7) can be replaced by

$$h(t) \le c[t^m + t^2],\ t \ge 0\,. \tag{4.14}$$

We get as in the proof of Lemma 4.1

$$\begin{aligned}
(4.15) \qquad & \int_{B_{R/2}} H(\varepsilon(u))\,dx \\
& \leq c\left[\int_{B_R} h'(|\varepsilon(u)|)|u-u_\infty||\nabla\eta|\,dx \right.\\
& \quad + \int_{B_R} h'(|\varepsilon(u)|)|\varepsilon(w)|\,dx + \left|\int_{B_R} u^k\partial_k u\cdot(u-u_\infty)\eta^2\,dx\right| \\
& \quad \left. + \left|\int_{B_R} u^k\partial_k u\cdot w\,dx\right|\right] =: c\sum_{i=1}^{4} T_i^*\,.
\end{aligned}$$

Let $T_R := B_R - B_{R/2}$ (with a slight abuse of notation compared to Section 2) and $\alpha \in (0,1)$. Then we get (h^* denoting the conjugate function to h)

$$\begin{aligned}
T_1^* &= \int_{T_R} \alpha h'(|\varepsilon(u)|)\frac{1}{\alpha}|u-u_\infty||\nabla\eta|\,dx \\
&\leq \int_{T_R} h^*\left(\alpha h'(|\varepsilon(u)|)\right)\,dx + \int_{T_R} h\left(\frac{1}{\alpha}|u-u_\infty||\nabla\eta|\right)\,dx \\
&\leq \alpha\int_{T_R} h^*\left(h'(|\varepsilon(u)|)\right)\,dx + c\int_{T_R} h\left(\frac{1}{\alpha R}\right)\,dx\,,
\end{aligned}$$

where we just used the boundedness of $|u-u_\infty|$ and Young's inequality for h and h^*. Recall that $h^*(h'(t)) + h(t) = th'(t)$ holds for all $t \geq 0$. Moreover, it follows from (1.6) that $th'(t) \leq ah(t)$ is true, hence $h^*(h'(t)) \leq ah(t)$. We find - choosing $\alpha = R^{-1/3}$ and quoting (4.13)

$$T_1^* \leq c\,R^{2/3} + c\,R^2 h\left(\frac{1}{R^{2/3}}\right)\,.$$

For $t \leq 1$ (4.14) implies $h(t) \leq c\,t^2$, thus we deduce

$$(4.16) \qquad T_1^* \leq c\,R^{2/3}\,.$$

The quantity T_2^* is handled in a similar way:

$$\begin{aligned}
T_2^* &\leq \int_{B_R} \alpha h'(|\varepsilon(u)|)\frac{1}{\alpha}|\varepsilon(w)|\,dx \\
&\overset{(4.14)}{\leq} \alpha\int_{B_R} h\left(|\varepsilon(u)|\right)\,dx + \alpha^{-2}\int_{B_R}|\varepsilon(w)|^2\,dx + \alpha^{-m}\int_{B_R}|\varepsilon(w)|^m\,dx
\end{aligned}$$

and the choice of w implies

$$\begin{aligned}
T_2^* \leq \alpha\int_{B_R} h\left(|\varepsilon(u)|\right)\,dx &+ \alpha^{-2}\int_{T_R}|\nabla\eta|^2|u-u_\infty|^2\,dx \\
&+\alpha^{-m}\int_{T_R}|\nabla\eta|^m|u-u_\infty|^m\,dx\,.
\end{aligned}$$

With $\alpha := R^{-1/3}$ inquality (4.13) gives (again exploiting only $|u - u_\infty| \in L^\infty(\mathbb{R}^2)$)

$$T_2^* \leq c\left[R^{2/3} + R^{+m/3}R^2R^{-m}\right] ,$$

and since we assume $R \geq 1$ we get

$$T_2^* \leq c\, R^{2/3} . \tag{4.17}$$

We next have

$$\begin{aligned}\int_{B_R} u^k\partial_k u^i(u^i - u^i_\infty)\eta^2\, dx &= -\int_{B_R} u^i\partial_k\left[u^k(u^i - u^i_\infty)\eta^2\right]\, dx \\ = \ & -\int_{B_R}(u^i - u^i_\infty)\partial_k\left[u^k(u^i - u^i_\infty)\eta^2\right]\, dx \\ = \ & -\int_{B_R}(u^i - u^i_\infty)u^k\partial_k(u^i - u^i_\infty)\eta^2\, dx \\ & -\int_{B_R}(u^i - u^i_\infty)u^k(u^i - u^i_\infty)\partial_k\eta^2\, dx\end{aligned}$$

and therefore

$$T_3^* = \frac{1}{2}\left|\int_{B_R}|u - u_\infty|^2\nabla\eta^2\cdot u\, dx\right| ,$$

hence

$$T_3^* \leq c\, R \sup_{\mathbb{R}^2 - B_{R/2}} |u - u_\infty|^2 . \tag{4.18}$$

Finally it holds by the properties of w

$$\begin{aligned}T_4^* &= \left|\int_{B_R} u^k\partial_k u^i w^i\, dx\right| = \left|\int_{B_R} u^k u^i\partial_k w^i\, dx\right| \\ &\leq c\int_{B_R}|\nabla w|\, dx \leq c\, R\|\nabla w\|_{L^2(B_R)} \\ &\leq c\, R\|\nabla\eta^2\cdot(u - u_\infty)\|_{L^2(T_R)} \leq c\, R \sup_{\mathbb{R}^2 - B_{R/2}}|u - u_\infty| .\end{aligned}$$

By combining this estimate with (4.15) - (4.18) we have shown the validity of

$$\int_{B_R} H(\varepsilon(u))\, dx \leq c\left[R^{2/3} + R\sup_{\mathbb{R}^2 - B_R}|u - u_\infty| + R\sup_{\mathbb{R}^2 - B_R}|u - u_\infty|^2\right] \tag{4.19}$$

valid for $R \geq 1$ and bounded solutions u, u_∞ denoting an arbitrary vector in $\mathbb{R}^2$. Note that in case $u^k\partial_k u = 0$ (4.19) just reduces to $\ldots \leq c\, R^{2/3}$. □

5 Estimates for the second derivatives of bounded solutions: proof of Theorem 1.2

In order to prove Theorem 1.2 we have to combine the inequalities from Section 4 with certain estimates for the second derivatives, which finally will give $\nabla^2 u \equiv 0$. We start with the derivation of suitable bounds for $\nabla^2 u$: consider a disc $B_r(x_0)$ and choose $\eta \in C_0^\infty(B_{\frac{3}{4}r}(x_0))$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{\frac{r}{2}}(x_0)$ and $|\nabla \eta| \leq c/r$ with radius r for the moment being arbitrary. We also assume the validity of the bound $|\nabla^2 \eta| \leq c/r^2$.
Let $\varphi \in C_0^\infty(B_{\frac{3}{4}r}(x_0); \mathbb{R}^2)$ and $k \in \{1, 2\}$. We multiply (1.1) with $\partial_k \varphi$ and use integration by parts to obtain ($\sigma := T(\varepsilon(u)) := DH(\varepsilon(u))$)

$$\int_{B_{\frac{3}{4}r}(x_0)} \partial_k \sigma : \varepsilon(\varphi)\, dx - \int_{B_{\frac{3}{4}r}(x_0)} \nabla \pi \cdot \partial_k \varphi\, dx - \int_{B_{\frac{3}{4}r}(x_0)} u^i \partial_i u \cdot \partial_k \varphi\, dx = 0\,.$$

Choosing $\varphi := \eta^2 \partial_k u$ this equation gives (from now on we again use the summation convention)

$$\text{(5.1)} \qquad \begin{aligned} &\int_{B_{\frac{3}{4}r}(x_0)} \partial_k \sigma : \varepsilon(\partial_k u) \eta^2\, dx \\ &= 2\int_{B_{\frac{3}{4}r}(x_0)} \sigma : \partial_k \left[\eta \nabla \eta \odot \partial_k u\right]\, dx - 2\int_{B_{\frac{3}{4}r}(x_0)} \pi \partial_k \left[\eta \nabla \eta \cdot \partial_k u\right]\, dx \\ &\quad + \int_{B_{\frac{3}{4}r}(x_0)} u^i \partial_i u \cdot \partial_k \left(\eta^2 \cdot \partial_k u\right)\, dx =: T_1 + T_2 + T_3\,. \end{aligned}$$

By definition we have

$$\begin{aligned} \partial_k \sigma \cdot \varepsilon(\partial_k u) &= D^2 H\left(\varepsilon(u)\right)\left(\partial_k \varepsilon(u), \partial_k \varepsilon(u)\right) \\ &\geq \min\left\{h''(|\varepsilon(u)|), \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}\right\} |\nabla \varepsilon(u)|^2\,, \end{aligned}$$

and (A3) shows

$$\text{(5.2)} \qquad \text{l.h.s. of (5.1)} \geq \int_{B_{\frac{3}{4}r}(x_0)} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \eta^2\, dx\,.$$

Furthermore it holds for arbitrary $\delta > 0$ using Young's inequality and estimate (1.6)

$$
\begin{aligned}
|T_1| \leq c\Bigg[& \int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)|\nabla u|(|\nabla\eta|^2 + |\nabla^2\eta|)\, dx \\
& + \int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)\eta|\nabla\eta||\nabla^2 u|\, dx\Bigg] \\
\leq\ & c\left[r^{-2}\int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)^2\, dx + r^{-2}\int_{B_{\frac{3}{4}r}(x_0)} |\nabla u|^2\, dx\right] \\
& + \delta\int_{B_{\frac{3}{4}r}(x_0)} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}|\nabla\varepsilon(u)|^2\eta^2\, dx \\
& + c(\delta)\int_{B_{\frac{3}{4}r}(x_0)} |\nabla\eta|^2 h(|\varepsilon(u)|)\, dx\,,
\end{aligned}
$$

and for δ small enough the δ-term can be absorbed in the r.h.s. of (5.2) so that we deduce from (5.1), (5.2) and the subsequent estimates

$$
\begin{aligned}
\text{(5.3)}\qquad & \int_{B_{\frac{3}{4}r}(x_0)} \eta^2\frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}|\nabla\varepsilon(u)|^2\, dx \\
& \leq\ c\, r^{-2}\Bigg[\int_{B_{\frac{3}{4}r}(x_0)} h(|\varepsilon(u)|)\, dx + \int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)^2\, dx \\
& \quad + \int_{B_{\frac{3}{4}r}(x_0)} |\nabla u|^2\, dx\Bigg] + c\Big[|T_2| + |T_3|\Big]\,.
\end{aligned}
$$

Korn's inequality from Lemma 2.3a) together with (1.5) easily gives (using the boundedness of u)

$$
\begin{aligned}
\text{(5.4)}\qquad \int_{B_{\frac{3}{4}r}(x_0)} |\nabla u|^2\, dx & \leq c\left[\int_{B_r(x_0)} |\varepsilon(u)|^2\, dx + r^{-2}\int_{B_r(x_0)} |u|^2\, dx\right] \\
& \leq\ c\left[\int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx + 1\right]\,.
\end{aligned}
$$

Next we look at T_2 observing that

$$
T_2 = 2\int_{\Delta_r} (\pi - \pi_0)\partial_k\left[\eta\nabla\eta\cdot\partial_k u\right]\, dx\,,
$$

where we have abbreviated $\Delta_r := B_{\frac{3}{4}r}(x_0) - B_{\frac{r}{2}}(x_0)$ and $\pi_0 := \fint_{\Delta_r} \pi \, dx$. We get (again for any $\delta > 0$)

$$
\begin{aligned}
|T_2| \leq c \Bigg[& \int_{\Delta_r} \eta |\nabla^2 u| |\pi - \pi_0| |\nabla \eta| \, dx \\
& + \int_{\Delta_r} |\pi - \pi_0| |\nabla u| (|\nabla \eta|^2 + |\nabla^2 \eta|) \, dx \Bigg] \\
\leq \; & c\,\delta \int_{B_{\frac{3}{4}r}(x_0)} \eta^2 |\nabla \varepsilon(u)|^2 \, dx + c(\delta) r^{-2} \int_{\Delta_r} |\pi - \pi_0|^2 \, dx \\
& + c\, r^{-2} \left\{ \int_{B_{\frac{3}{4}r}(x_0)} |\nabla u|^2 \, dx + \int_{\Delta_r} |\pi - \pi_0|^2 \, dx \right\},
\end{aligned}
$$

and if δ is small, the δ-term can be put into the l.h.s. of (5.3). Using also (5.4) it follows

$$
\begin{aligned}
(5.5) \qquad & \int_{B_{\frac{3}{4}r}(x_0)} \eta^2 \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \, dx \\
\leq \; & c\, r^{-2} \Bigg[\int_{B_r(x_0)} h(|\varepsilon(u)|) \, dx + \int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)^2 \, dx + 1 \\
& + \int_{\Delta_r} |\pi - \pi_0|^2 \, dx \Bigg] + c\, |T_3| \, .
\end{aligned}
$$

We have the identity

$$
\begin{aligned}
& \int_{B_{\frac{3}{4}r}(x_0)} u^i \partial_i u^j \partial_k (\eta^2 \partial_k u^j) \, dx = - \int_{B_{\frac{3}{4}r}(x_0)} \partial_k (u^i \partial_i u^j) \eta^2 \partial_k u^j \, dx \\
= \; & - \int_{B_{\frac{3}{4}r}(x_0)} \partial_k u^i \partial_i u^j \partial_k u^j \eta^2 \, dx - \int_{B_{\frac{3}{4}r}(x_0)} u^i \partial_k \partial_i u^j \partial_k u^j \eta^2 \, dx \\
= \; & - \int_{B_{\frac{3}{4}r}(x_0)} \partial_k u^i \partial_i u^j \partial_k u^j \eta^2 \, dx - \frac{1}{2} \int_{B_{\frac{3}{4}r}(x_0)} u^i \partial_i |\nabla u|^2 \eta^2 \, dx \, ,
\end{aligned}
$$

and since we are in the $2D$- case the first integral on the r.h.s. is equal to zero. We therefore have

$$
\begin{aligned}
|T_3| = \; & \frac{1}{2} \left| \int_{B_{\frac{3}{4}r}(x_0)} u^i \partial_i |\nabla u|^2 \eta^2 \, dx \right| \\
= \; & \frac{1}{2} \left| \int_{B_{\frac{3}{4}r}(x_0)} \nabla \eta^2 \cdot u |\nabla u|^2 \, dx \right| \\
\leq \; & c\, r^{-1} \int_{B_{\frac{3}{4}r}(x_0)} |\nabla u|^2 \, dx \, .
\end{aligned}
$$

To the last integral we apply (5.4) and deduce from (5.5)

$$
\begin{aligned}
(5.6)\qquad & \int_{B_{\frac{r}{2}}(x_0)} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}|\nabla\varepsilon(u)|^2\,dx \\
&\leq\; c\,r^{-2}\Bigg[\int_{B_r(x_0)} h(|\varepsilon(u)|)\,dx + \int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)^2\,dx + 1 \\
&\qquad + \int_{\Delta_r} |\pi-\pi_0|^2\,dx\Bigg] + c\,r^{-1}\left[\int_{B_r(x_0)} h\left(|\varepsilon(u)|\right)\,dx + 1\right].
\end{aligned}
$$

Note that in case $u^k\partial_k u = 0$ the last term in (5.6) does not occur. In a next step we discuss the pressure term: by Lemma 2.2 we can construct $w \in \overset{\circ}{W}{}^1_2(\Delta_r;\mathbb{R}^2)$ such that

$$
(5.7)\qquad \left\{\begin{array}{l} \operatorname{div} w = \pi-\pi_0 \text{ on } \Delta_r\,, \\ \|\nabla w\|_{L^2(\Delta_r)} \leq c\|\pi-\pi_0\|_{L^2(\Delta_r)}\,. \end{array}\right.
$$

Equation (1.1) gives

$$
\int_{\Delta_r} \sigma:\varepsilon(w)\,dx + \int_{\Delta_r} u^k\partial_k u\cdot w\,dx = \int_{\Delta_r} \operatorname{div} w(\pi-\pi_0)\,dx\,,
$$

and therefore we get from (5.7) with Young's inequality

$$
(5.8)\qquad \left\{\begin{array}{l} \displaystyle\int_{\Delta_r} |\pi-\pi_0|^2\,dx \leq c\left[\int_{\Delta_r}|\sigma|^2\,dx + |S|\right]\,, \\ \displaystyle S := \int_{\Delta_r} u^k\partial_k u\cdot w\,dx\,. \end{array}\right.
$$

Noting that

$$
S = \int_{\Delta_r} u^k\partial_k u^i w^i\,dx = \int_{\Delta_r} u^k\partial_k\left(u^i-u^i_\infty\right)w^i\,dx = -\int_{\Delta_r} u^k\left(u^i-u^i_\infty\right)\partial_k w^i\,dx\,,
$$

we find (recall (5.7))

$$
|S| \leq c\|u-u_\infty\|_{L^\infty(\Delta_r)}\int_{\Delta_r}|\nabla w|\,dx \leq \delta\int_{\Delta_r}|\nabla w|^2\,dx + c(\delta)r^2\|u-u_\infty\|^2_{L^\infty(\Delta_r)}\,,
$$

and for δ small enough this together with (5.8) implies

$$
(5.9)\qquad \int_{\Delta_r}|\pi-\pi_0|^2\,dx \leq c\left[\int_{\Delta_r}|\sigma|^2\,dx + r^2\|u-u_\infty\|^2_{L^\infty(\Delta_r)}\right].
$$

Inserting (5.9) into (5.6) it is shown that

$$\text{(5.10)} \qquad \int_{B_{\frac{r}{2}}(x_0)} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2 \, dx$$

$$\le \; c\, r^{-2} \left[\int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx + \int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)^2 \, dx + 1 \right]$$

$$+ c \left\{ \|u - u_\infty\|^2_{L^\infty(\Delta_r)} + \frac{1}{r} \int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx + \frac{1}{r} \right\} ,$$

where $\{\dots\}$ does not occur in case $u^k \partial_k u = 0$. Let us remark that from (5.10) we could already deduce $\nabla^2 u \equiv 0$ by passing to the limit $r \to \infty$, provided we are in the situation of Theorem 1.2 (using the estimates (4.1) and (4.19)) and if we could neglect the unpleasant term involving $h'(|\varepsilon(u)|)^2$. Unfortunately we have to discuss this quantity in an next step. For any $L > 0$ it holds (recall (1.6))

$$\text{(5.11)} \qquad \int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)^2 \, dx \le c \left[r^2 h'(L)^2 + \frac{1}{L^2} \int_{B_{\frac{3}{4}r}(x_0)} h(|\varepsilon(u)|)^2 \, dx \right] .$$

Consider a "new" cut-off function η now satisfying $\eta \equiv 1$ on $B_{\frac{3}{4}r}(x_0)$, $\operatorname{spt} \eta \subset B_r(x_o)$, $0 \le \eta \le 1$ and $|\nabla \eta| \le c/r$. Sobolev's inequality implies

$$\int_{B_{\frac{3}{4}r}(x_0)} h(|\varepsilon(u)|)^2 \, dx \le \int_{B_r(x_0)} \left(\eta h(|\varepsilon(u)|) \right)^2 \, dx$$

$$\le \; c \left[\int_{B_r(x_0)} |\nabla \eta| h(|\varepsilon(u)|)\, dx + \int_{B_r(x_0)} h'(|\varepsilon(u)|) |\nabla \varepsilon(u)| \, dx \right]^2$$

$$\le \; c\, r^2 \left(\int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx \right)^2 + c \left(\int_{B_r(x_0)} h'(|\varepsilon(u)|) |\nabla \varepsilon(u)| \, dx \right)^2 ,$$

moreover we have

$$\left(\int_{B_r(x_0)} h'(|\varepsilon(u)|) |\nabla \varepsilon(u)| \, dx \right)^2 \le c \int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx \int_{B_r(x_0)} \omega \, dx ,$$

where we have abbreviated $\omega := \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |\nabla \varepsilon(u)|^2$. Returning to (5.11) we get the inequality

$$\text{(5.12)} \qquad \int_{B_{\frac{3}{4}r}(x_0)} h'(|\varepsilon(u)|)^2 \, dx \le c \Bigg[r^2 h'(L)^2$$

$$+ \frac{1}{L^2} \frac{1}{r^2} \left(\int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx \right)^2$$

$$+ \frac{1}{L^2} \int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx \int_{B_r(x_0)} \omega \, dx \Bigg] .$$

Case 1: $u^k \partial_k u = 0$

Now we just have the information that u is a bounded solution, and the combination of (5.10) (without $\{\ldots\}$!) and (5.12) gives

$$\int_{B_{r/2}(x_0)} \omega\, dx \le \frac{c}{r^2 L^2} \int_{B_r(x_0)} h(|\varepsilon(u)|) \int_{B_r(x_0)} \omega\, dx \\ + c\, r^{-2}\Bigg[1 + \int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx + r^2 h'(L)^2 \\ + \frac{1}{L^2 r^2}\left(\int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx\right)^2\Bigg]. \tag{5.13}$$

Note that (5.13) is true for all $L > 0$ and any disc $B_r(x_0)$. We let $L := \frac{1}{\gamma r}$ for some $\gamma > 0$. (5.13) then takes the form

$$\int_{B_{r/2}(x_0)} \omega\, dx \le c\gamma^2 \int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx \int_{B_r(x_0)} \omega\, dx \\ + c\Bigg[r^{-2} + r^{-2}\int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx + h'\left(\frac{1}{\gamma r}\right)^2 \\ + \gamma^2 r^{-2}\left(\int_{B_r(x_0)} h(|\varepsilon(u)|)\, dx\right)^2\Bigg]. \tag{5.13*}$$

We apply (4.1) and deduce from (5.13)*

$$\int_{B_{\frac{r}{2}}(x_0)} \omega\, dx \le c\,\gamma^2 (r+1) \int_{B_r(x_0)} \omega\, dx + c\left[r^{-2} + r^{-1} + h'\left(\frac{1}{\gamma r}\right)^2 + \gamma^2 r^{-2}(r+1)^2\right].$$

For a positive number β we define

$$\gamma := \sqrt{\beta}\Big/\sqrt{c}\sqrt{1+r}$$

and obtain

$$\int_{B_{\frac{r}{2}}(x_0)} \omega\, dx \le \beta \int_{B_r(x_0)} \omega\, dx + c(\beta)\left[r^{-1} + r^{-2}\right] + c\, h'\left(\frac{1}{\sqrt{\beta}}\sqrt{r^{-1} + r^{-2}}\right)^2. \tag{5.14}$$

From the proof of Lemma 2.5 it is immediate that we have the inequality

$$h'(t) \le c\left[t^{m-1} + t\right],\ t \ge 0, \tag{5.15}$$

and (5.15) clearly implies the bound

$$h'\left(\frac{1}{\sqrt{\beta}}\sqrt{r^{-1}+r^{-2}}\right)^2 \le c(\beta)\left[1+r^{-s}\right]$$

with exponent s (w.l.o.g.) ≥ 2. Inserting this into (5.14) it is shown that

$$\int_{B_{\frac{r}{2}}(x_0)} \omega\,dx \le \beta \int_{B_r(x_0)} \omega\,dx + c(\beta)\left[1+r^{-s}\right] \tag{5.16}$$

for all discs $B_r(x_0)$ and any $\beta > 0$. Noting the validity of

$$1+r^{-s} \le c\left[\int_{B_r(x_0)} 1\,dx + r^{-s-2}\int_{B_r(x_0)} 1\,dx\right]$$

we deduce from (5.16) with the help of Lemma 2.1

$$\int_{B_r(x_0)} \omega\,dx \le c\left[r^2+r^{-s}\right]\,. \tag{5.17}$$

Now let $x_0 = 0$ and consider $r \ge 1$. Then (5.17) shows

$$\int_{B_r(x_0)} \omega\,dx \le c\,r^2\,,$$

and if we insert this estimate in (5.13)* choosing $\gamma = 1/r$, we immediately arrive at

$$\int_{B_{r/2}} \omega\,dx \le c\left[1+(1+r^{-2})\int_{B_r} h(|\varepsilon(u)|)\,dx + r^{-4}\left(\int_{B_r} h(|\varepsilon(u)|)\,dx\right)^2\right] \tag{5.18}$$

valid for all $r \ge 1$. Quoting (4.19) we obtain from (5.18) the upper bound

$$\int_{B_t} \omega\,dx \le c\,t^{2/3},\ t \ge 1\,. \tag{5.19}$$

With (5.19) we again go back to (5.13)* using (4.19) for the integrals involving h and get

$$\int_{B_{r/2}} \omega\,dx \le c\gamma^2 r^{4/3} + c\left[r^{-2}+r^{-4/3}+h'\left(\frac{1}{\gamma r}\right)^2+\gamma^2 r^{-2/3}\right],\ r \ge 1\,,$$

thus the choice $\gamma = r^{-1+\delta}$ for some small positive δ immediately yields by passing to limit $r \to \infty$

$$\int_{\mathbb{R}^2} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}|\nabla\varepsilon(u)|^2\,dx = 0\,.$$

On account of (A3) and $h''(0) > 0$ we find $\nabla^2 u = 0$, hence u is affine, but the boundedness of u shows that u must be constant.

Case 2: $u_k\partial_k u$ not necessarily zero

Now we have to take care about the expression

$$\{\ldots\} := \|u-u_\infty\|^2_{L^\infty(\Delta_r)} + \frac{1}{r}\int_{B_r(x_0)} h(|\varepsilon(u)|)\,dx + \frac{1}{r}$$

from (5.10), which means that in place of (5.13)* we get the inequality (valid for all $\gamma > 0$ and any disc $B_r(x_0)$)

$$\int_{B_{\frac{r}{2}}(x_0)} \omega\,dx \le (\text{r.h.s. of } (5.13)^*) + c\{\ldots\}\,. \tag{5.20}$$

On the r.h.s. of (5.20) we bound all integrals involving h with the help of (4.1) and $\|u-u_\infty\|_{L^\infty(\Delta_r)}$ is estimated through a constant. As a result we get in place of (5.16) (following the arguments outlined after (5.13)*)

$$\int_{B_{r/2}(x_0)} \omega\,dx \le \beta\int_{B_r(x_0)} \omega\,dx + c(\beta)[1+r^{-s}] + c[1+r^{-1}]\,,$$

hence with new $c(\beta)$

$$\begin{aligned}\int_{B_{r/2}(x_0)} \omega\,dx &\le \beta\int_{B_r(x_0)} \omega\,dx + c(\beta)[1+r^{-s}]\\ &\le \beta\int_{B_r(x_0)} \omega\,dx + c(\beta)\left[\int_{B_r(x_0)} 1\,dx + r^{-s-2}\int_{B_r(x_0)} 1\,dx\right].\end{aligned}$$

The arbitrariness of β and $B_r(x_0)$ then again yields (5.17) by an application of Lemma 2.1. Next let $x_0 = 0$ and consider $r \ge 1$. As in case 1 we insert (5.17) into the r.h.s. of (5.20) and choose $\gamma = 1/r$. In place of (5.18) we get

$$\int_{B_{r/2}} \omega\,dx \le c\left[1 + (1+r^{-2})\int_{B_r} h(|\varepsilon(u)|)\,dx + r^{-4}\left(\int_{B_r} h(|\varepsilon(u)|)\,dx\right)^2\right] + c\{\ldots\} \tag{5.21}$$

We here know that

$$\alpha(r) := \sup_{\mathbb{R}^2 - B_r} |u-u_\infty| \to 0,\ r\to\infty\,,$$

and by quoting (4.19) it is immediate that

$$\{\ldots\} \to 0,\ r\to\infty\,.$$

For large t inequality (4.19) states that

$$\begin{aligned}&\int_{B_t} h(|\varepsilon(u)|)\,dx \le c\Theta(t)\,,\\ &\Theta(t) := t^{2/3} + t\alpha(t) + t\alpha(t)^2\,,\end{aligned} \tag{5.22}$$

and it is easy to see that (5.21) implies the same bound for $\int_{B_r} \omega\, dx$, i.e.

$$\int_{B_r} \omega\, dx \le c\,\Theta(r),\ r \ge 1\,. \tag{5.23}$$

Finally, we again return to (5.20) using (5.22) and (5.23) on the r.h.s. with the result

$$\int_{B_{r/2}} \omega\, dx \le c\gamma^2\Theta(r)^2 + c\left[r^{-2} + r^{-2}\Theta(r) + h'\left(\frac{1}{\gamma r}\right)^2 + \gamma^2 r^{-2}\Theta(r)^2\right] + c\{\ldots\}\,, \tag{5.24}$$

and the r.h.s. of (5.24) disappears as $r \to \infty$ for the choice $\gamma := \frac{1}{r}\min\{r^{1/4}, \frac{1}{\sqrt{\alpha(r)}}\}$: in fact we have as $r \to \infty$

$$\gamma r = \min\left\{r^{1/4}, \frac{1}{\sqrt{\alpha(r)}}\right\} \longrightarrow \infty,\ h'\left(\frac{1}{\gamma r}\right)^2 \longrightarrow 0\,,$$

and

$$\begin{aligned}
\gamma^2\Theta(r)^2 \le c\,\gamma^2 &\left[r^{4/3} + r^2\alpha(r)^2 + r^2\alpha(r)^4\right] \\
\le\ \ c &\left[r^{-2}r^{4/3}r^{1/2} + \frac{1}{\alpha(r)}\alpha(r)^2 + \frac{1}{\alpha(r)}\alpha(r)^4\right] \\
=\ \ c &\left[r^{-1/6} + \alpha(r) + \alpha(r)^3\right] \longrightarrow 0\,.
\end{aligned}$$

As in case 1 we deduce $u = \text{const}$, and the proof of Theorem 1.2 is complete. □

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