Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 288

On the marginals of probability contents on lattices

Heinz König

Saarbrücken 2011

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 288
submitted: November 28, 2011

On the marginals of probability contents on lattices

Heinz König

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
hkoenig@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

To appear in Mathematika

ON THE MARGINALS OF PROBABILITY CONTENTS ON LATTICES

HEINZ KÖNIG

ABSTRACT. The paper extends the fundamental existence assertion for probability contents and measures with given marginals: The extension is from *algebras* to *lattices*, and thus is in accord with an actual trend in measure and integration. The proof of the basic theorem is a rapid application of a former Hahn-Banach type separation theorem.

We start from the well-known theorem of Strassen [6] Theorem 6 on the existence of probability measures with given marginals. We also refer to Jacobs [1] Appendix B. In the version of Fremlin [2] Proposition 457D, that is under the usual product formation and in terms of probability contents on a nonvoid set X, the theorem reads as follows.

THEOREM 1. *Let* $\mathfrak{P}$ *and* $\mathfrak{Q}$ *be algebras in* X*, and* $\varphi : \mathfrak{P} \to [0, \infty[$ *and* $\psi : \mathfrak{Q} \to [0, \infty[$ *be probability contents. For an algebra* $\mathfrak{A}$ *in* X *with* $\mathfrak{P}, \mathfrak{Q} \subset \mathfrak{A}$ *and a content* $\vartheta : \mathfrak{A} \to [0, \infty]$ *then*

there exists a probability content $\gamma : \mathfrak{A} \to [0, \infty[$ *with* $\gamma \leqq \vartheta$
which extends φ *and* ψ
$\Longleftrightarrow \varphi(A) + \psi(B) \leqq 1 + \vartheta(A \cap B)$ *for all* $A \in \mathfrak{P}$ *and* $B \in \mathfrak{Q}$.

As to the transition from contents to measures, we restrict ourselves to the obvious remark that γ is upward and downward σ continuous when ϑ is downward σ continuous at $\varnothing$. Besides [1] Appendix B we also refer to the results listed in [5] Section 3.

For the subsequent extension we define as usual $\varphi : \mathfrak{S} \to [0, \infty]$ to be a *content* on a *lattice* $\mathfrak{S}$ in X if $\varnothing \in \mathfrak{S}$ and φ is isotone with $\varphi(\varnothing) = 0$ and

modular: $\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B)$ for all $A, B \in \mathfrak{S}$,

with *submodular* and *supermodular* defined to mean $\leqq$ and $\geqq$ instead of $=$; and we define $\varphi : \mathfrak{S} \to [0, \infty[$ to be a *probability content* if in addition $X \in \mathfrak{S}$ and $\varphi(X) = 1$. Then our extension reads as follows.

THEOREM 2. *Let* $\mathfrak{P}$ *and* $\mathfrak{Q}$ *be lattices in* X *which contain* $\varnothing$ *and* X*, and* $\varphi : \mathfrak{P} \to [0, \infty[$ *and* $\psi : \mathfrak{Q} \to [0, \infty[$ *be isotone and supermodular with* $\varphi(\varnothing) = \psi(\varnothing) = 0$ *and* $\varphi(X) = \psi(X) = 1$*. For a lattice* $\mathfrak{A}$ *in* X *with* $\mathfrak{P}, \mathfrak{Q} \subset \mathfrak{A}$ *and a content* $\vartheta : \mathfrak{A} \to [0, \infty]$ *then*

there exists a probability content $\gamma : \mathfrak{A} \to [0, \infty[$ *with* $\gamma \leqq \vartheta$
such that $\varphi \leqq \gamma|\mathfrak{P}$ *and* $\psi \leqq \gamma|\mathfrak{Q}$
$\Longleftrightarrow \varphi(A) + \psi(B) \leqq 1 + \vartheta(A \cap B)$ *for all* $A \in \mathfrak{P}$ *and* $B \in \mathfrak{Q}$.

1991 *Mathematics Subject Classification.* 28A12.
Key words and phrases. Probability contents and measures, marginals, Hahn-Banach type theorems.

Of course $\varphi \leqq \gamma|\mathfrak{P} \Leftrightarrow \varphi = \gamma|\mathfrak{P}$ when $\mathfrak{P}$ is an algebra and φ is a probability content, and the same for ψ. Thus the new assertion contains the former one. Also γ is downward σ continuous at $\varnothing$ when ϑ is so.

The proof of Theorem 2 is quite short. The basic point is the Hahn-Banach type separation result [4] Theorem 1.2 which follows. Its proof in [4] combines the Hahn-Banach version [4] Theorem 1.1 with [3] Theorem 11.11 for the Choquet integral.

Separation Theorem. *On a lattice $\mathfrak{S}$ in X with $\varnothing \in \mathfrak{S}$ let*

$\alpha : \mathfrak{S} \to [0, \infty]$ *be isotone with* $\alpha(\varnothing) = 0$ *and supermodular,*
$\beta : \mathfrak{S} \to [0, \infty]$ *be isotone with* $\beta(\varnothing) = 0$ *and submodular,*

and $\alpha \leqq \beta$. *Then there exists a content* $\gamma : \mathfrak{S} \to [0, \infty]$ *such that* $\alpha \leqq \gamma \leqq \beta$.

An important consequence is [4] Theorem 1.3: *Each content* $\vartheta : \mathfrak{S} \to [0, \infty]$ *on a lattice* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *can be extended to a content* $\Theta : \mathfrak{P}(X) \to [0, \infty]$. In fact, this follows from the Separation Theorem applied to the familiar envelopes

$\vartheta_\star : \mathfrak{P}(X) \to [0, \infty]$ defined $\vartheta_\star(A) = \sup\{\vartheta(S) : S \in \mathfrak{S} \text{ with } S \subset A\}$,
$\vartheta^\star : \mathfrak{P}(X) \to [0, \infty]$ defined $\vartheta^\star(A) = \inf\{\vartheta(S) : S \in \mathfrak{S} \text{ with } S \supset A\}$.

In this context we note that the isotone set functions $\vartheta : \mathfrak{S} \to [0, \infty]$ with $\vartheta(\varnothing) = 0$ fulfil

ϑ supermodular $\Rightarrow$ $\vartheta_\star$ supermodular,
ϑ submodular $\Rightarrow$ $\vartheta^\star$ submodular.

It follows that the particular Separation Theorem for the domain $\mathfrak{P}(X)$ implies the full theorem for all lattices $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$.

The proof of Theorem 2 requires one more lemma.

Lemma. *Let $\mathfrak{A}$ be a lattice in X and $\vartheta : \mathfrak{A} \to [0, \infty]$ be isotone and modular. For $A, B, U, V \in \mathfrak{A}$ then*

$$\vartheta\big((A \cup B) \cap (U \cap V)\big) + \vartheta\big((A \cap B) \cap (U \cup V)\big) \leqq \vartheta(A \cap U) + \vartheta(B \cap V).$$

Proof of the Lemma. We can assume that $\vartheta(A \cap U), \vartheta(B \cap V) < \infty$. The left side of the assertion is

$$\begin{aligned}
&= \vartheta\big((A \cap U \cap V) \cup (B \cap U \cap V)\big) + \vartheta\big(A \cap B \cap U) \cup (A \cap B \cap V)\big) \\
&= \vartheta(A \cap U \cap V) + \vartheta(B \cap U \cap V) - \vartheta(A \cap B \cap U \cap V) \\
&+ \vartheta(A \cap B \cap U) + \vartheta(A \cap B \cap V) - \vartheta(A \cap B \cap U \cap V) \\
&= \vartheta\big((A \cap U) \cap B\big) + \vartheta\big((A \cap U) \cap V\big) - \vartheta\big((A \cap U) \cap (B \cap V)\big) \\
&+ \vartheta\big(A \cap (B \cap V)\big) + \vartheta\big(U \cap ((B \cap V)\big) - \vartheta\big((A \cap U) \cap (B \cap V)\big) \\
&= \vartheta\big((A \cap U) \cap (B \cup V)\big) + \vartheta\big((A \cup U) \cap (B \cap V)\big),
\end{aligned}$$

and this is $\leqq \vartheta(A \cap U) + \vartheta(B \cap V)$. $\square$

Proof of Theorem 2. The implication $\Longrightarrow$ is clear. For the proof of $\Longleftarrow$ let $\Theta : \mathfrak{P}(X) \to [0, \infty]$ be a content which extends ϑ. 1) For $A \subset X$ and for $P \in \mathfrak{P}$ with $P \subset A$ and $Q \in \mathfrak{Q}$ the assumption shows that $\varphi(P) \leqq 1 - \psi(Q) + \vartheta(P \cap Q) \leqq 1 - \psi(Q) + \Theta(A \cap Q)$. We define $\alpha, \beta : \mathfrak{P}(X) \to [0, \infty]$ to be

$\alpha(A) = \sup\{\varphi(P) : P \in \mathfrak{P} \text{ with } P \subset A\} = \varphi_\star(A)$,
$\beta(A) = \inf\{1 - \psi(Q) + \Theta(A \cap Q) : Q \in \mathfrak{Q}\}$.

It is clear that α and β are isotone with $\alpha(\varnothing)=\beta(\varnothing)=0$, and the above shows that $\alpha \leqq \beta$. From $1 \leqq \alpha(X) \leqq \beta(X) \leqq 1$ then $\alpha(X)=\beta(X)=1$.

2) It is obvious that α is supermodular. We show that β is submodular: For $A, B \subset X$ and $U, V \in \mathfrak{Q}$ we obtain from the Lemma

$$\big(1-\psi(U)+\Theta(A\cap U)\big)+\big(1-\psi(V)+\Theta(B\cap V)\big)$$
$$\geqq 1-\psi(U\cup V)+\Theta\big((A\cap B)\cap(U\cup V)\big)+1-\psi(U\cap V)+\Theta\big((A\cup B)\cap(U\cap V)\big)$$
$$\geqq \beta(A\cap B)+\beta(A\cup B),$$

and hence the assertion.

3) Now the Separation Theorem furnishes a content $\Gamma : \mathfrak{P}(X) \to [0,\infty[$ with $\alpha \leqq \Gamma \leqq \beta$. Thus $\Gamma(X)=1$, so that Γ is a probability content. From $\alpha \leqq \Gamma$ we obtain $\varphi \leqq \Gamma|\mathfrak{P}$. And $\Gamma \leqq \beta$ means that $\Gamma(A) \leqq 1-\psi(Q)+\Theta(A\cap Q)$ for $A \subset X$ and $Q \in \mathfrak{Q}$. Thus on the one hand $Q := X$ furnishes $\Gamma(A) \leqq \Theta(A)$ for $A \subset X$. On the other hand we obtain for $Q \in \mathfrak{Q}$ and $A := Q'$ that $1-\Gamma(Q)=\Gamma(Q') \leqq 1-\psi(Q)$ or $\psi(Q) \leqq \Gamma(Q)$. It follows that $\gamma := \Gamma|\mathfrak{A}$ is as required. $\square$

In conclusion we want to transform our theorem into the traditional version in terms of marginals. The notations will be as follows. Let $H : X \to Y$ be a map between nonvoid sets X and Y. For a set system $\mathfrak{A}$ in X one defines the image set system $\vec{H}\mathfrak{A} := \{B \subset Y : H^{-1}(B) \in \mathfrak{A}\}$ in Y. Then

$\mathfrak{A}$ lattice in $X \Rightarrow \vec{H}\mathfrak{A}$ lattice in Y,
$\varnothing \in \mathfrak{A} \Rightarrow \varnothing \in \vec{H}\mathfrak{A}$ and $X \in \mathfrak{A} \Rightarrow Y \in \vec{H}\mathfrak{A}$,
$\mathfrak{A}$ algebra in $X \Rightarrow \vec{H}\mathfrak{A}$ algebra in Y.

For a set function $\alpha : \mathfrak{A} \to [0,\infty]$ one defines the image set function $\vec{H}\alpha : \vec{H}\mathfrak{A} \to [0,\infty]$ to be $\vec{H}\alpha(B)=\alpha\big(H^{-1}(B)\big)$. Then α content $\Rightarrow \vec{H}\alpha$ content, etc.

After this we fix nonvoid sets X and Y, with the product set $Z = X \times Y$ and the canonical projections $I : Z \to X$ and $J : Z \to Y$. We assume

$\mathfrak{P}$ lattice in X which contains $\varnothing$ and X,
$\mathfrak{Q}$ lattice in Y which contains $\varnothing$ and Y,
$\mathfrak{A}$ lattice in Z such that $\mathfrak{P} \subset \vec{I}\mathfrak{A}$ and $\mathfrak{Q} \subset \vec{J}\mathfrak{A}$;

the last two relations mean $A \times Y = I^{-1}(A) \in \mathfrak{A}\ \forall A \in \mathfrak{P}$ and $X \times B = J^{-1}(B) \in \mathfrak{A}\ \forall B \in \mathfrak{Q}$, and hence combine to $\mathfrak{P} \times \mathfrak{Q} \subset \mathfrak{A}$. Then a probability content $\gamma : \mathfrak{A} \to [0,\infty[$ produces the probability contents $\vec{I}\gamma|\mathfrak{P}$ on $\mathfrak{P}$ and $\vec{J}\gamma|\mathfrak{Q}$ on $\mathfrak{Q}$, the so-called *marginals* of γ. In these terms the transformed theorem reads as follows.

THEOREM 3. *Let $\varphi : \mathfrak{P} \to [0,\infty[$ and $\psi : \mathfrak{Q} \to [0,\infty[$ be isotone and supermodular with $\varphi(\varnothing)=\psi(\varnothing)=0$ and $\varphi(X)=\psi(Y)=1$, and $\vartheta : \mathfrak{A} \to [0,\infty]$ be a content. Then*

there exists a probability content $\gamma : \mathfrak{A} \to [0,\infty[$ with $\gamma \leqq \vartheta$
such that $\varphi \leqq \vec{I}\gamma|\mathfrak{P}$ and $\psi \leqq \vec{J}\gamma|\mathfrak{Q}$
$\iff \varphi(A)+\psi(B) \leqq 1+\vartheta(A\times B)$ for all $A \in \mathfrak{P}$ and $B \in \mathfrak{Q}$.

Proof. By assumption $\tilde{\mathfrak{P}} := \{A \times Y : A \in \mathfrak{P}\}$ and $\tilde{\mathfrak{Q}} := \{X \times B : B \in \mathfrak{Q}\}$ are lattices in Z which contain $\varnothing$ and Z and fulfil $\tilde{\mathfrak{P}}, \tilde{\mathfrak{Q}} \subset \mathfrak{A}$. And

$\tilde{\varphi} : \tilde{\mathfrak{P}} \to [0,\infty[$ defined to be $\tilde{\varphi}(A \times Y)=\varphi(A)$ for $A \in \mathfrak{P}$ and
$\tilde{\psi} : \tilde{\mathfrak{Q}} \to [0,\infty[$ defined to be $\tilde{\psi}(X \times B)=\psi(B)$ for $B \in \mathfrak{Q}$

are isotone and supermodular with $\tilde{\varphi}(\varnothing) = \tilde{\psi}(\varnothing) = 0$ and $\tilde{\varphi}(Z) = \tilde{\psi}(Z) = 1$. For a probability content $\gamma : \mathfrak{A} \to [0, \infty[$ we have

$$\tilde{\varphi} \leqq \gamma|\tilde{\mathfrak{P}} \text{ or } \tilde{\varphi}(A \times Y) \leqq \gamma\big(I^{-1}(A)\big)\ \forall A \in \mathfrak{P} \iff \varphi \leqq \vec{I}\gamma|\mathfrak{P},$$
$$\tilde{\psi} \leqq \gamma|\tilde{\mathfrak{Q}} \text{ or } \tilde{\psi}(X \times B) \leqq \gamma\big(J^{-1}(B)\big)\ \forall B \in \mathfrak{Q} \iff \psi \leqq \vec{J}\gamma|\mathfrak{Q}.$$

Now in Theorem 2 the condition for $\gamma \leqq \vartheta$ combined with $\tilde{\varphi} \leqq \gamma|\tilde{\mathfrak{P}}$ and $\tilde{\psi} \leqq \gamma|\tilde{\mathfrak{Q}}$ reads

$$\tilde{\varphi}(A \times Y) + \tilde{\psi}(X \times B) \leqq 1 + \vartheta\big((A \times Y) \cap (X \times B)\big) \text{ for } A \in \mathfrak{P} \text{ and } B \in \mathfrak{Q},$$

that is $\varphi(A) + \psi(B) \leqq 1 + \vartheta(A \times B)$ for $A \in \mathfrak{P}$ and $B \in \mathfrak{Q}$. Thus Theorem 2 turns at once into the present assertion. □

As before we also have $\varphi \leqq \vec{I}\gamma|\mathfrak{P} \Leftrightarrow \varphi = \vec{I}\gamma|\mathfrak{P}$ when $\mathfrak{P}$ is an algebra and φ is a probability content, and the same for ψ. And of course γ is downward σ continuous at $\varnothing$ when ϑ is so.

Acknowledgement. The author is most grateful to David Fremlin for an advice which gave rise to the present substantial improvement of the previous version of the article.

Note added 19 November 2011. The author likewise wants to thank the anonymous referee who reminded him of two previous articles of Jürgen Kindler: Kindler's paper in Journ.Math.An.Appl.120(1986) (cited in [3] [4]) contains the present Separation Theorem (even with reference to earlier work of Przemysław Kranz). And the final result in his rich paper in Math.Nachr.134(1987) (cited in [3]) obtains a version of the present Theorem 2, which however is restricted in view of tightness assumptions.

References

[1] K.Jacobs, Measure and Integral. Academic Press 1978.
[2] D.H.Fremlin, Measure Theory, Vol.4. Torres Fremlin 2003, reprint 2006.
[3] H.König, Measure and Integration. Springer 1997, reprint 2009.
[4] H.König, Upper envelopes of inner premeasures. Ann.Inst.Fourier 50,2(2000), 401-422.
[5] D.Ramachandran, Perfect measures and related topics. In: Handbook of Measure Theory (ed.E.Pap), Elsevier 2002, pp.766-786.
[6] V.Strassen, The existence of probability measures with given marginals. Ann.Math. Statist. 36(1965), 423-439.

Universität des Saarlandes, Fakultät für Mathematik und Informatik, D-66041 Saarbrücken, Germany
E-mail address: hkoenig@math.uni-sb.de